Business Processes Modelling

MPB (6 cfu, 295AA)

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10 - Liveness

Object

$$N \vdash \psi$$

We give a formal account of a key property of Petri nets

Free Choice Nets (book, optional reading)
https://www7.in.tum.de/~esparza/bookfc.html

Digression

How to disprove an implication?

$$P \not\Rightarrow Q$$

$$P \wedge \neg Q$$

Let's practice with some formula

$$N = (P, T, F, M_0)$$

for any markings M and M' it follows that M' is reachable from M_0

$$\forall M, M', \quad M \in [M_0\rangle \wedge M' \in [M\rangle \quad \Rightarrow \quad M' \in [M_0\rangle$$

with M reachable from M_0 and M' reachable from M

any marking that is reachable from a reachable marking is also reachable

Disclaim

When we say:
for any reachable marking M

we mean:
for any marking M reachable from M_0

Boundedness

A place p can be:

safe / **bounded**
will never contain more than a certain amount of tokens

unbounded
can arrive to contain more tokens than any fixed quantity

Petri nets: liveness

Liveness

A transition t can be:

live

can always be enabled in the future

dead

can never be enabled in the future

non live

can become dead (or is dead already)

non dead

can be enabled at least once

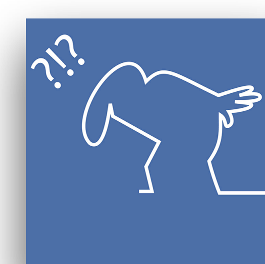
Liveness

A place p can be:

live
can always be marked in the future

dead
can never be marked in the future
(it is empty and will always remain empty)

non live
can become dead (or is dead already)
non dead
can become marked at least once



Liveness, intuitively

At any point in time of the computation, we cannot exclude that t will fire in the future

or, equivalently,

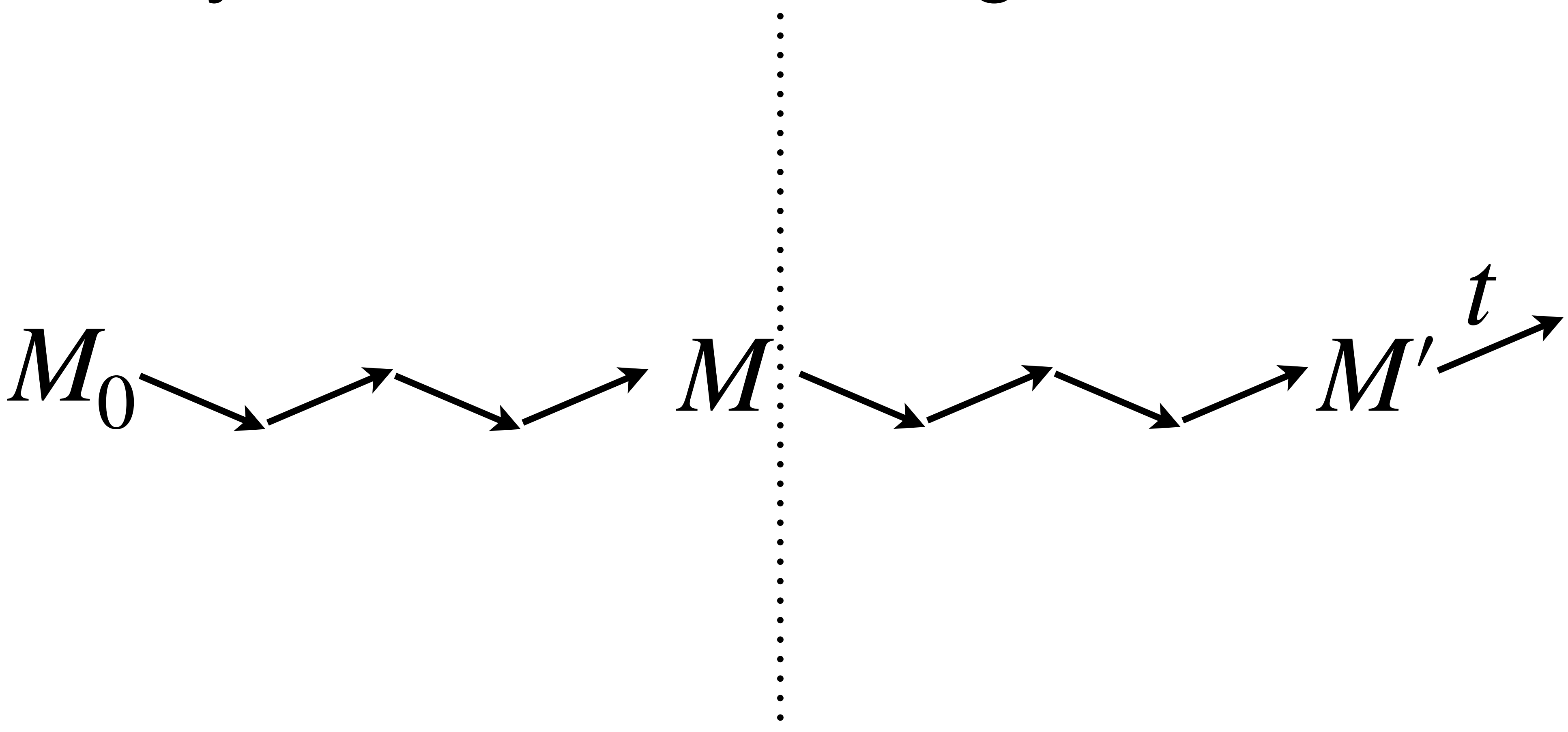
at any point in time of the computation, it is still possible to enable t in the future

A transition t is **live** if from any reachable marking M another marking M' can be reached where t is enabled

A Petri net is **live** if all of its transitions are live

Liveness illustrated

For any reachable marking M…



… we can find a way to enable t

Liveness, formally

$$(P, T, F, M_0)$$

for any transition t there is a marking M' reachable from M

$$\forall t \in T, \quad \forall M \in [\, M_0 \,\rangle, \quad \exists M' \in [\, M \,\rangle, \quad M' \xrightarrow{t}$$

for any reachable marking M that enables t

Liveness: pay attention!

Liveness of t should not be confused with the following property:

starting from the initial marking M_0 it is possible to reach a marking M that enables t

$$\exists M \in [M_0\rangle.\ M \xrightarrow{t}$$

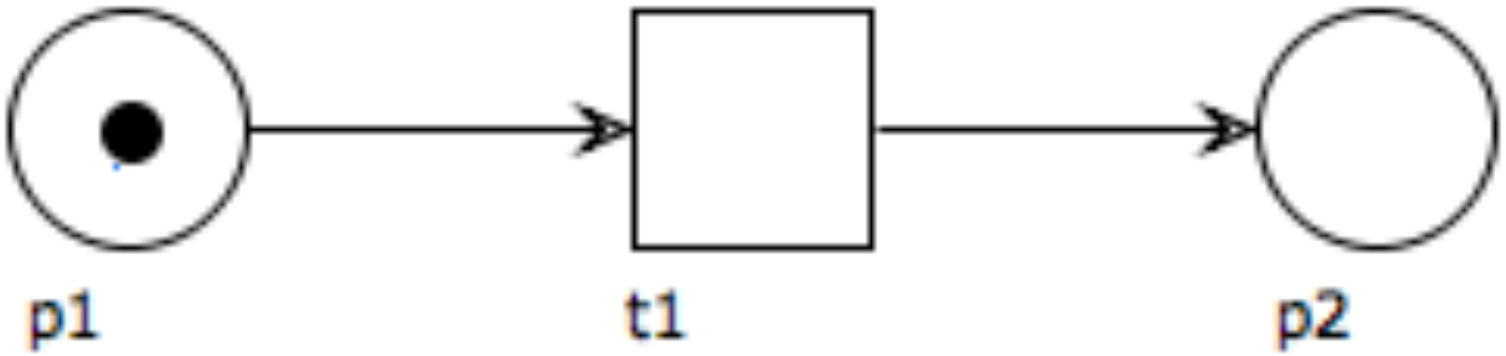
(this property just ensures that t is not "dead" at M_0)

Dead transition

Given a marking M

A transition t is **dead** at M
if t will never be enabled in the future
(i.e., t is not enabled at any marking reachable from M)

Example



$M_0 = p_1$

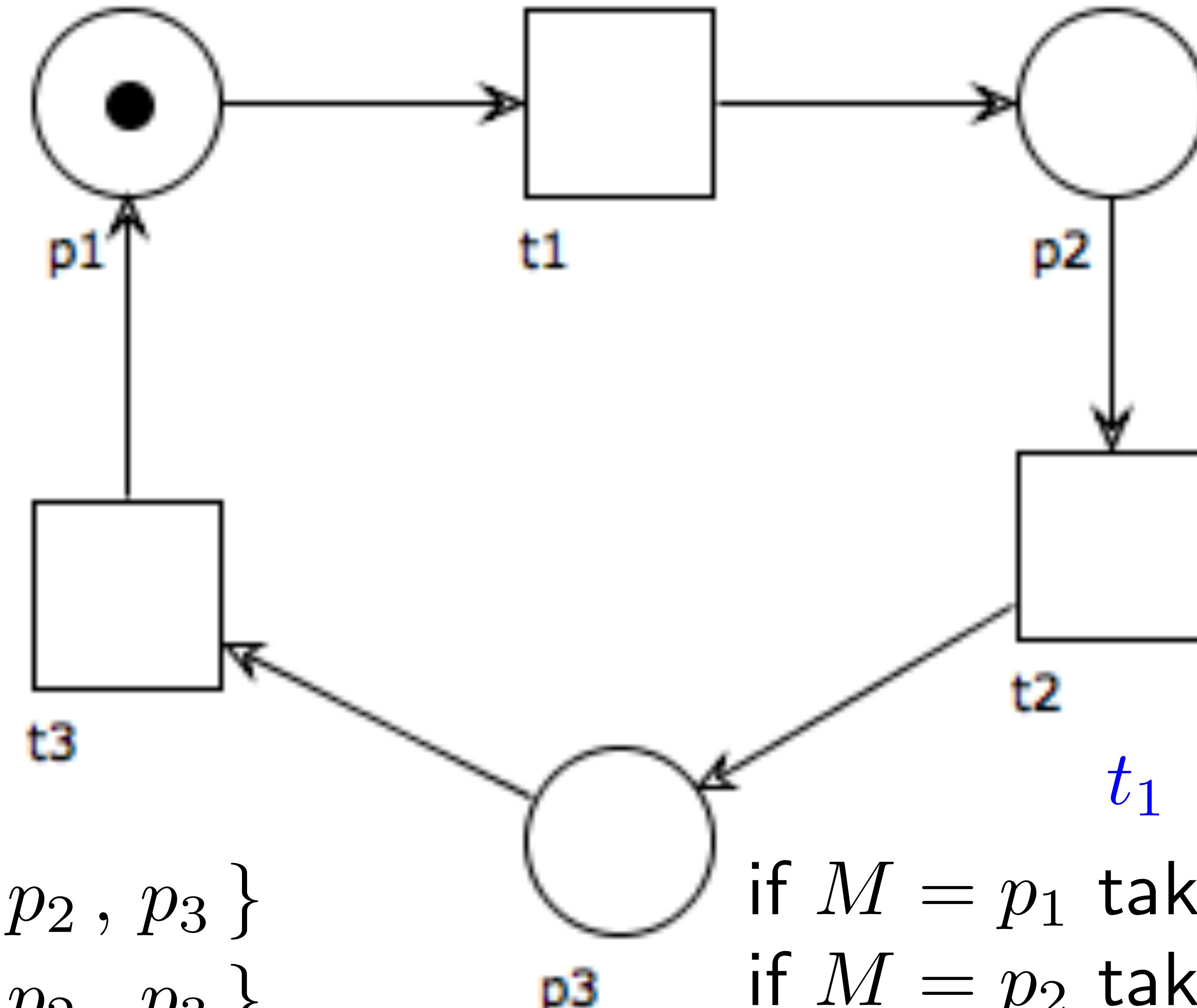
$[M_0\rangle = \{\, p_1 \,,\, p_2 \,\}$

$p_2 \not\rightarrow$

t_1 non live!

t_1 non dead!

Example: Live



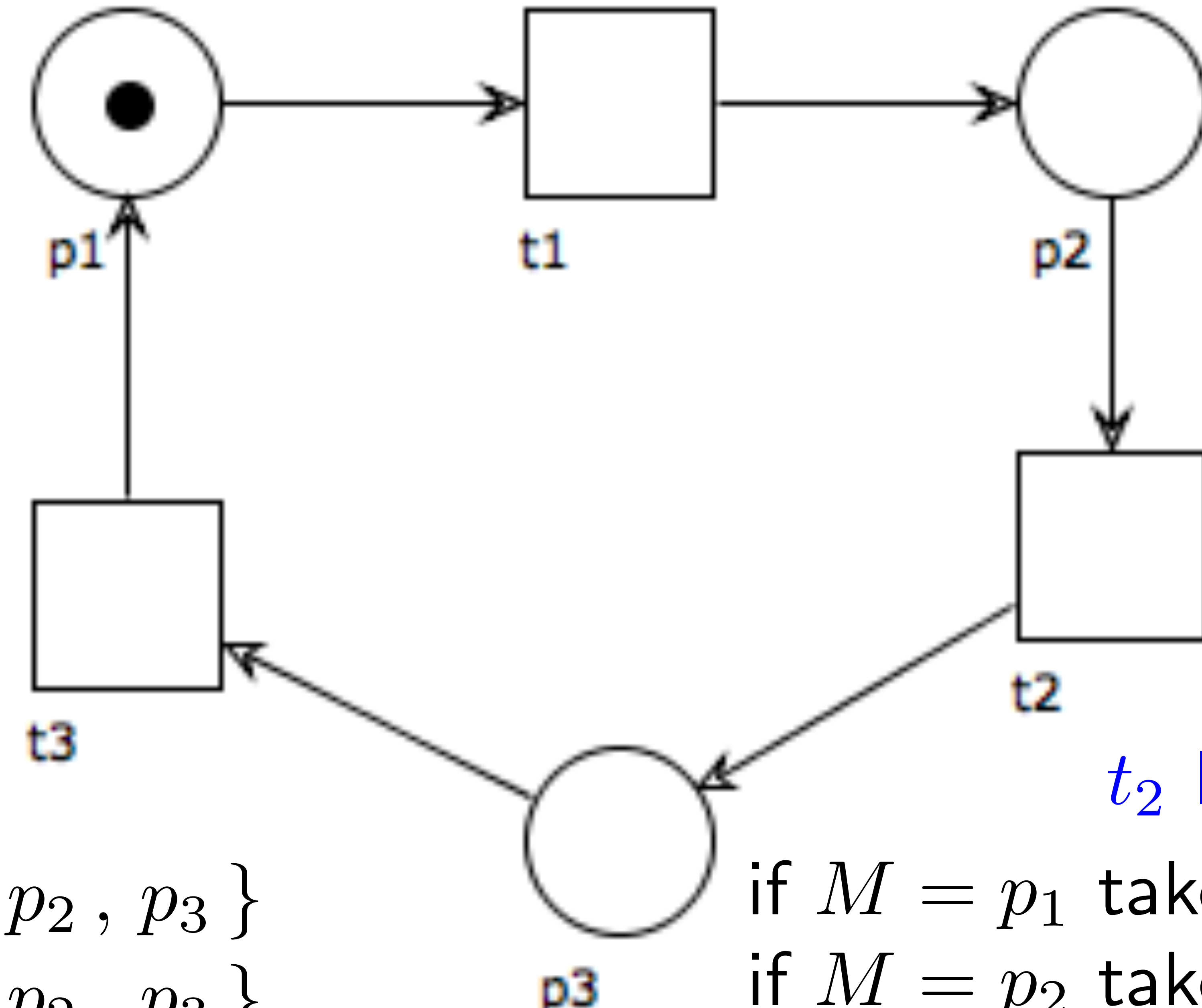
$M_0 = p_1$

$[M_0\rangle = \{\, p_1 \,,\, p_2 \,,\, p_3 \,\}$
$[p_2\rangle = \{\, p_1 \,,\, p_2 \,,\, p_3 \,\}$
$[p_3\rangle = \{\, p_1 \,,\, p_2 \,,\, p_3 \,\}$

t_1 live

if $M = p_1$ take $M' = p_1 \xrightarrow{t_1}$
if $M = p_2$ take $M' = p_1 \xrightarrow{t_1}$
if $M = p_3$ take $M' = p_1 \xrightarrow{t_1}$

Example: Live



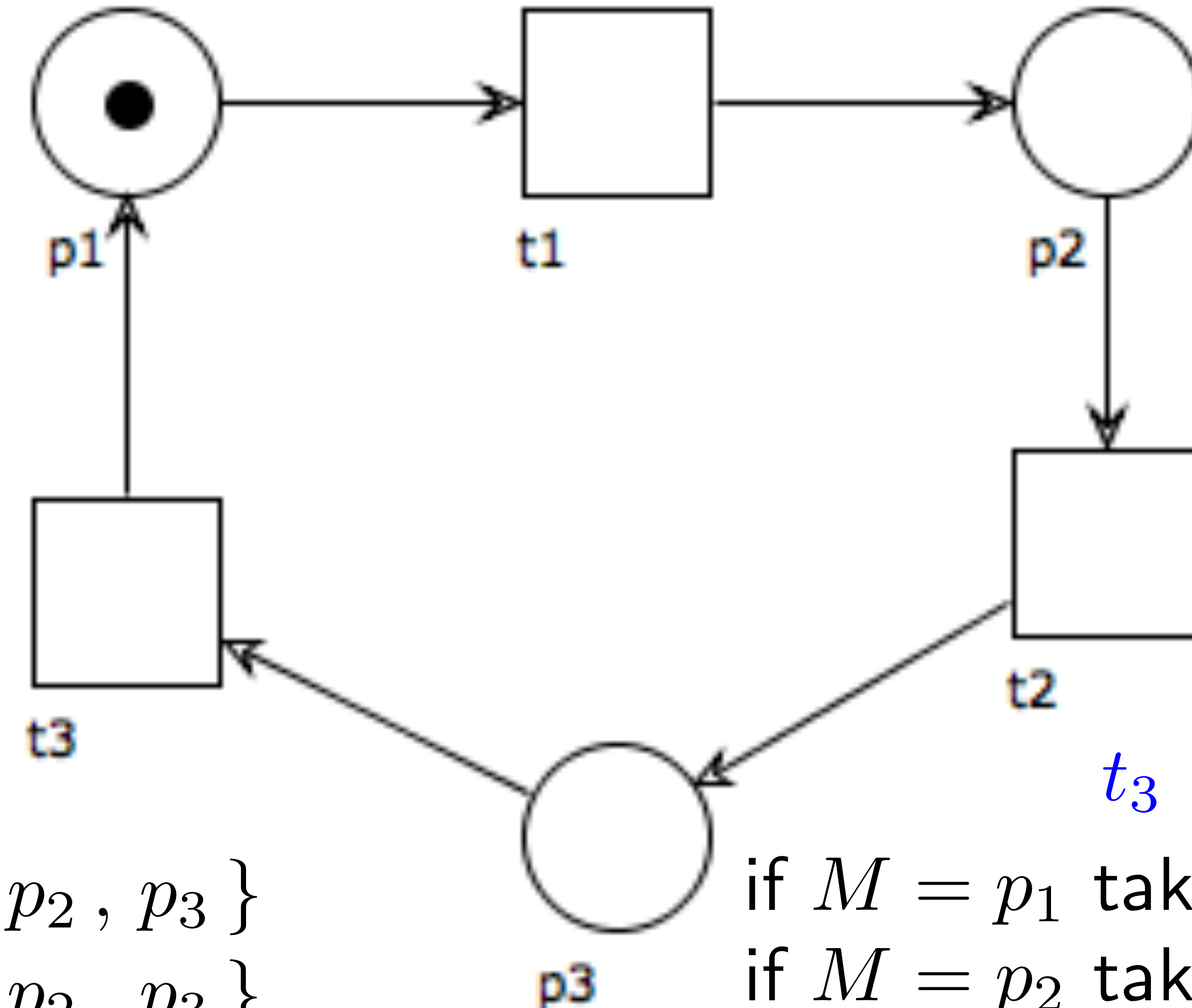
$M_0 = p_1$

$[M_0\rangle = \{\, p_1 \,,\, p_2 \,,\, p_3 \,\}$
$[p_2\rangle = \{\, p_1 \,,\, p_2 \,,\, p_3 \,\}$
$[p_3\rangle = \{\, p_1 \,,\, p_2 \,,\, p_3 \,\}$

t_2 live

if $M = p_1$ take $M' = p_2 \xrightarrow{t_2}$
if $M = p_2$ take $M' = p_2 \xrightarrow{t_2}$
if $M = p_3$ take $M' = p_2 \xrightarrow{t_2}$

Example: Live



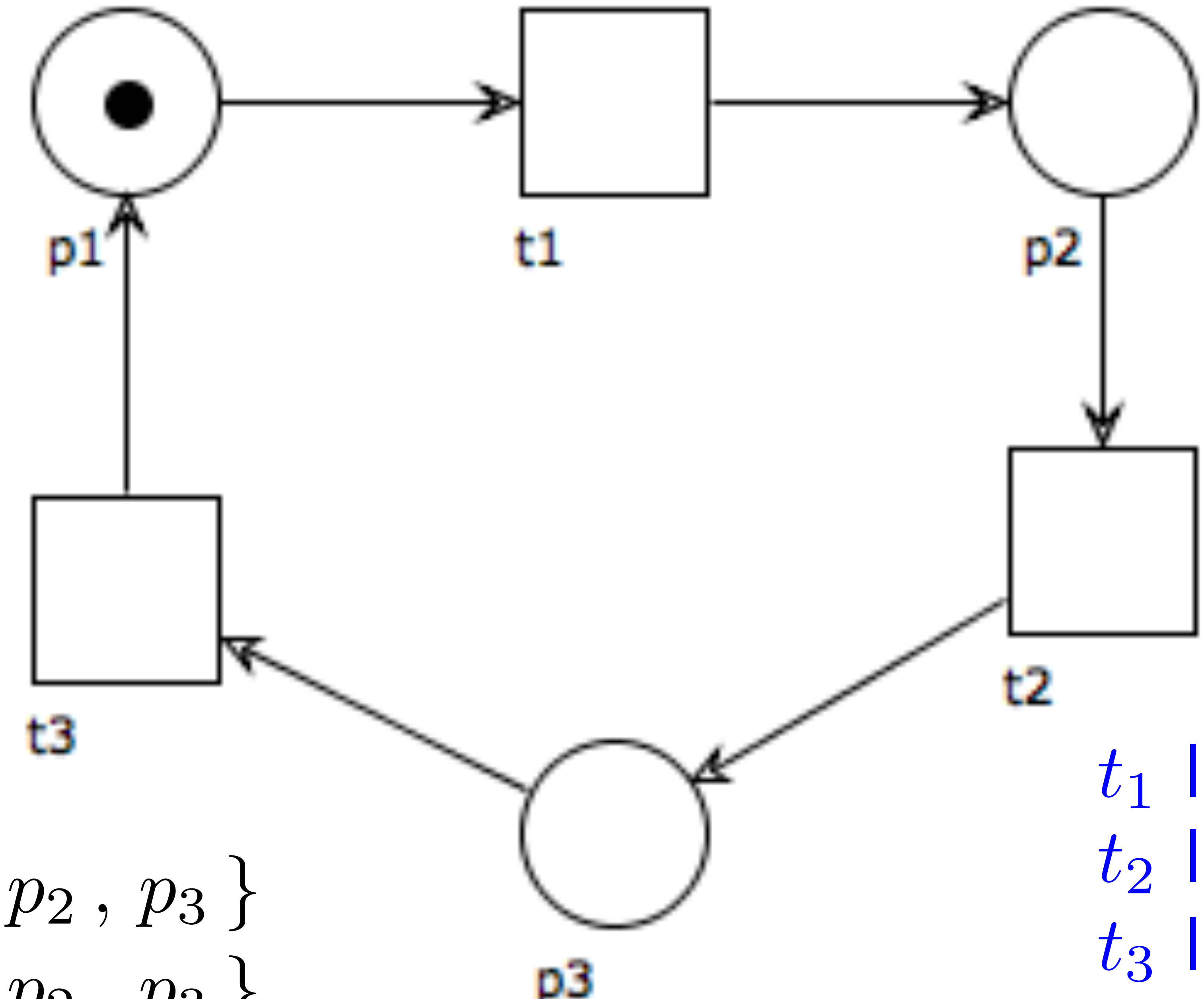
$M_0 = p_1$

$[M_0\rangle = \{\, p_1 \,,\, p_2 \,,\, p_3 \,\}$
$[p_2\rangle = \{\, p_1 \,,\, p_2 \,,\, p_3 \,\}$
$[p_3\rangle = \{\, p_1 \,,\, p_2 \,,\, p_3 \,\}$

t_3 live

if $M = p_1$ take $M' = p_3 \xrightarrow{t_3}$
if $M = p_2$ take $M' = p_3 \xrightarrow{t_3}$
if $M = p_3$ take $M' = p_3 \xrightarrow{t_3}$

Example: Live



$M_0 = p_1$

$[M_0\rangle = \{\, p_1 \,,\, p_2 \,,\, p_3 \,\}$

$[p_2\rangle = \{\, p_1 \,,\, p_2 \,,\, p_3 \,\}$

$[p_3\rangle = \{\, p_1 \,,\, p_2 \,,\, p_3 \,\}$

t_1 live

t_2 live

t_3 live

N live

Liveness, a formal recap

$$N = (P, T, F, M_0) \qquad\qquad t \in T$$

$$\mathsf{Live}(t, N) \quad \equiv \quad \forall M \in [M_0\rangle, \quad \exists M' \in [M\rangle, \quad M' \xrightarrow{t}$$

$$\begin{aligned}
\mathsf{NonLive}(t, N) \quad &\equiv \quad \neg\mathsf{Live}(t, N) \\
&\equiv \quad \neg(\forall M \in [M_0\rangle, \quad \exists M' \in [M\rangle, \quad M' \xrightarrow{t}) \\
&\equiv \quad \exists M \in [M_0\rangle, \quad \forall M' \in [M\rangle, \quad M' \stackrel{t}{\not\to}
\end{aligned}$$

Deadness, a formal recap

$$N = (P, T, F, M_0) \qquad\qquad t \in T$$

$$\mathsf{Dead}(t, N) \quad \equiv \quad \forall M \in [M_0\rangle, \quad M \stackrel{t}{\not\to}$$

$$\begin{aligned}
\mathsf{NonDead}(t, N) \quad &\equiv \quad \neg\mathsf{Dead}(t, N) \\
&\equiv \quad \neg(\forall M \in [M_0\rangle, \quad M \stackrel{t}{\not\to}) \\
&\equiv \quad \exists M \in [M_0\rangle, \quad M \stackrel{t}{\to}
\end{aligned}$$

Non-live vs Dead

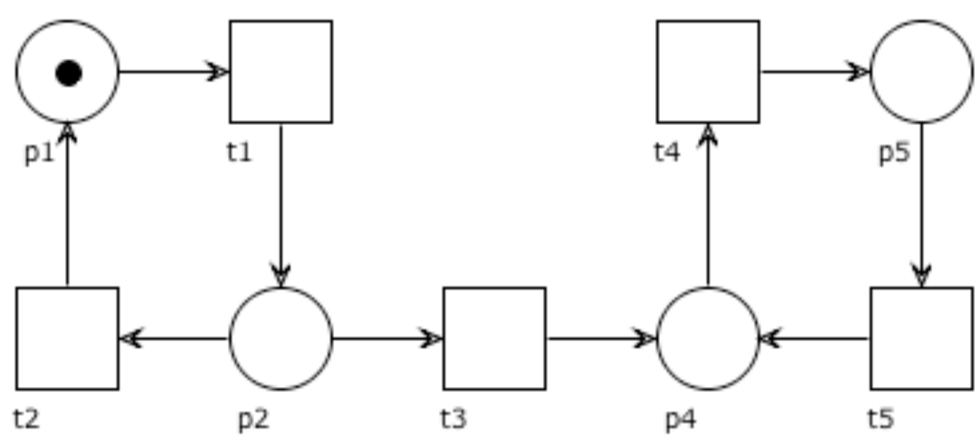
$$\mathsf{NonLive}(t, N) \quad \equiv \quad \exists M \in [M_0\rangle, \quad \mathsf{Dead}(t, N' = (P, T, F, M))$$

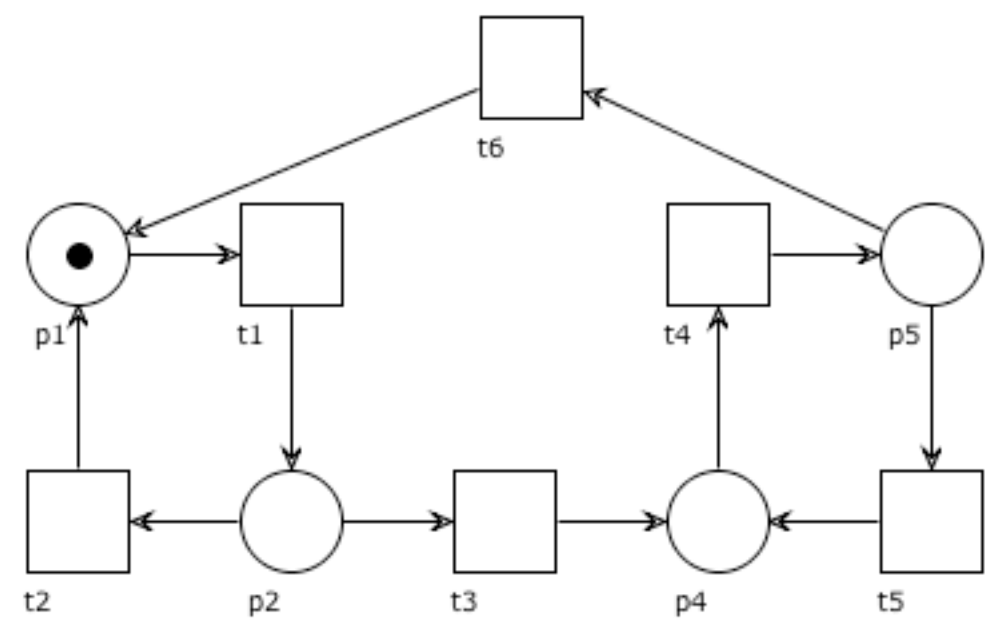
$$\mathsf{NonLive}(t, N) \quad \equiv \quad \exists M \in [M_0\rangle, \quad \forall M' \in [M\rangle, \quad M' \stackrel{t}{\not\to}$$

$$\mathsf{Dead}(t, N') \quad \equiv \quad \forall M' \in [M\rangle, \quad M' \stackrel{t}{\not\to}$$

a system is not live
iff it has a non-live transition
iff it has a transition that **can become** dead

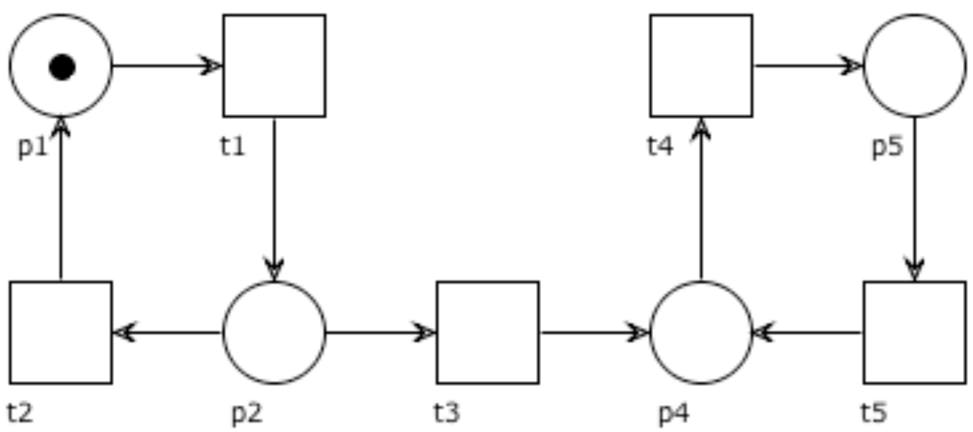
Liveness: example

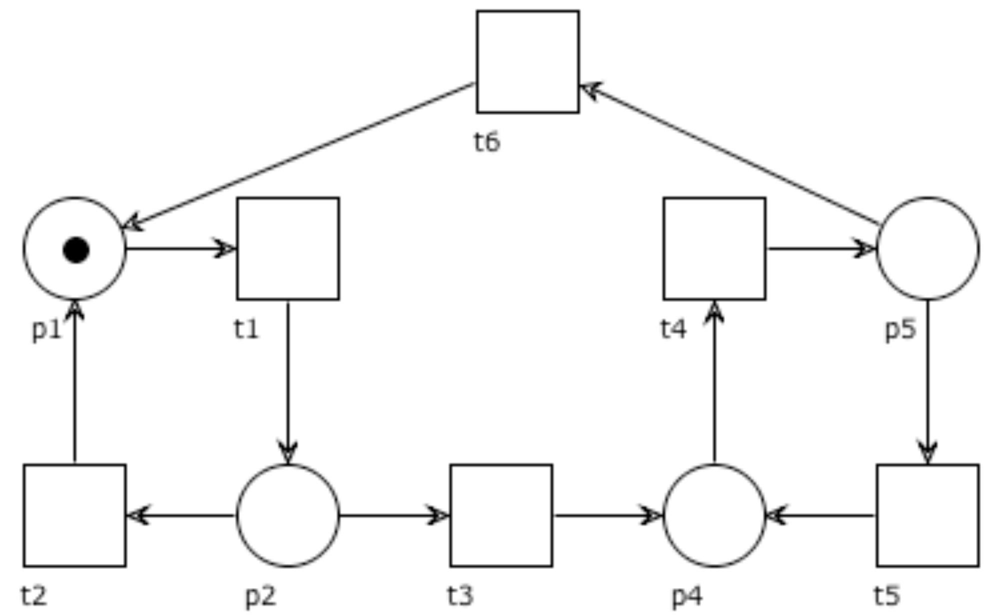




Which transitions are live?
Which are not?
Which are dead?
Is the net live?

Liveness: example





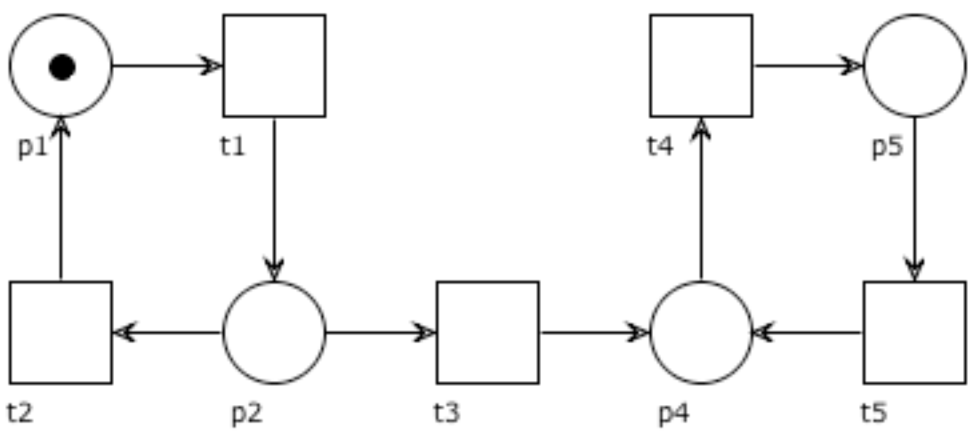
t4, t5	Which transitions are live?	all
t1, t2, t3	Which are not?	none
none	Which are dead?	none
No	Is the net live?	Yes

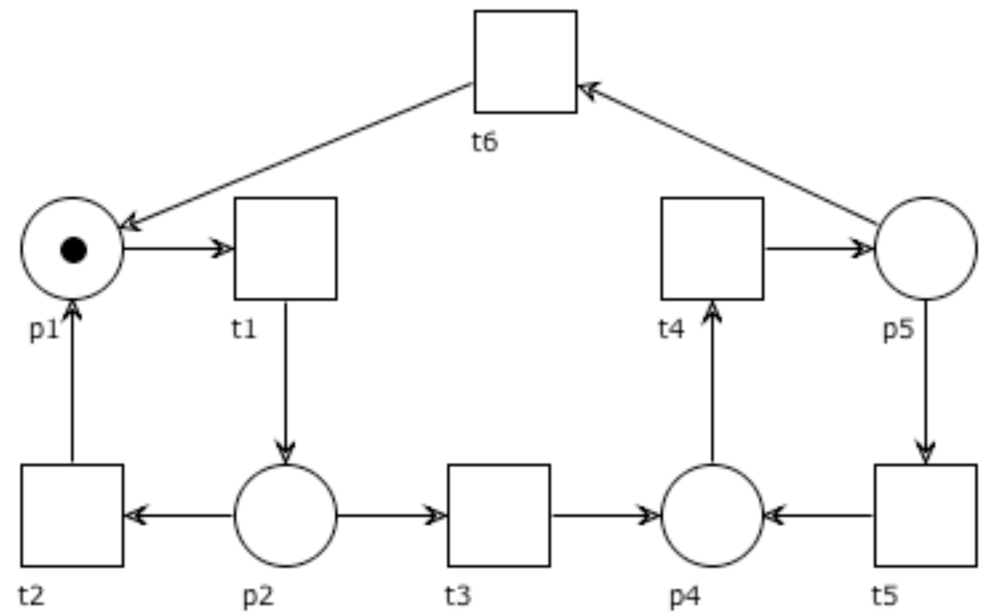
Liveness on the occurrence graph

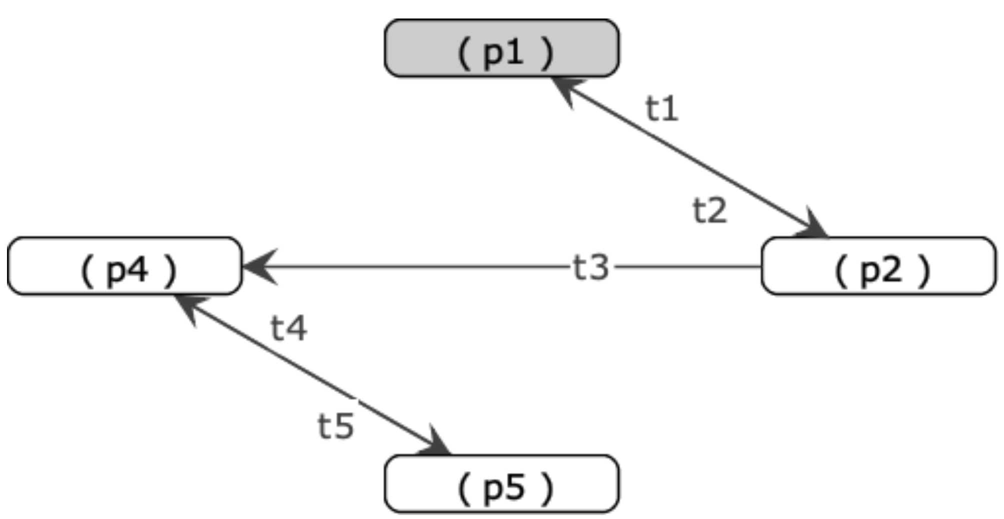
A transition t is live
iff
From any node of the occurrence graph we can reach a node with an outgoing arc labelled by t

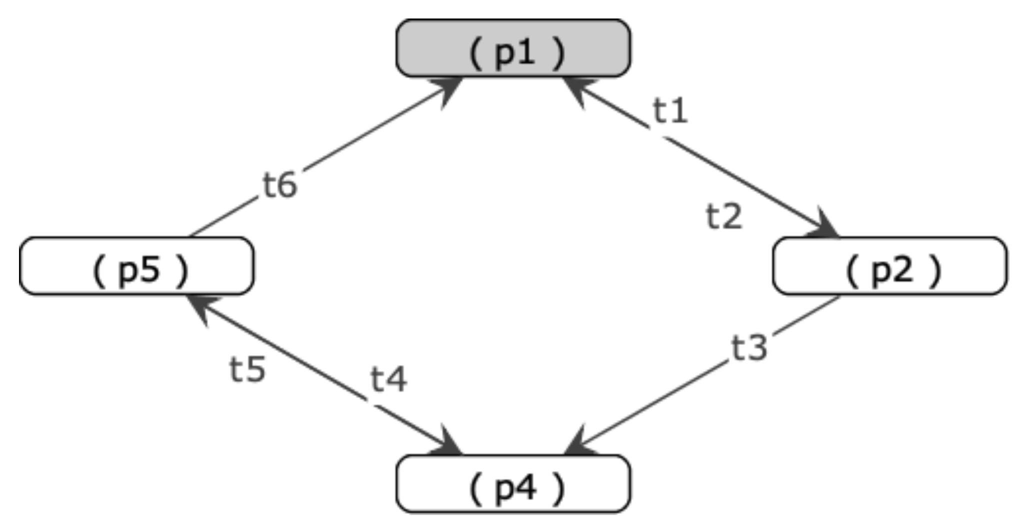
A transition t is dead (at M_0)
iff
There is no t-labelled arc in the occurrence graph

Liveness: example









Marked place

Given a marking M

We say that a place p is **marked** (at M)
if $M(p) > 0$
(i.e., there is a token in p in the marking M)

We say that p is **unmarked**
if $M(p) = 0$
(i.e., there is no token in p in the marking M)

Place-liveness, intuitively

A place p is **live** if
every time it becomes unmarked
there is still the possibility to be marked in the future
(or if it always stays marked)

A Petri net is **place-live** if all of its places are live

Live place

Definition: Let (P, T, F, M_0) be a net system.

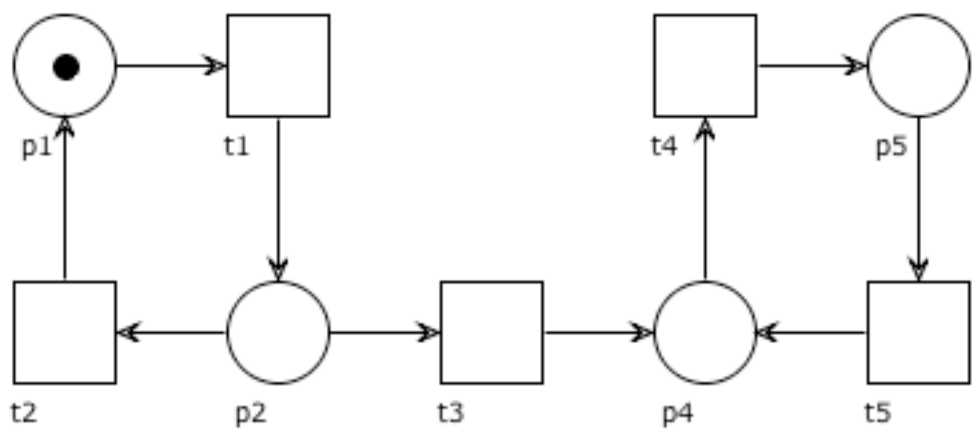
A place $p \in P$ is **live** if $\forall M \in [\, M_0 \,\rangle .\, \exists M' \in [\, M \,\rangle .\, M'(p) > 0$

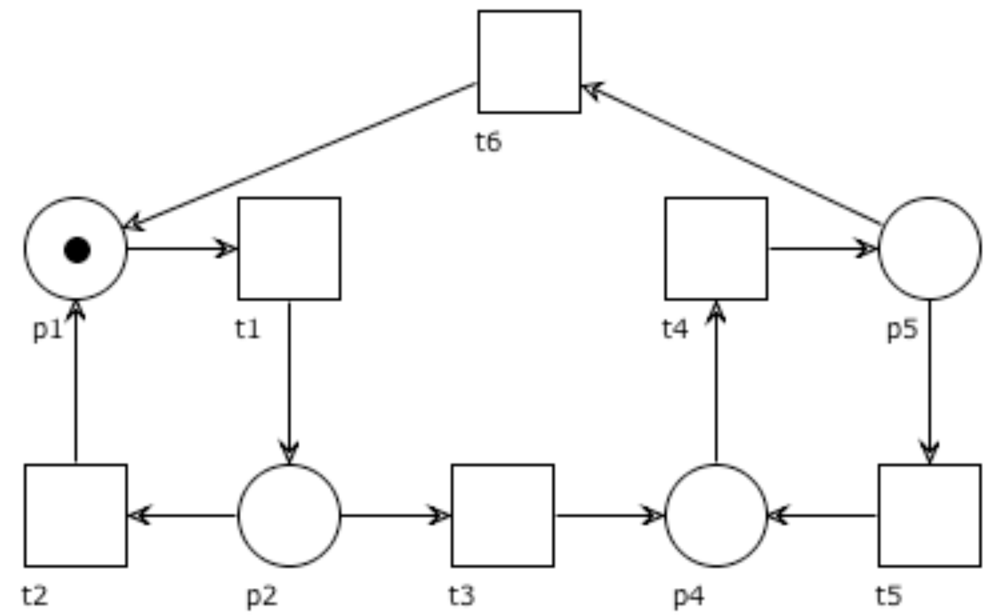
Place-liveness, formally

$$(P, T, F, M_0)$$

$$\forall p \in P. \quad \forall M \in [M_0\rangle. \quad \exists M' \in [M\rangle. \quad M'(p) > 0$$

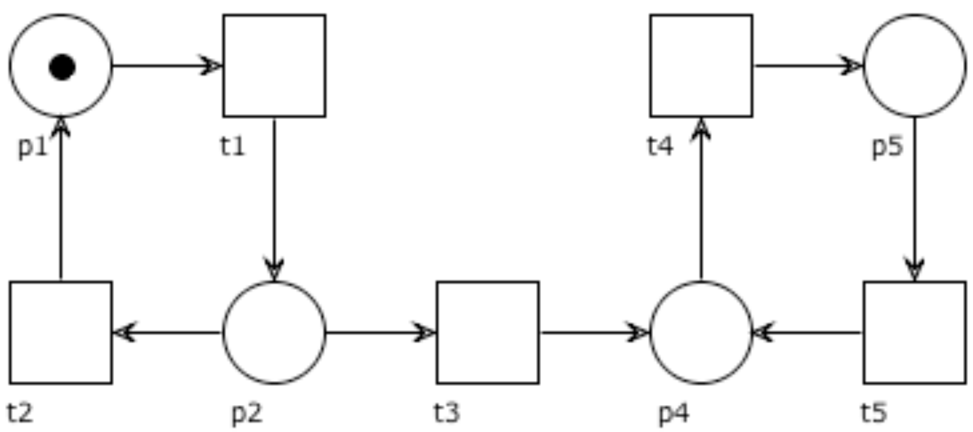
Place Liveness: example

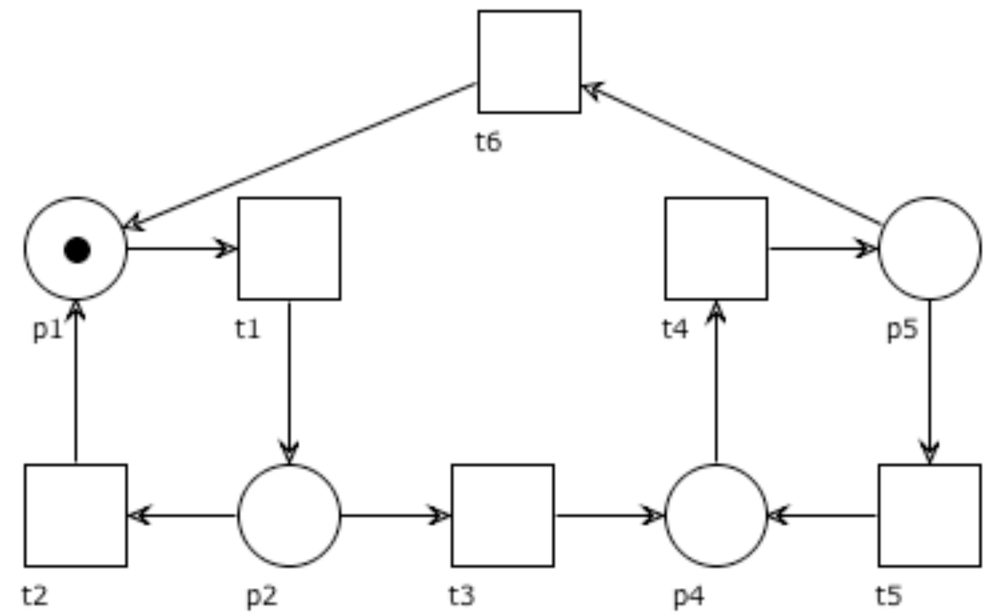




Which places are live?
Which are not?
Which are dead?
Is the net place live?

Place Liveness: example





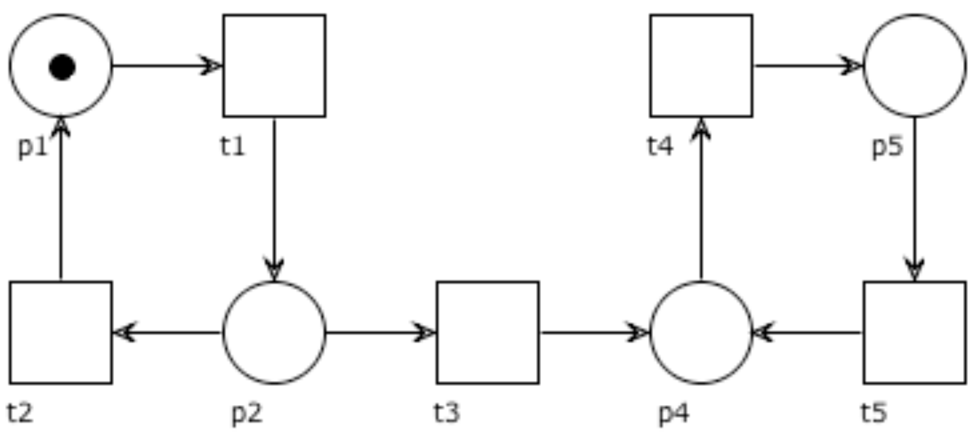
p4, p5	Which places are live?	all
p1, p2	Which are not?	none
none	Which are dead?	none
No	Is the net place live?	Yes

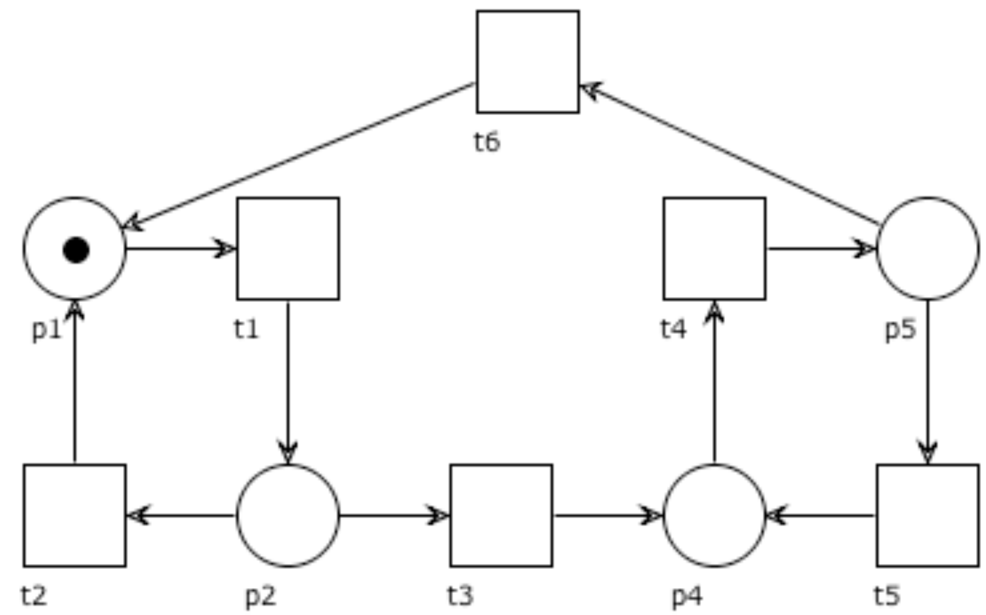
Place liveness on the occurrence graph

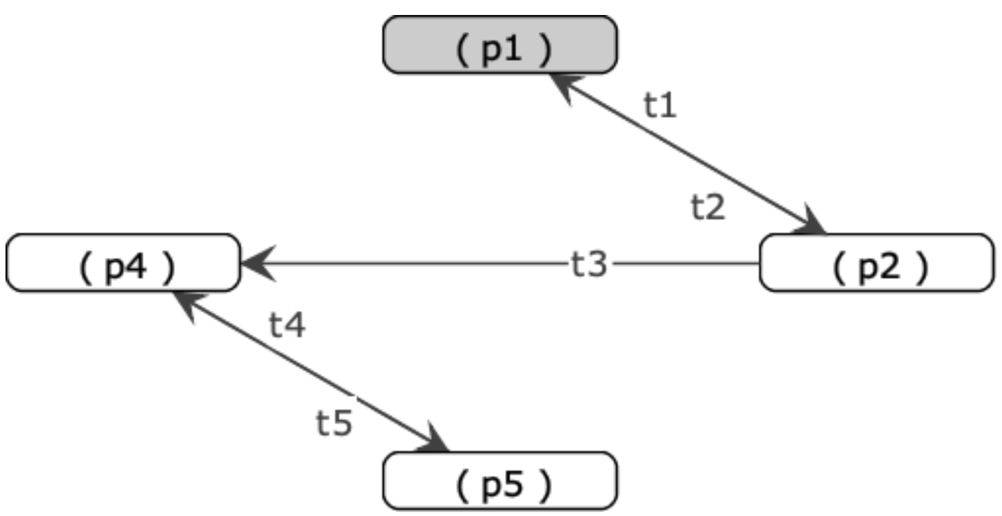
A place p is live
iff
From any node of the occurrence graph we can reach a node with a token in p

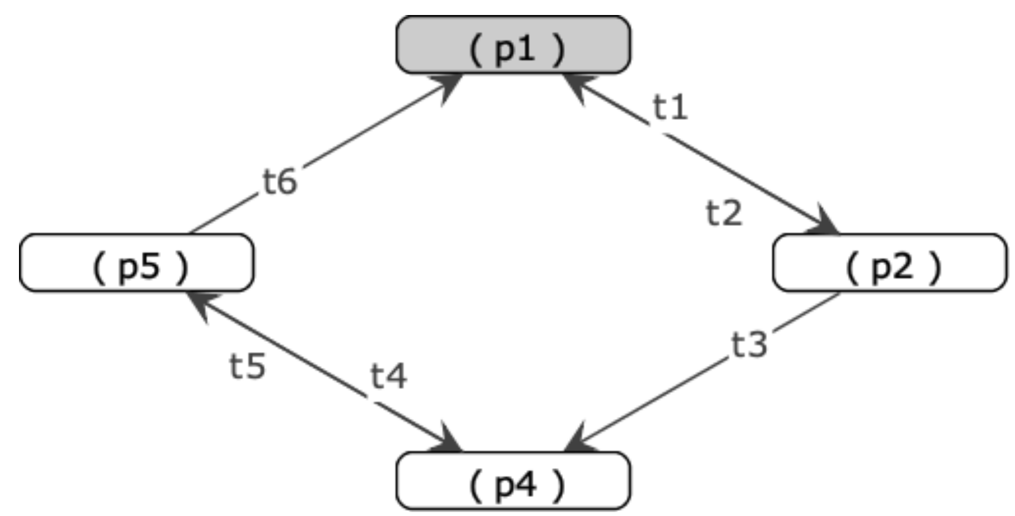
A place p is dead (at M_0)
iff
All the nodes of the occurrence graph have no token in p

Place Liveness: example









Question time

$$N = (P, T, F, M_0) \qquad\qquad p \in P$$

$$\begin{aligned} \mathsf{PLive}(p, N) &\equiv \forall M \in [M_0\rangle, \quad \exists M' \in [M\rangle, \quad M'(p) > 0 \\ \mathsf{NonPLive}(p, N) &\equiv \neg\mathsf{PLive}(p, N) \end{aligned}$$

write the explicit formula for NonPLive(p,N)

Question time

$$N = (P, T, F, M_0) \qquad\qquad p \in P$$

$$\begin{aligned}
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&\equiv \quad \exists M \in [M_0\rangle, \quad \forall M' \in [M\rangle, \quad M'(p) = 0
\end{aligned}$$

write the explicit formula for NonPLive(p,N)

Question time

$$N = (P, T, F, M_0) \qquad\qquad p \in P$$

$$\begin{aligned} \mathsf{Dead}(p, N) \quad &\equiv \quad \forall M \in [M_0\rangle, \quad M(p) = 0 \\ \mathsf{NonDead}(p, N) \quad &\equiv \quad \neg\mathsf{Dead}(p, N) \end{aligned}$$

write the explicit formula for NonDead(p,N)

Question time

$$N = (P, T, F, M_0) \qquad\qquad p \in P$$

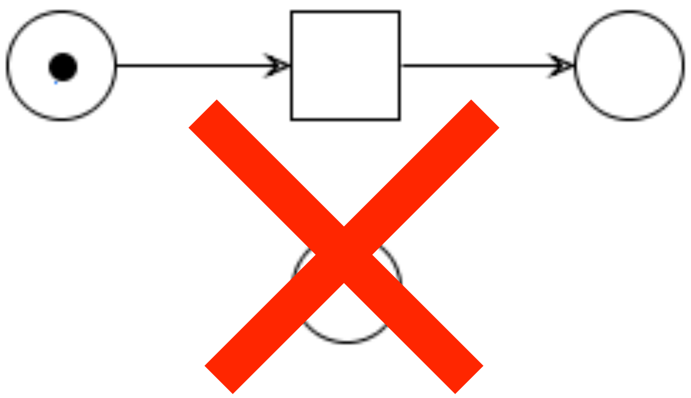
$$\begin{aligned}
\mathsf{Dead}(p, N) &\equiv \forall M \in [M_0\rangle, \quad M(p) = 0 \\
\mathsf{NonDead}(p, N) &\equiv \neg\mathsf{Dead}(p, N) \\
&\equiv \exists M \in [M_0\rangle, \quad M(p) > 0
\end{aligned}$$

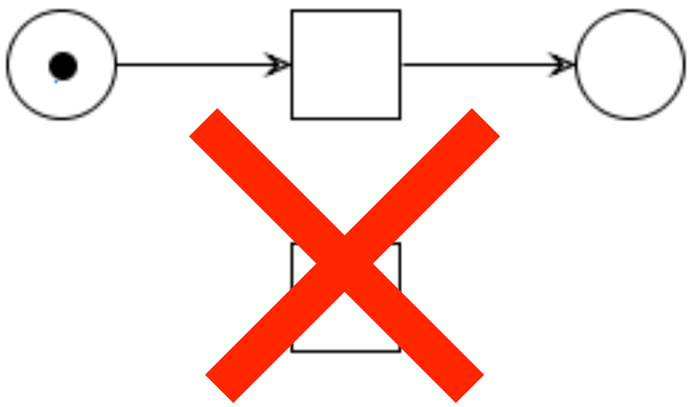
write the explicit formula for NonDead(p,N)

Disclaim

A node (place or transition) is called **isolated** if its pre- and post-sets are empty

We only consider nets
without isolated nodes





Dead nodes, intuitively

Given a marking M

A transition t is **dead** at M
if t will never be enabled in the future
(i.e., t is not enabled in any marking reachable from M)

A place p is **dead** at M
if p will never be marked in the future
(i.e., p is unmarked in any marking reachable from M)

Dead nodes

Definition: Let (P, T, F) be a net

A transition $t \in T$ is **dead** at M if $\forall M' \in [\, M \,\rangle .\, M' \not\xrightarrow{t}$

A place $p \in P$ is **dead** at M if $\forall M' \in [\, M \,\rangle .\, M'(p) = 0$

Non-live vs Dead

If a transition is dead at some reachable marking M
then it is non-live

If a place is dead at some reachable marking M
then it is non-live

being non-live implies possibly becoming dead
(but not necessarily in the current marking)

Some obvious facts

a system is not live iff it has a transition that can become dead at some reachable marking

a system is not place-live iff it has a place that can become dead at some reachable marking

If a place / transition is dead at M, then it remains dead at any marking reachable from M
(**the set of dead nodes can only increase during a run**)

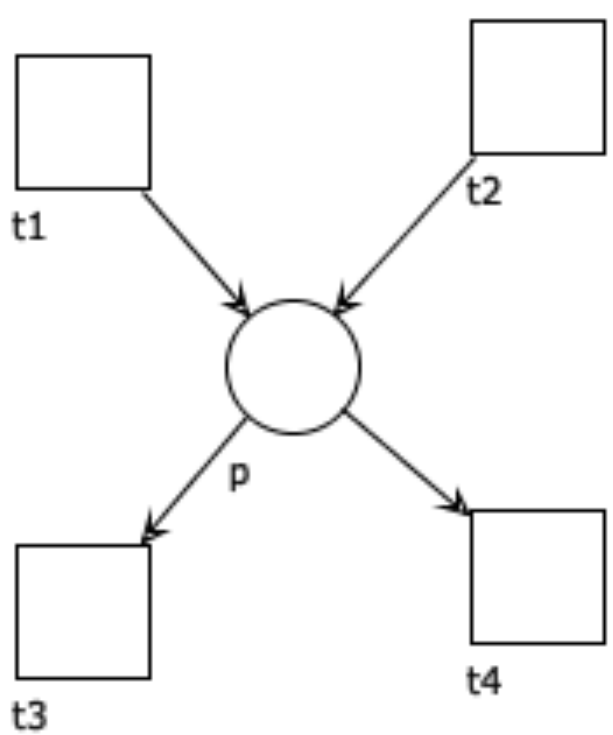
True or false?

Every transition in the pre- or post-set of a dead place is also dead

True or false?

Every transition in the pre- or post-set of a dead place is also dead

True



suppose p is dead
if p remains empty then t3 and t4 cannot fire
so they are dead

suppose p is dead
if t1 or t2 could fire then some token would arrive in p
since no token can arrive in p
it means that t1 and t2 will never fire

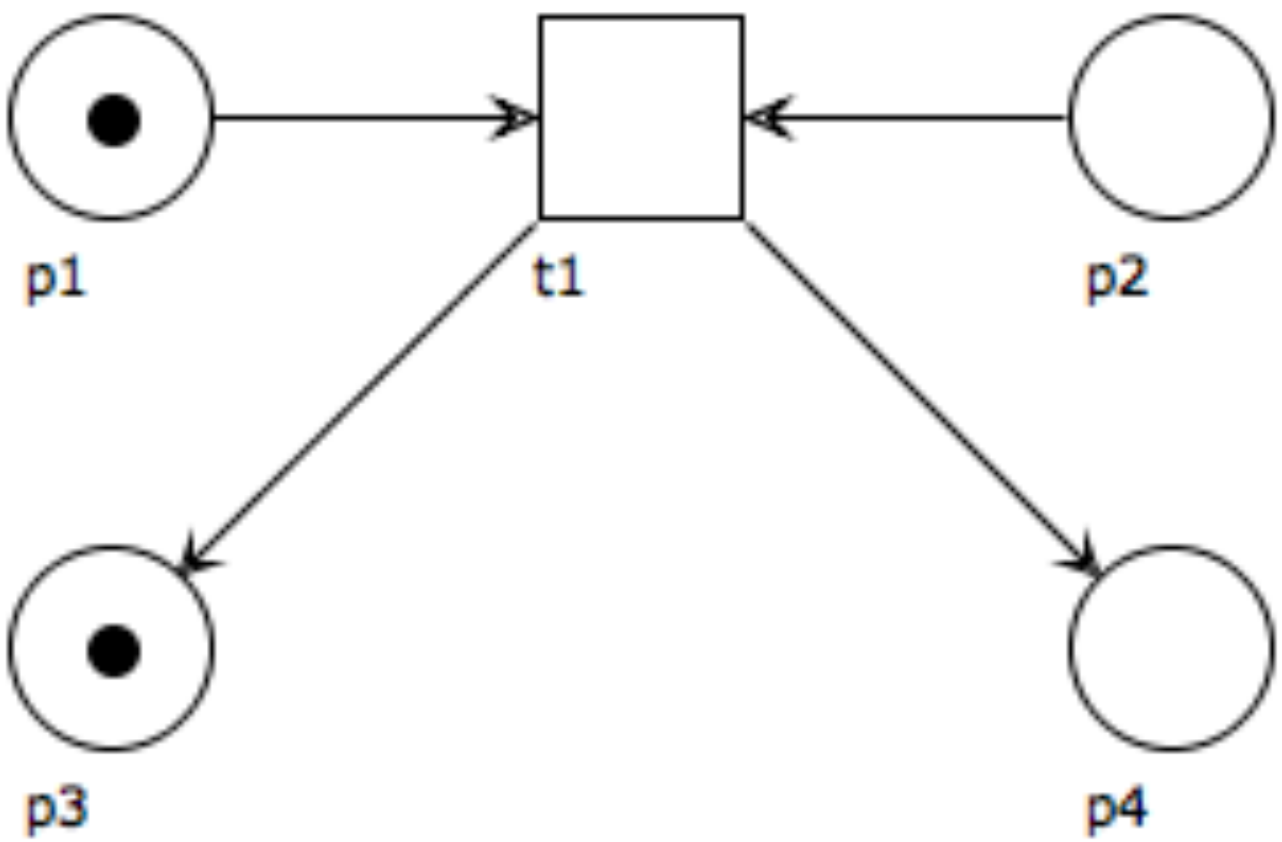
True or false?

Every place in the pre- or post-set of a dead transition is also dead

True or false?

Every place in the pre- or post-set of a dead transition is also dead

False



t1 is dead but p1 and p3 are not dead

Liveness implies place-liveness

Proposition: Live systems are also place-live

Assume the net is live

Take any $p \in P$ and $M \in [M_0\rangle$

We want to find $M' \in [M_0\rangle$ s.t. $M'(p) > 0$

Take any $t \in \bullet p \cup p\bullet$

By liveness: there are $M'', M''' \in [M_0\rangle$ s.t. $M'' \xrightarrow{t} M'''$

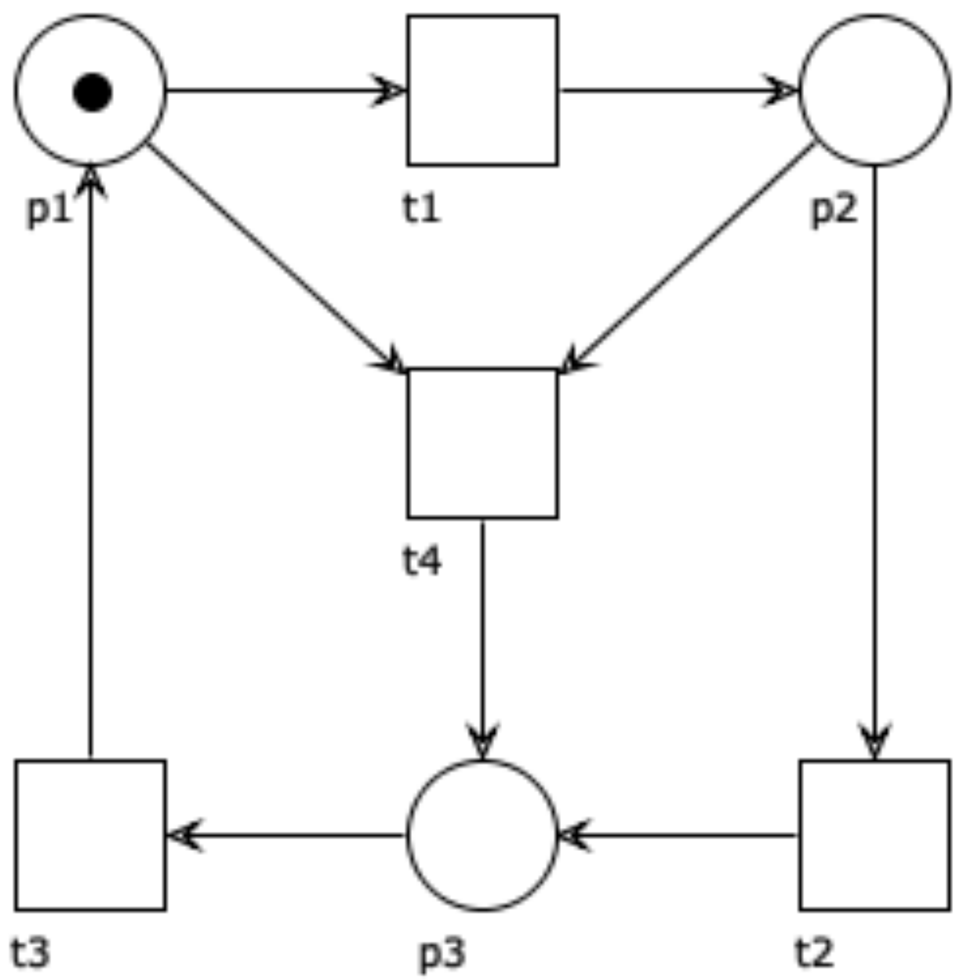
Then $M''(p) > 0$ or $M'''(p) > 0$

Exercise

Draw a net that
is place-live but not live

Exercise

Draw a net that
is place-live but not live



Petri nets:
Deadlock-freedom

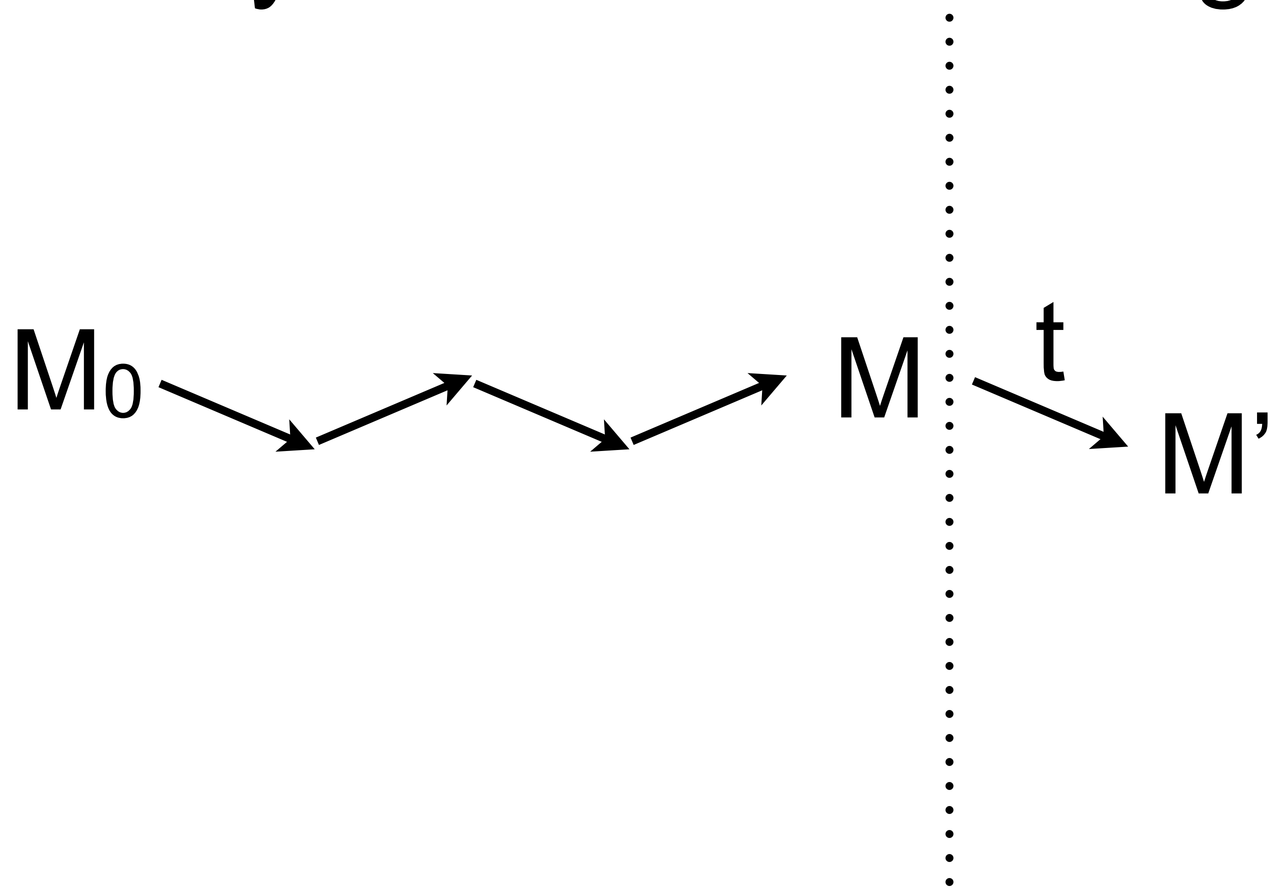
Deadlock-freedom

A Petri net is **deadlock free**, if every reachable marking enables some transition

In other words, we are guaranteed that at any point in time of the computation, some transition can be fired

Deadlock-freedom illustrated

For any reachable marking M…



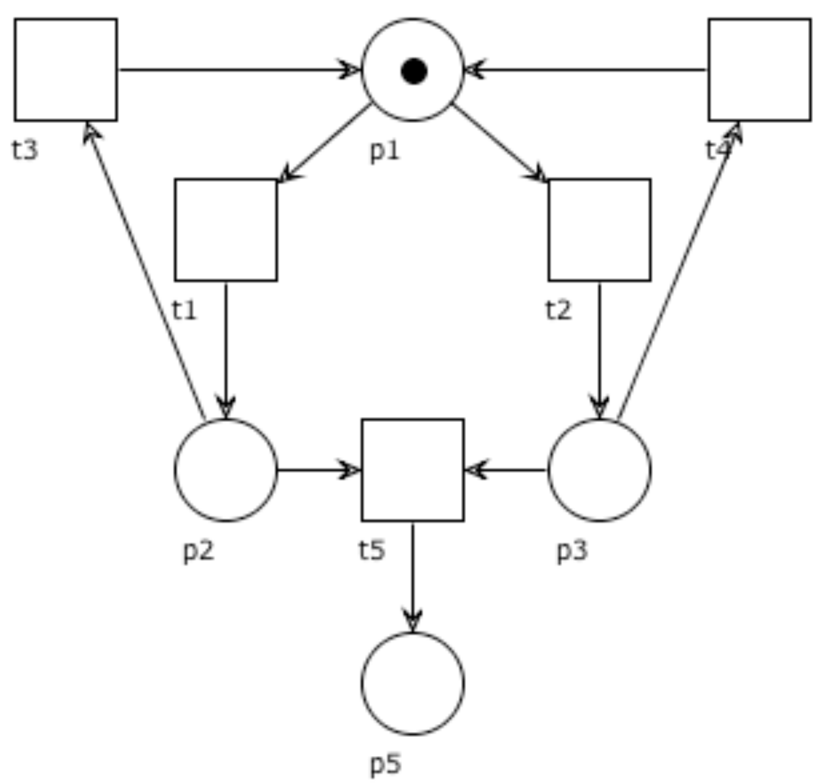
… we can still fire some transition next

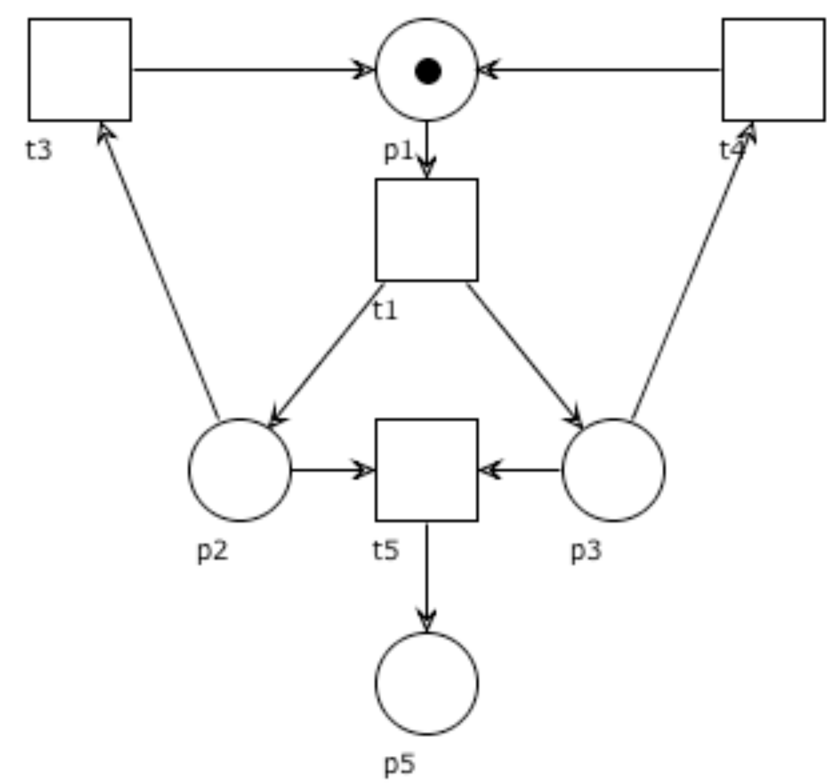
Deadlock freedom, formally

$$(P, T, F, M_0)$$

$$\forall M \in [\, M_0 \,\rangle, \quad \exists t \in T, \quad M \xrightarrow{t}$$

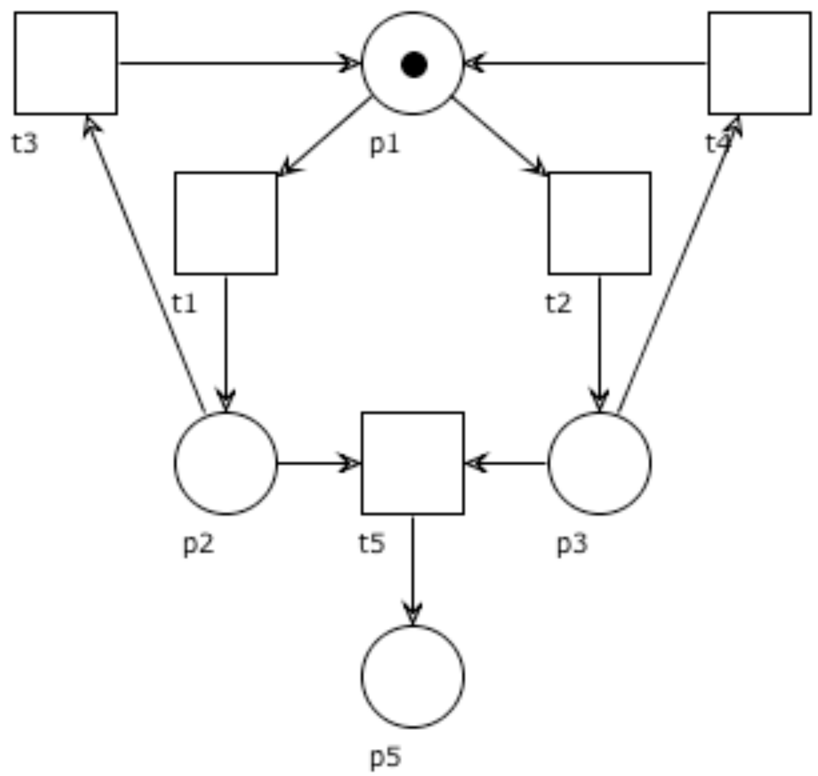
Deadlock-freedom: example

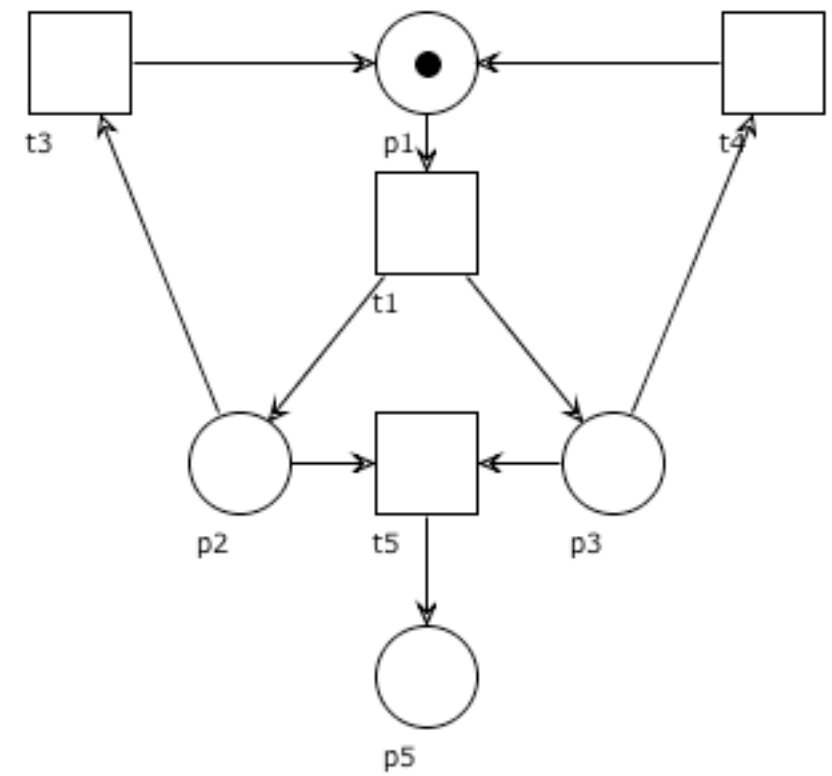




Is the net deadlock-free?

Deadlock-freedom: example





Yes Is the net deadlock-free? No

$$[p_1\rangle = \{\, p_1 \,,\, p_2 \,,\, p_3 \,\}$$

$$p_1 \rightarrow \qquad p_2 \rightarrow \qquad p_3 \rightarrow$$

$$[p_1\rangle = \{\, p_1 \,,\, p_2 + p_3 \,,\, p_5 \,,\, \ldots \}$$

$$p_5 \not\rightarrow$$

Deadlock freedom on the occurrence graph

A net is deadlock free
iff
Every node of the occurrence graph has an outgoing arc

Question time

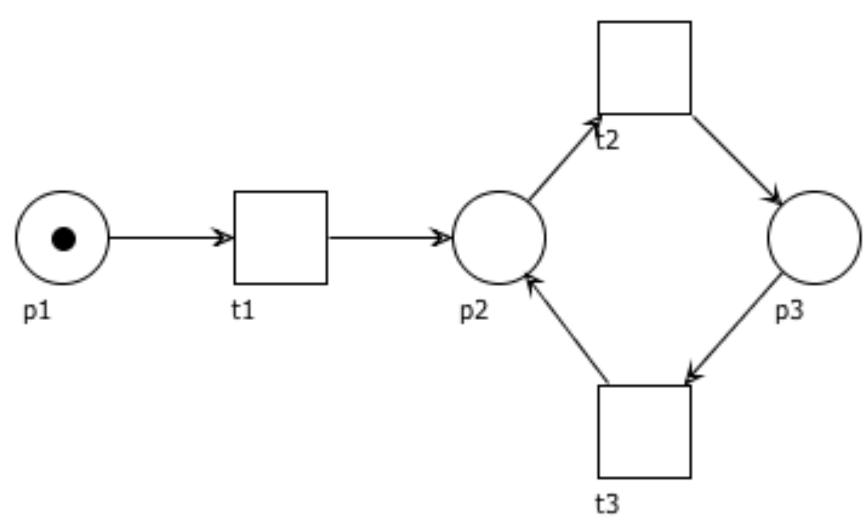
Does liveness imply deadlock-freedom?
(Can you exhibit a live Petri net that is not deadlock-free?)

Does deadlock-freedom imply liveness?
(Can you exhibit a deadlock-free net that is not live?)

Question time

Does liveness imply deadlock-freedom? **YES**
(Can you exhibit a live Petri net that is not deadlock-free?)
NO

Does deadlock-freedom imply liveness? **NO**
(Can you exhibit a deadlock-free net that is not live?) **YES**



Liveness implies deadlock freedom

Lemma If (P, T, F, M_0) is live, then it is deadlock-free

By contradiction, let $M \in [\, M_0 \,\rangle$, with $M \not\to$

Let $t \in T$ (T cannot be empty).
By liveness, $\exists M' \in [\, M \,\rangle$ with $M' \xrightarrow{t}$.

Since $M \not\to$, we have $[\, M \,\rangle = \{\, M \,\}$.
Therefore $M = M' \xrightarrow{t}$, which is absurd.

Digression: for next exercises

Contraposition

$$P \Rightarrow Q \qquad \equiv \qquad (\neg Q) \Rightarrow \neg P$$

True or false?

If a system is not place-live, then it is not live

True or false?

If a system is not place-live, then it is not live

(true, by contraposition)

liveness implies place-liveness

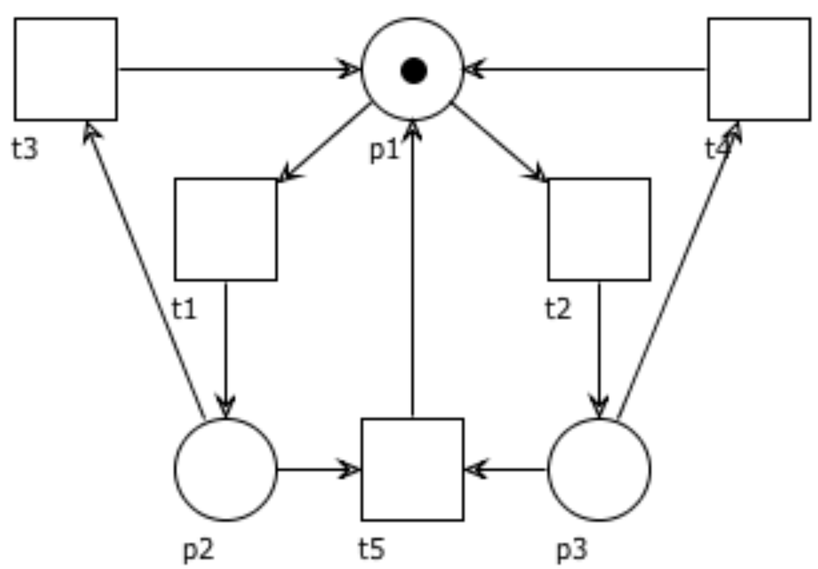
True or false?

If a system is not live, then it is not place-live

True or false?

If a system is not live, then it is not place-live

(false, see below)



t_5 is dead (and thus non-live)

p_1, p_2 and p_3 are live

True or false?

If a system is place-live, then it is deadlock-free

True or false?

If a system is place-live, then it is deadlock-free
(true, see below)

Take $M \in [\, M_0 \,\rangle$.

If all places are marked at M, then $\exists t \in T$ such that $M \xrightarrow{t}$.

Otherwise, take $p \in P$ such that $M(p) = 0$.
Since the system is place-live, then $\exists M' \in [\, M \,\rangle$ such that $M'(p) > 0$.
Obviously $M' \neq M$, therefore $\exists \sigma \neq \epsilon$ such that $M \xrightarrow{\sigma} M'$.
Thus, M is not a deadlock state.

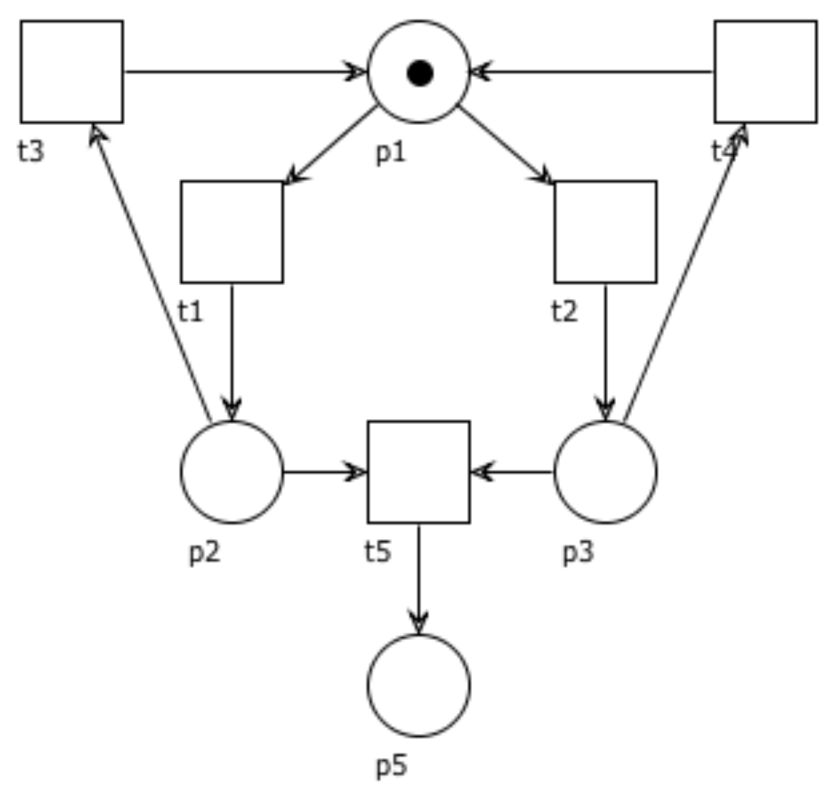
True or false?

If a system is deadlock-free, then it is place-live

True or false?

If a system is deadlock-free, then it is place-live

(false, see below)

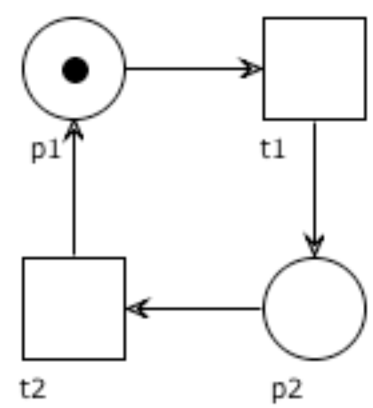


p_5 is dead (and thus non-live)

the system is deadlock-free

Exercises

For each of the following nets, say if they are bounded, safe, live, deadlock-free



Exercises

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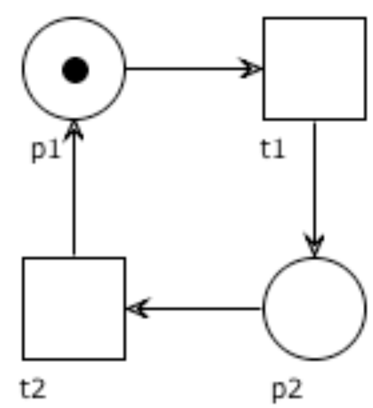
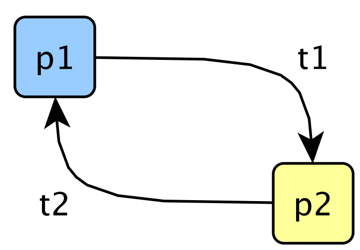
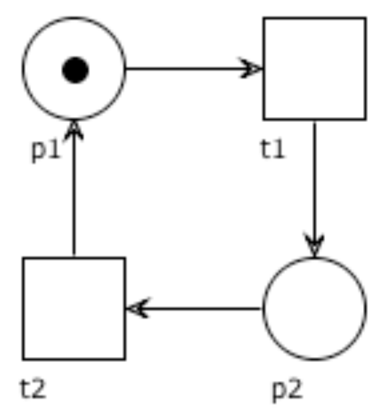




Exercises

For each of the following nets, say if they are **bounded**, **safe**, **live**, **deadlock-free**



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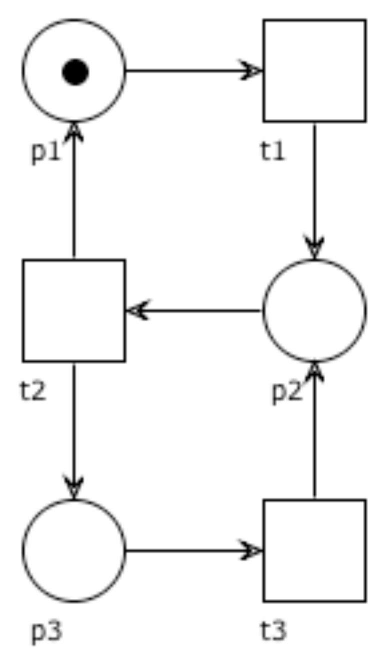

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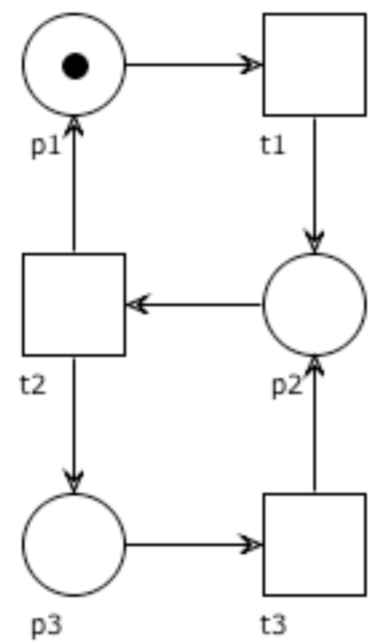

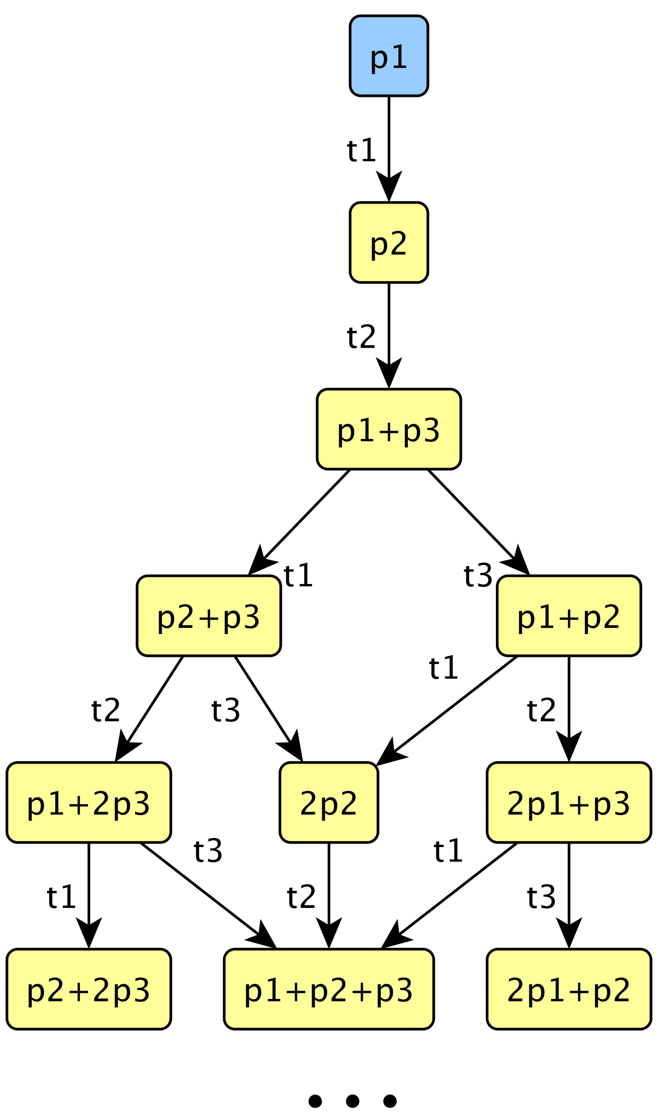

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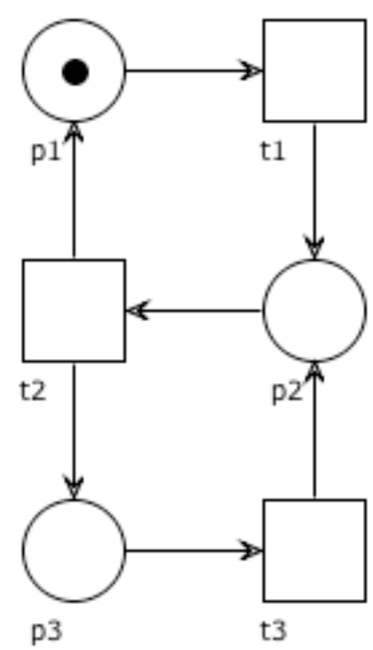

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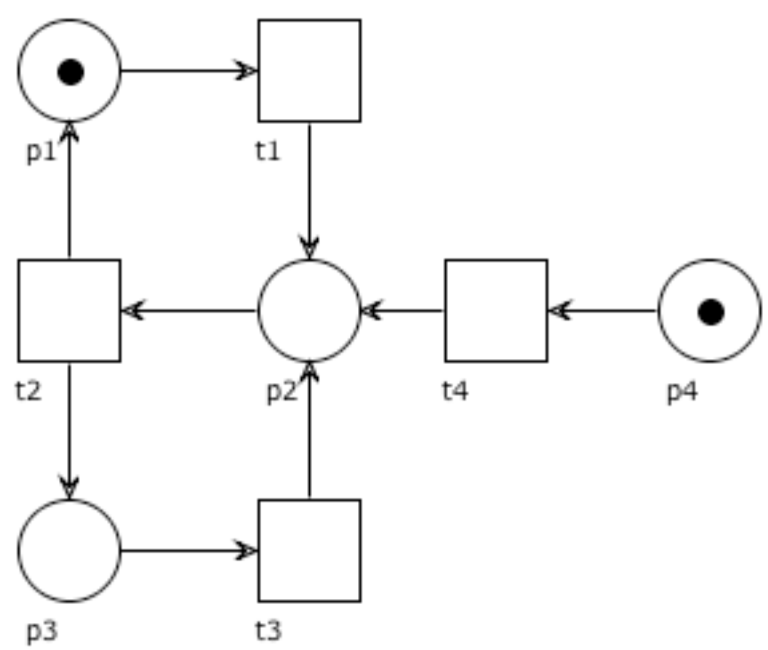

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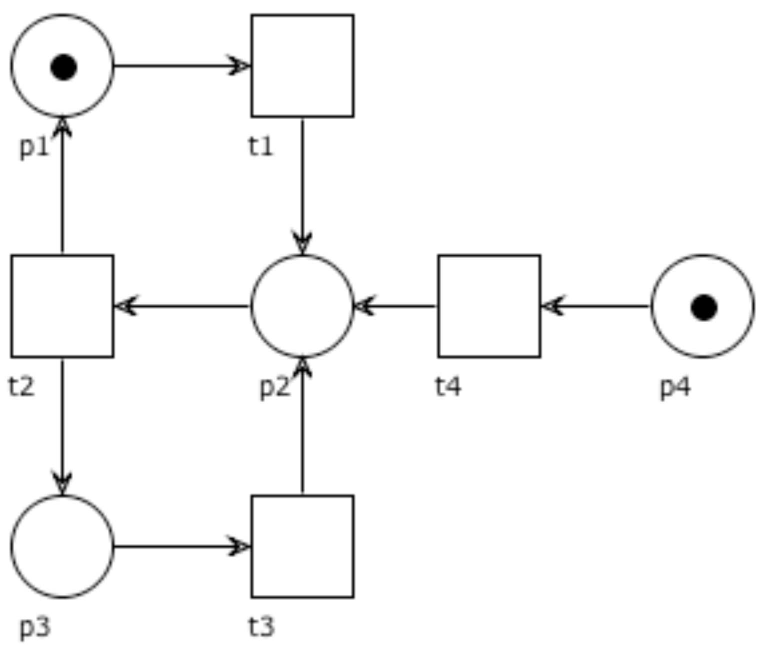

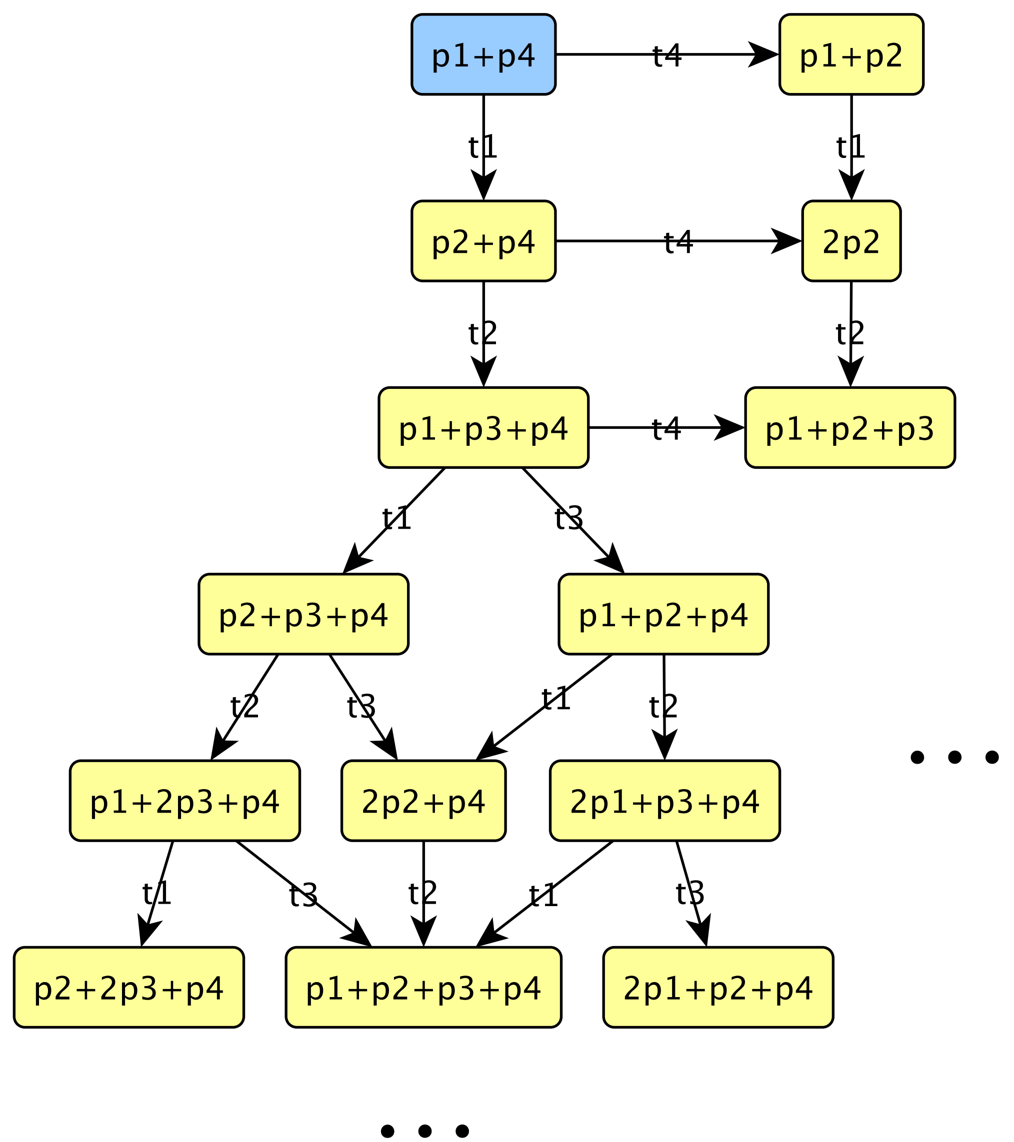

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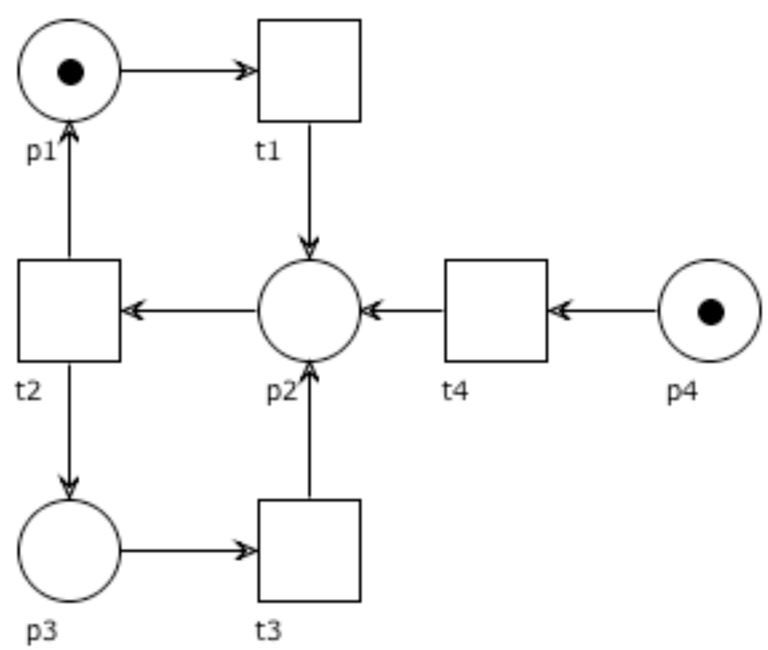

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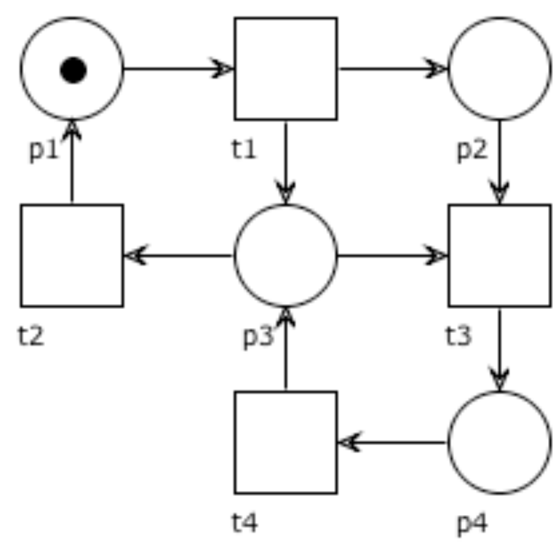

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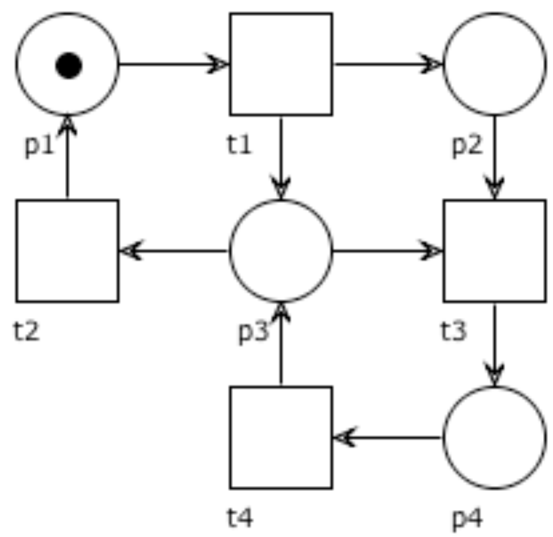

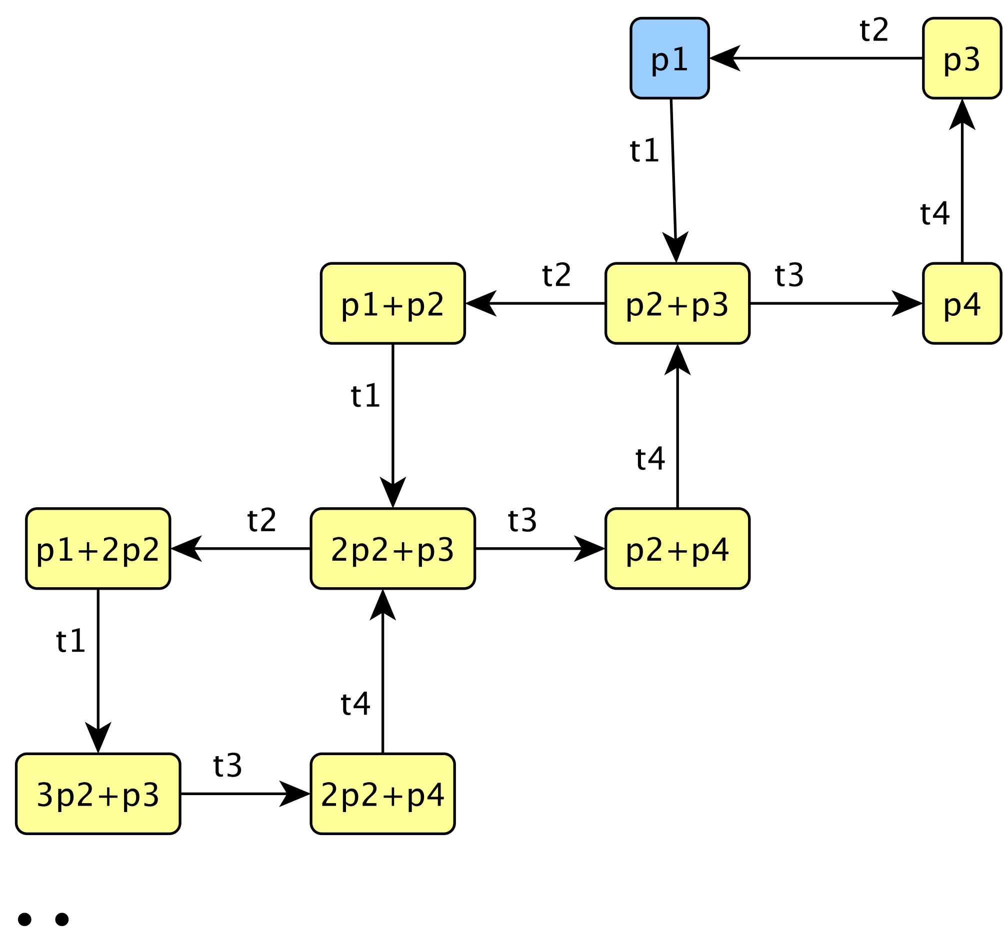

Exercises

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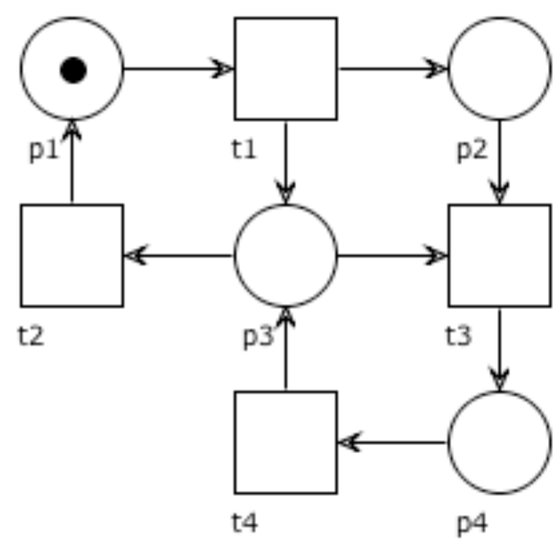

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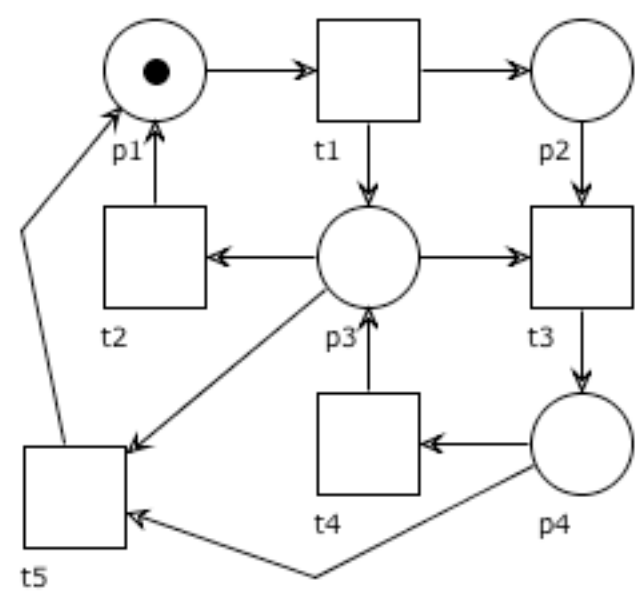

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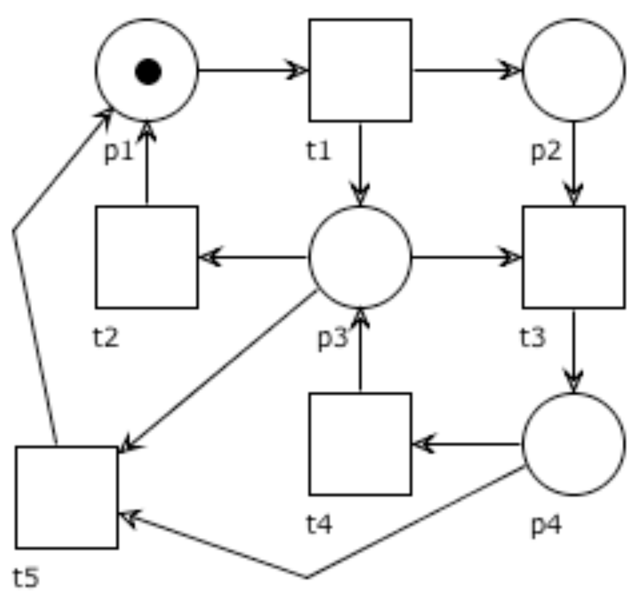
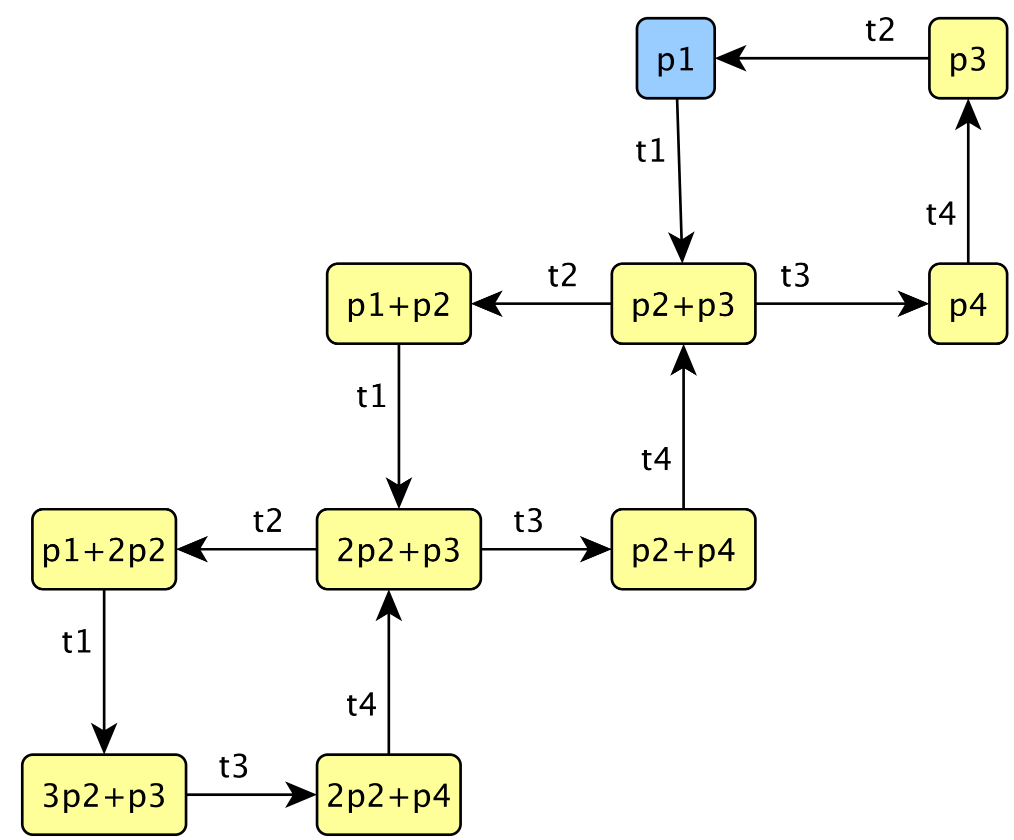
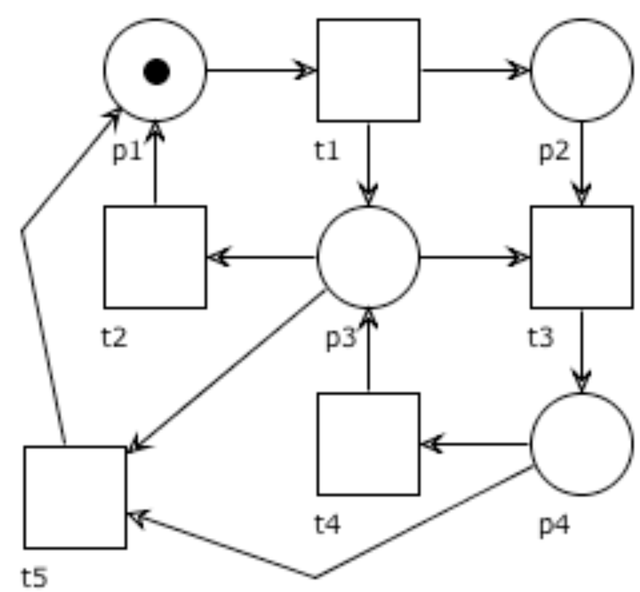




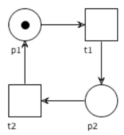
. . .

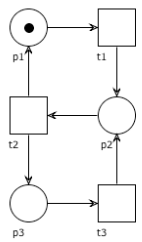
Exercises

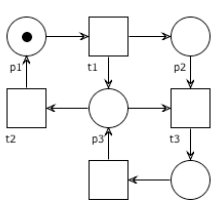
For each of the following nets, say if they are bounded, safe, live, **deadlock-free**



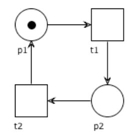
Live



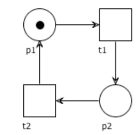


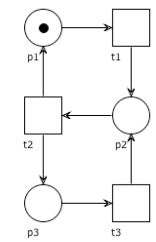


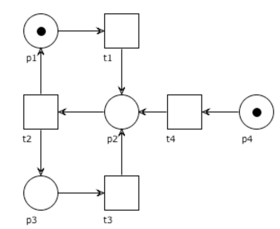
Bounded

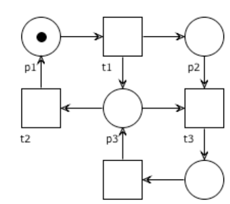


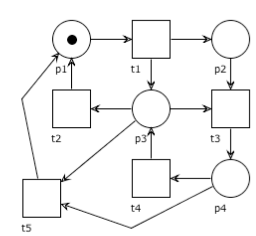
Deadlock free











None

Recap

live => place-live
non place-live => non live
t dead => non live
p dead => non place-live
p dead => non live

live => deadlock-free
possible deadlock => non live
place live => deadlock-free
possible deadlock => non place-live