Methods for the specification and verification of business processes

MPB (6 cfu, 295AA)

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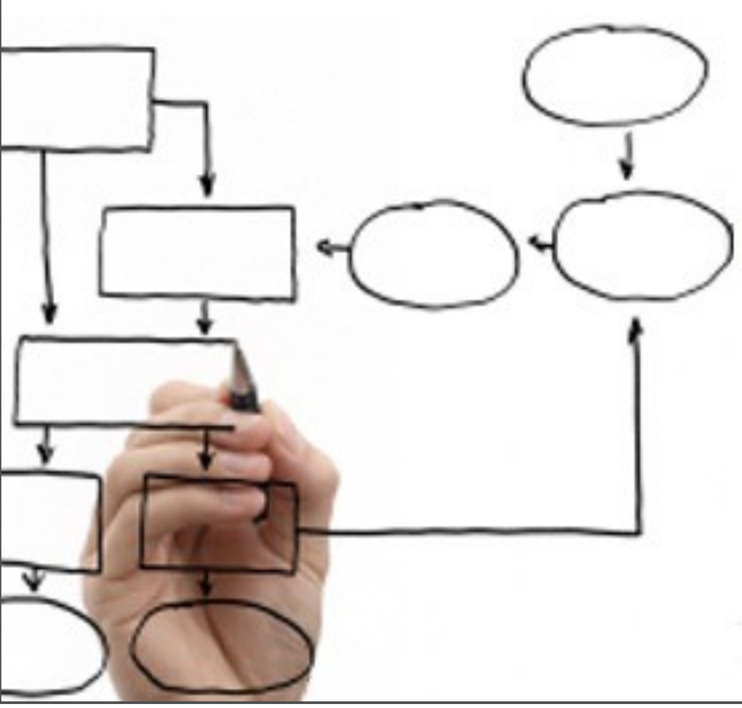


09 - Incidence matrices

Object

We give a matrix-based representation of Petri nets and their computations

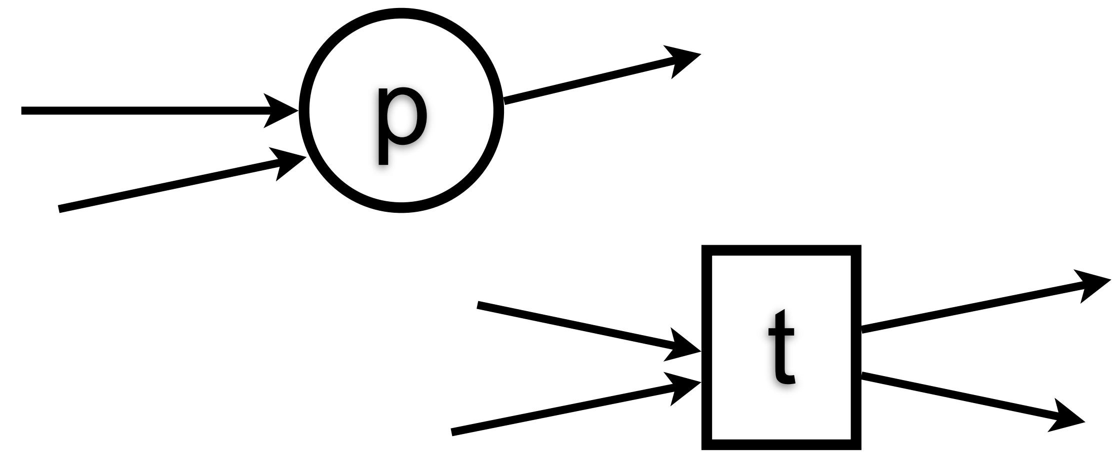
Key point

The change of the numbers of tokens on a place p caused by the firing of the transition t does not depend on the current marking

It is entirely determined by the net (i.e., by the flow relation)

Let us have a look at the relative changes for every place and transition...

How p relates to t

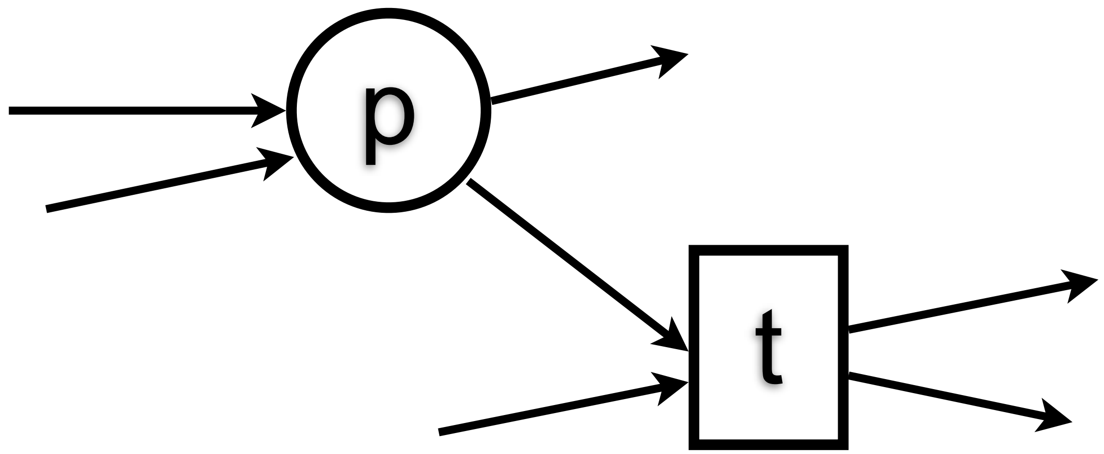


$(p, t) \notin F$ and $(t, p) \notin F$

Place p and transition t are completely unrelated:

- p has no influence on the enabling of t
- firing t does not change the number of tokens in p

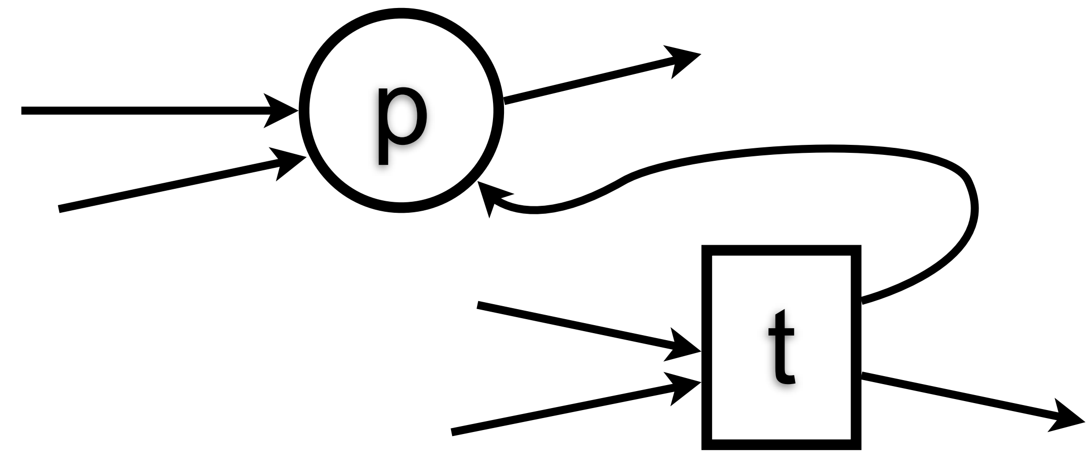
How p relates to t



$\underline{(p,t) \in F}$ and $(t,p) \notin F$

- one token in p is needed to enable t
- firing t reduces by one the number of tokens in p

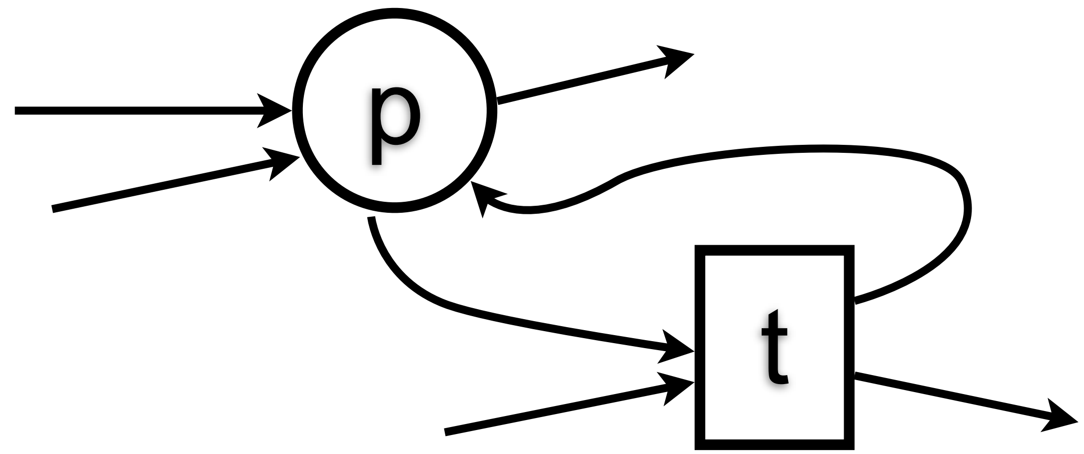
How p relates to t



$(p,t) \notin F$ and $\underline{(t,p) \in F}$

- firing t increases by one the number of tokens in p

How p relates to t



$\underline{(p, t) \in F}$ and $\underline{(t, p) \in F}$

- one token in p is needed to enable t
- firing t does not change the number of tokens in p

Incidence matrix

Let $N = (P, T, F)$ be a net.

Its **incidence matrix** $\mathbf{N} : (P \times T) \to \{-1, 0, 1\}$ is defined as:

$$\mathbf{N}(p,t) = \begin{cases} 0 & \text{if } (p,t) \notin F \wedge (t,p) \notin F \text{ or } (p,t) \in F \wedge (t,p) \in F \\ -1 & \text{if } (p,t) \in F \wedge (t,p) \notin F \\ +1 & \text{if } (p,t) \notin F \wedge (t,p) \in F \end{cases}$$

Matrix view

m columns, one for each transition

n rows,
one for
each place

	t_1												
p_1													
p_2	-1												
p_3													
...													
	+1												
p_n													

Matrix view

m columns, one for each transition

n rows, one for each place

	t_1	t_2											
p_1		+1											
p_2	-1												
p_3		+1											
...													
	+1												
p_n		-1											

Matrix view

m columns, one for each transition

n rows, one for each place

	t_1	t_2	t_3					...					
p_1		+1	-1										
p_2	-1		+1										
p_3		+1											
			+1										
...								...					
	+1												
			-1										
p_n		-1	+1										

Matrix view

m columns, one for each transition

n rows, one for each place

	t_1	t_2	t_3					...					t_m
p_1		+1	-1										-1
p_2	-1		+1										
p_3		+1											
			+1										
...								...					+1
	+1												
			-1										
													+1
p_n		-1	+1										-1

Column vector t_j

$\mathbf{t_j} : P \rightarrow \{-1, 0, 1\}$ such that $\mathbf{t_j}(p) = \mathbf{N}(p, t_j)$

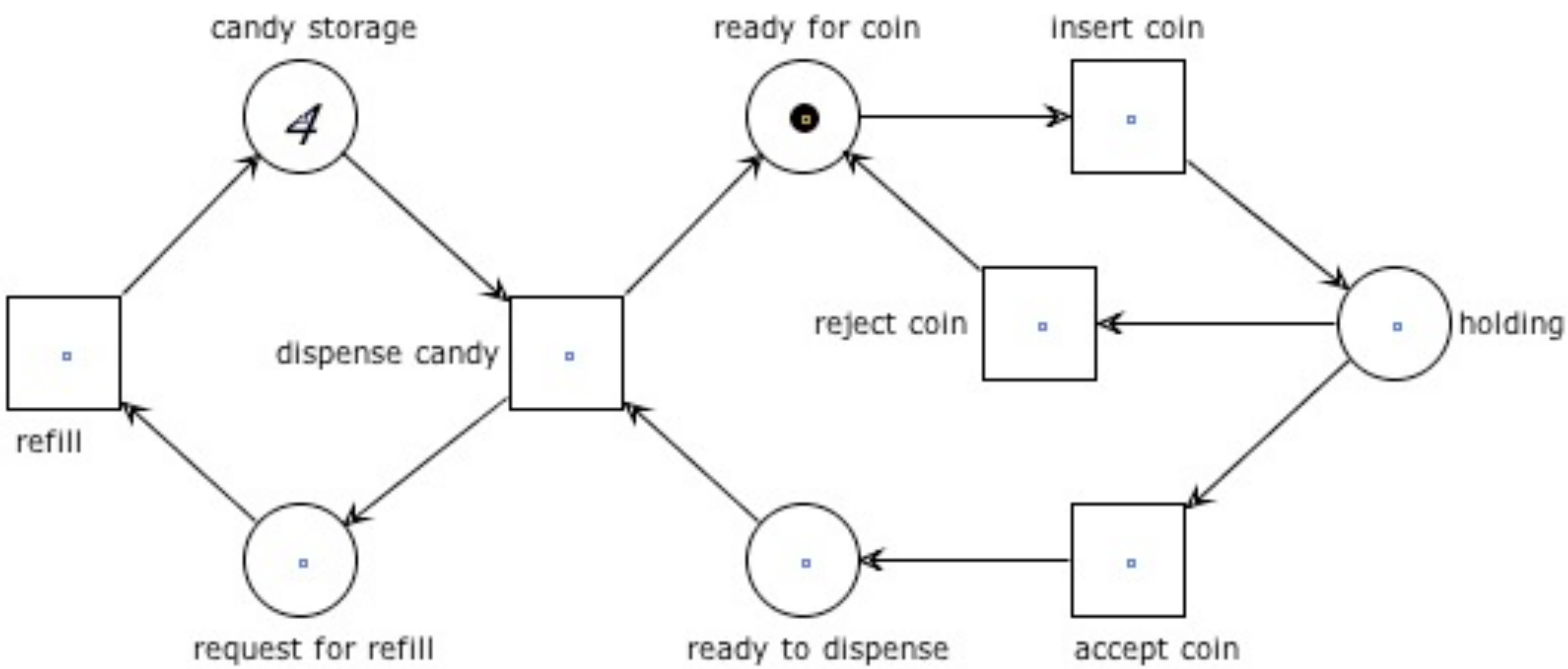
	t_1	t_2	t_3					...					t_m
p_1		+1	-1										-1
p_2	-1		+1										
p_3		+1											
			+1										
...								...					+1
	+1												
			-1										
													+1
p_n		-1	+1										-1

Row vector p_i

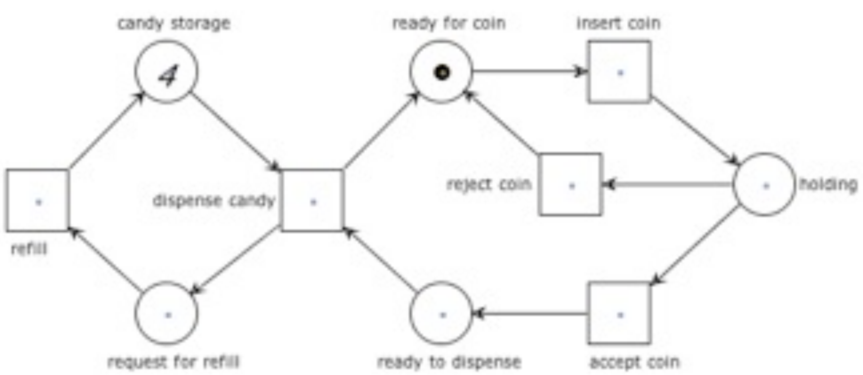
	t_1	t_2	t_3					...					t_m
p_1		+1	-1										-1
p_2	-1		+1										
p_3		+1											
			+1										
...								...					+1
	+1												
			-1										
													+1
p_n		-1	+1										-1

$$\mathbf{p_i} : T \to \{-1, 0, 1\}$$
such that
$$\mathbf{p_i}(t) = \mathbf{N}(p_i, t)$$

Example: vending machine



Example: vending machine



	refill t_1	dispense candy t_2	insert coin t_3	accept coin t_4	reject coin t_5
candy storage p_1	1	-1			
request for refill p_2	-1	1			
ready for coin p_3		1	-1		1
holding p_4			1	-1	-1
ready to dispense p_5		-1		1	

Vectors: notation

Let $E = \{e_1, e_2, ..., e_n\}$ be a finite set of elements.

Any mapping $v : E \to \mathbb{Q}$ (or to $\mathbb{N}$, $\mathbb{Z}$,...) can be regarded as a vector:

$$\mathbf{v} = [\, v(e_1),\, v(e_2),\, ...,\, v(e_n)\,]$$

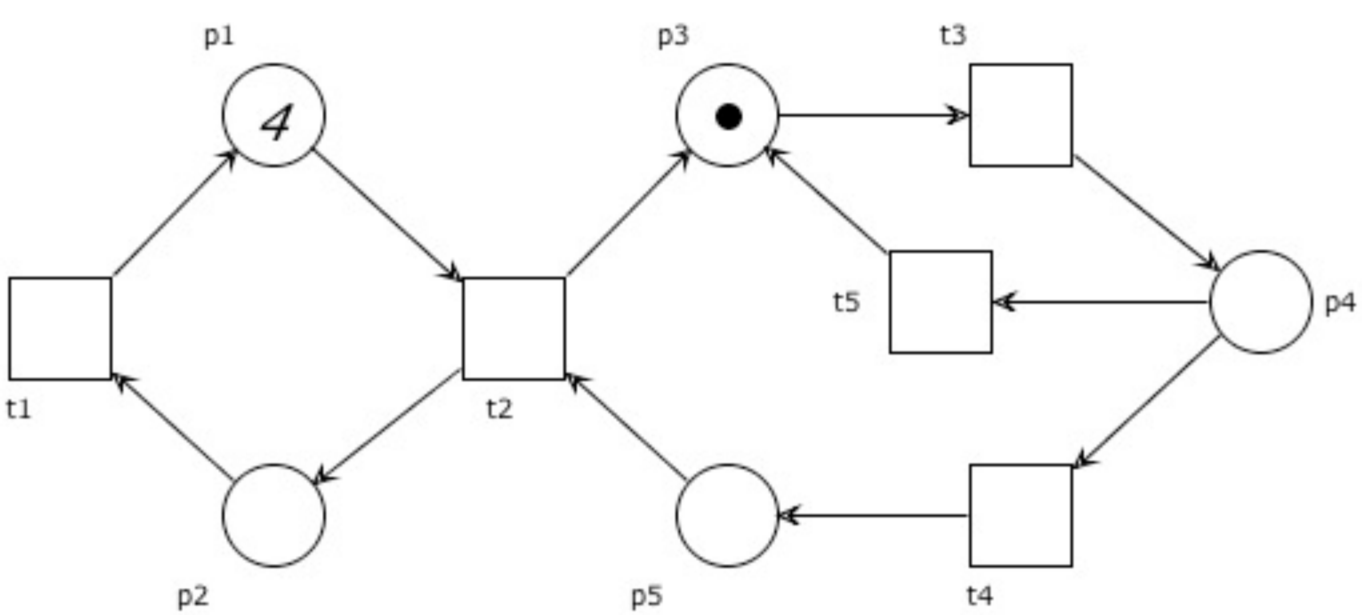
We **do not** use different symbols for row and columns vectors:

$$\mathbf{v} = \begin{bmatrix} v(e_1) \\ v(e_2) \\ \vdots \\ v(e_n) \end{bmatrix}$$

Marking as a vector

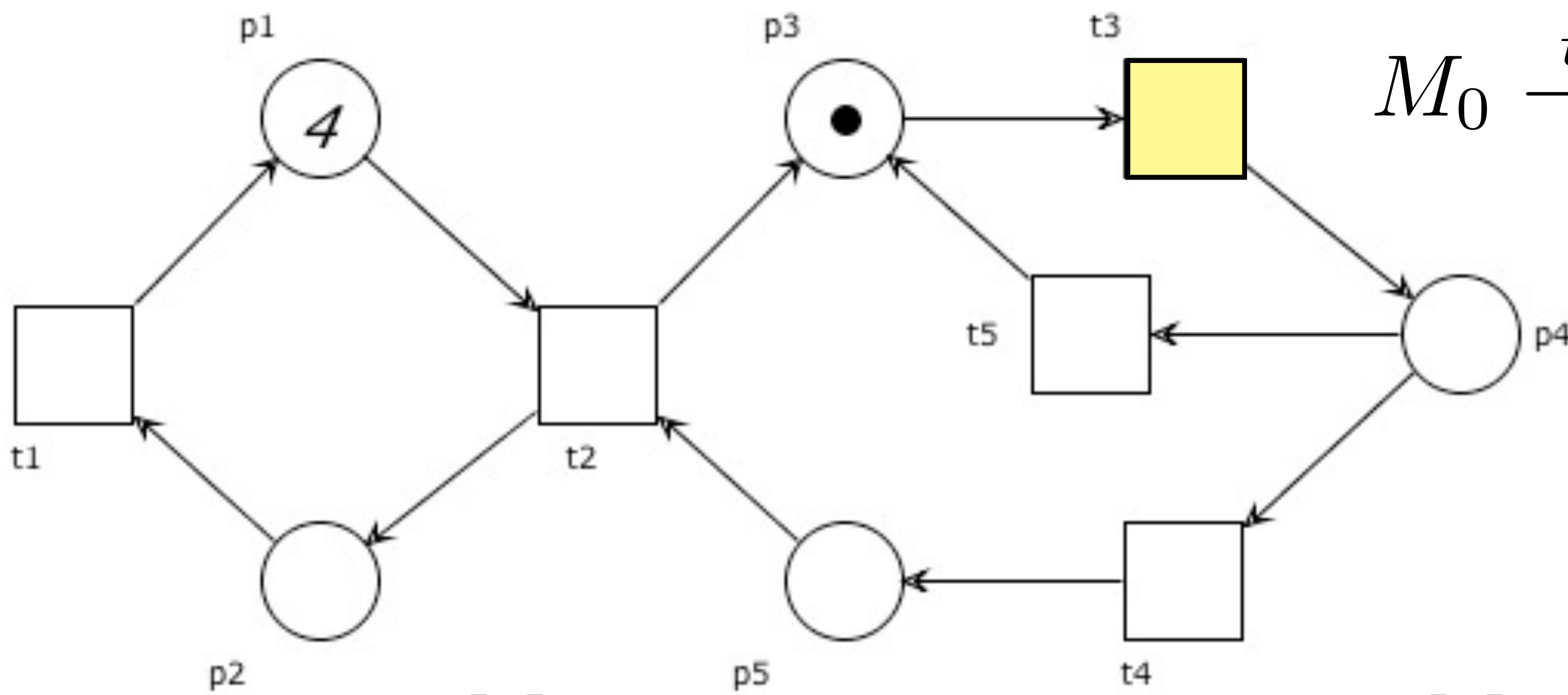
Any marking $M : P \to \mathbb{N}$ corresponds to a vector:

$$M = [\ M(p_1) \quad M(p_2) \quad ... \quad M(p_n)\ ]$$



$M_0 = [\ 4\ \ 0\ \ 1\ \ 0\ \ 0\ ]$

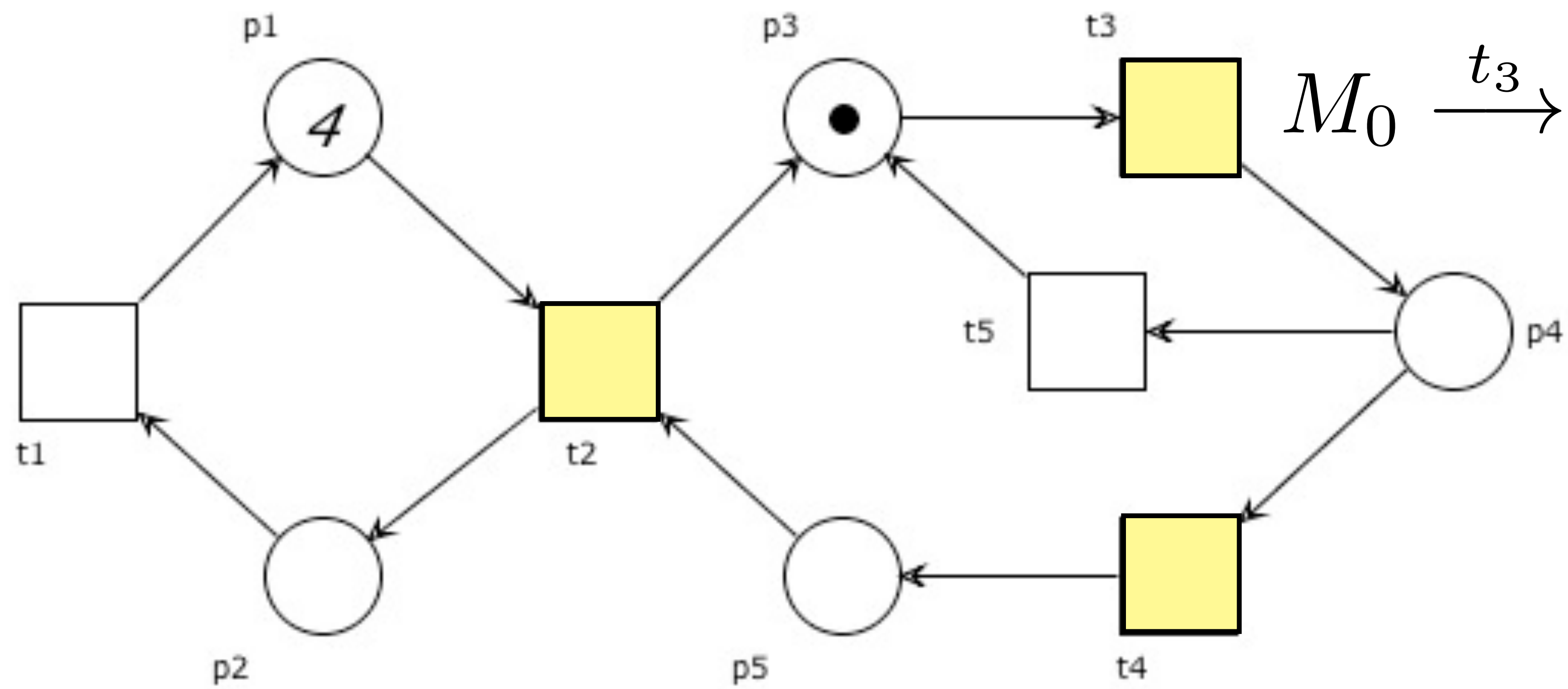
Firing, in vector notation



$$M_0 \xrightarrow{t_3} M_1 = 4p_1 + p_4$$

$$\overset{M_0}{\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + \overset{t_3}{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}} = \overset{M_1}{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

Firing, in vector notation



$$M_0 \xrightarrow{t_3} M_1 \xrightarrow{t_4} M_2 \xrightarrow{t_2} M_3$$

$$\overset{M_0}{\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + \overset{t_3}{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}} = \overset{M_1}{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}} + \overset{t_4}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}} = \overset{M_2}{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} + \overset{t_2}{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}} = \overset{M_3}{\begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$$

Vectors: notation

Let $\mathbf{v}, \mathbf{w}$ be two vectors over E

We write $\mathbf{v} \leq \mathbf{w}$ if $v(e) \leq w(e)$ for any $e \in E$

We write $\mathbf{v} < \mathbf{w}$ if $v \leq w$ and $v(e) < w(e)$ for some $e \in E$

We write $\mathbf{v} \prec \mathbf{w}$ if $v(e) < w(e)$ for any $e \in E$

We let $\mathbf{0}$ denote any vector of any length whose entries are all 0

Products

Let $\mathbf{x}, \mathbf{y}$ be two vectors of equal length n (written $|\mathbf{x}| = |\mathbf{y}| = n$)

We define their scalar product by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

$$\left[\, x_1 \; x_2 \; ... \; x_n \,\right] \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + ... + x_n y_n$$

Products: example

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = (0 \cdot 1) + (1 \cdot 1) + (-1 \cdot 2) + (0 \cdot 0) \cdot (1 \cdot 1) = 0 + 1 - 2 + 0 + 1 = 0$$

Products

Let $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_k}, \mathbf{y}$ be all vectors of equal length

Let X be a $(k \times n)$-matrix whose i-th row is $\mathbf{x_i}$

We define the product $X \cdot \mathbf{y}$ as the (column) vector where

$$(X \cdot \mathbf{y})_i = \mathbf{x_i} \cdot \mathbf{y}$$

$$\begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \vdots \\ \mathbf{x_k} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \mathbf{x_1} \cdot \mathbf{y} \\ \mathbf{x_2} \cdot \mathbf{y} \\ \vdots \\ \mathbf{x_k} \cdot \mathbf{y} \end{bmatrix}$$

Products: example

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -1+1 \\ 1-2+1 \\ 2-1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Products

Let $\mathbf{x}, \mathbf{y_1}, \mathbf{y_2}, ..., \mathbf{y_k}$ be all vectors of equal length

Let Y be a $(n \times k)$-matrix whose i-th column is $\mathbf{y_i}$

We define the product $\mathbf{x} \cdot Y$ as the (row) vector where

$$(\mathbf{x} \cdot Y)_i = \mathbf{x} \cdot \mathbf{y_i}$$

$$[\, x_1 \; x_2 \; ... \; x_n \,] \cdot [\, \mathbf{y_1} \; \mathbf{y_2} \; ... \; \mathbf{y_k} \,] = [\, \mathbf{x} \cdot \mathbf{y_1} \quad \mathbf{x} \cdot \mathbf{y_2} \quad ... \quad \mathbf{x} \cdot \mathbf{y_k} \,]$$

Products

Let $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_k}, \mathbf{y_1}, \mathbf{y_2}, ..., \mathbf{y_h}$ be all vectors of equal length

Let X be a $(k \times n)$-matrix whose i-th row is $\mathbf{x_i}$

Let Y be a $(n \times h)$-matrix whose j-th column is $\mathbf{y_j}$

We define the product $X \cdot Y$ as the $(k \times h)$-matrix where

$$(X \cdot Y)_{i,j} = \mathbf{x_i} \cdot \mathbf{y_j}$$

$$\begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \vdots \\ \mathbf{x_k} \end{bmatrix} \cdot \left[\, \mathbf{y_1}\ \mathbf{y_2}\ ...\ \mathbf{y_h} \,\right] = \begin{bmatrix} \mathbf{x_1} \cdot \mathbf{y_1} & \mathbf{x_1} \cdot \mathbf{y_2} & ... & \mathbf{x_1} \cdot \mathbf{y_h} \\ \mathbf{x_2} \cdot \mathbf{y_1} & \mathbf{x_2} \cdot \mathbf{y_2} & ... & \mathbf{x_2} \cdot \mathbf{y_h} \\ \vdots & \vdots & & \vdots \\ \mathbf{x_k} \cdot \mathbf{y_1} & \mathbf{x_k} \cdot \mathbf{y_2} & ... & \mathbf{x_k} \cdot \mathbf{y_h} \end{bmatrix}$$

Vector perspective

Let $P = \{ p_1, ..., p_n \}$ and $T = \{ t_1, ..., t_m \}$

The net (P,T,F) can be seen as a matrix (n x m)

A marking is a vector of length n

But we miss an ingredient:
can a firing be seen as a vector?
can a firing sequence be seen as a vector?
(of limited length)

Parikh vectors of transition sequences

Let $N = (P, T, F)$ be a net and $\sigma \in T^*$ a finite sequence of transitions.

The **Parikh vector**

$$\vec{\sigma} : T \to \mathbb{N}$$

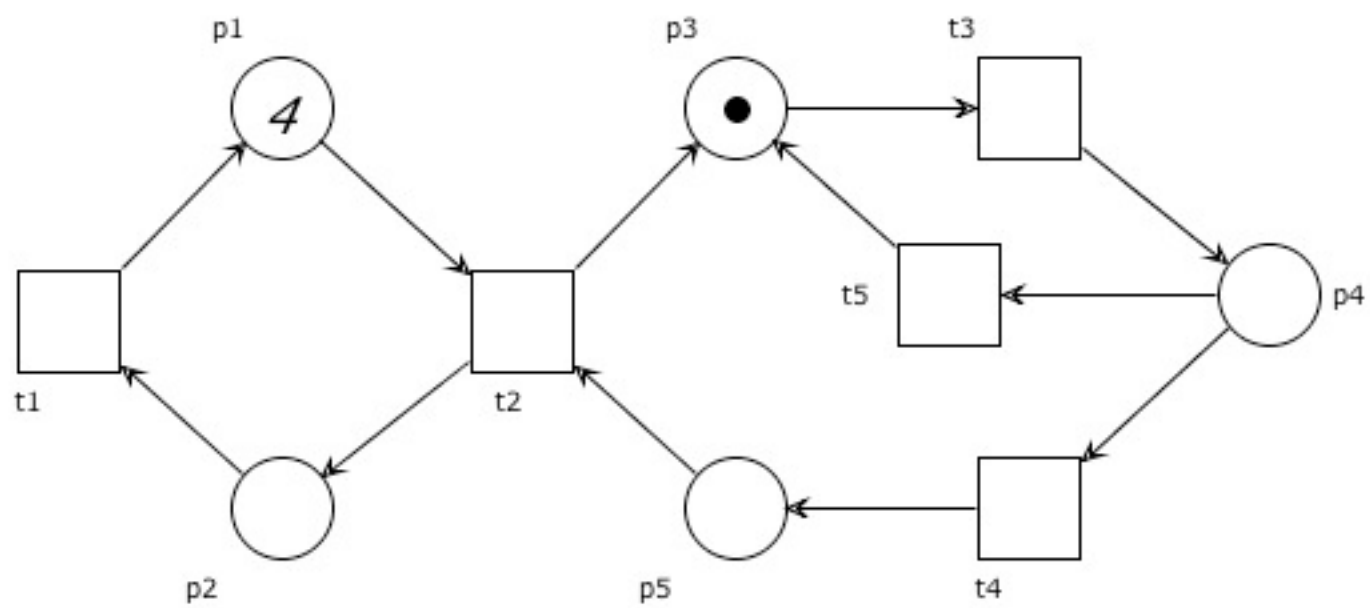
of σ maps every $t \in T$ to the number of its occurrences in σ.

Parikh vector of a firing

As a special case, for a sequence $\sigma = t$ (one single transition):

$$\vec{t} = [\ \underset{t_1}{0}\ \ \dots\ \ 0\ \ \underset{t}{1}\ \ 0\ \ \dots\ \ \underset{t_m}{0}\ ]$$

Parikh vector: example



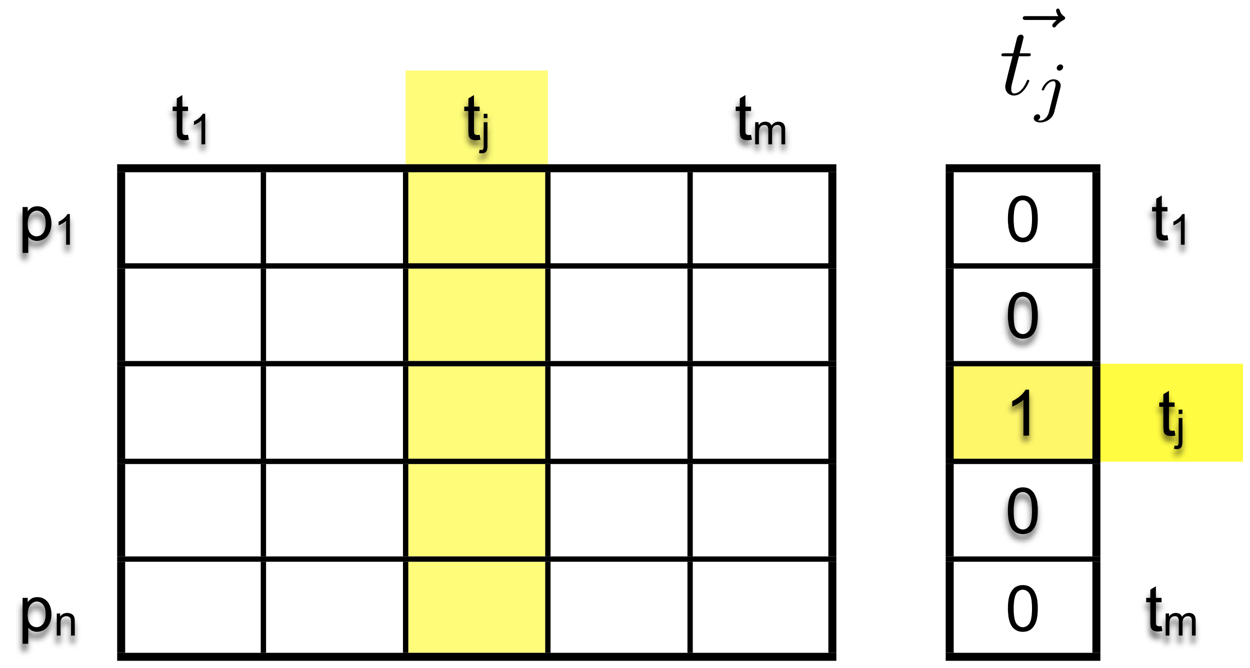
$$M_0 = 4p_1 + p_3$$

$$M_0 \xrightarrow{\sigma = t_3 t_5 t_3 t_4 t_2} 3p_1 + p_2 + p_3 \qquad \vec{\sigma} = [\,0 \quad 1 \quad 2 \quad 1 \quad 1\,]$$

$$M_0 \xrightarrow{\sigma' = t_3 t_4 t_2 t_3 t_4 t_2 t_3 t_5 t_3} 2p_1 + 2p_2 + p_4 \qquad \vec{\sigma'} = [\,0 \quad 2 \quad 4 \quad 2 \quad 1\,]$$

First fact

$$\mathbf{N} \cdot \vec{t_j} = \mathbf{t_j}$$



Second fact

$$\mathbf{N} \cdot \vec{t_j} = \mathbf{t_j}$$

If $M \xrightarrow{t} M'$ then $M' = M + \mathbf{t}$

Third fact

$$\mathbf{N} \cdot \vec{t_j} = \mathbf{t_j}$$

If $M \xrightarrow{t} M'$ then $M' = M + \mathbf{t}$

If $M \xrightarrow{t} M'$ then $M' = M + \mathbf{N} \cdot \vec{t}$

Marking equation lemma

Lemma: If $M \xrightarrow{\sigma} M'$ then $M' = M + \mathbf{N} \cdot \vec{\sigma}$

The proof is by induction on the length of σ

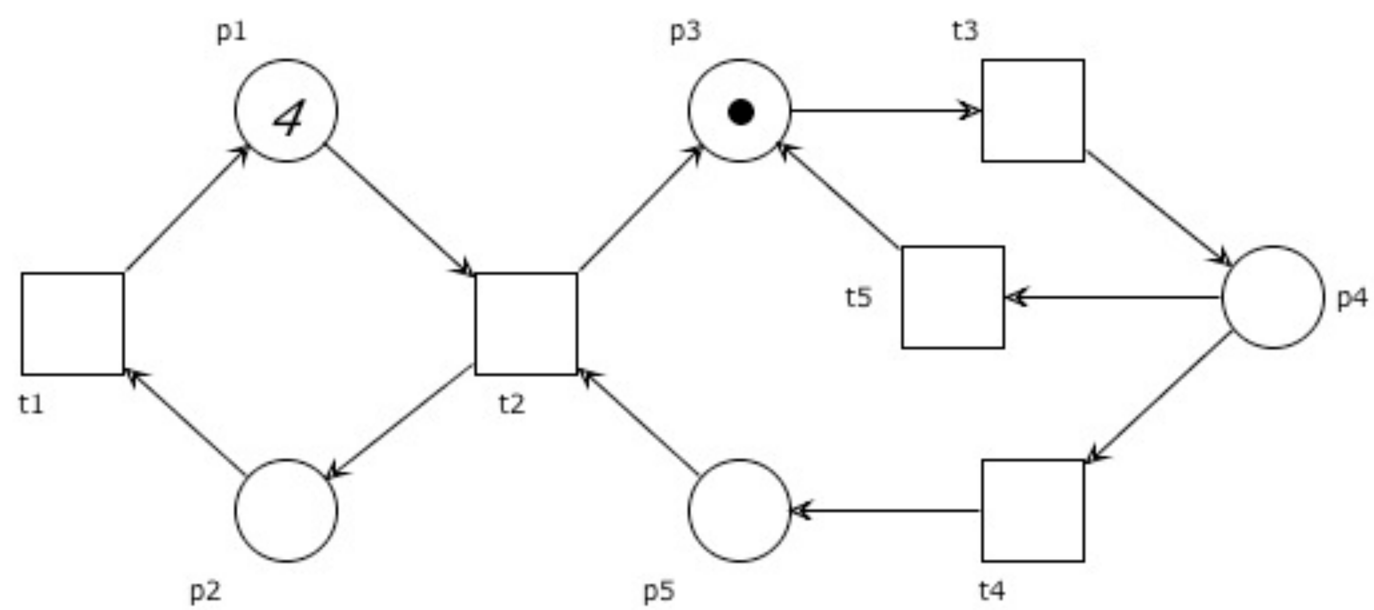
base $(\sigma = \epsilon)$**:** and therefore $M' = M$. The equality hold trivially, because $\vec{\sigma} = \mathbf{0}$

induction $(\sigma = \sigma' t$ **for some sequence** σ' **and transition** $t)$**:**

Let $M \xrightarrow{\sigma'} M'' \xrightarrow{t} M'$. We have:

$$\begin{aligned} M' &= M'' + \mathbf{t} \\ &= M'' + \mathbf{N} \cdot \vec{t} \\ &= M + \mathbf{N} \cdot \vec{\sigma'} + \mathbf{N} \cdot \vec{t} \\ &= M + \mathbf{N} \cdot (\vec{\sigma'} + \vec{t}) \\ &= M + \mathbf{N} \cdot (\overrightarrow{\sigma' t}) \\ &= M + \mathbf{N} \cdot \vec{\sigma} \end{aligned}$$

Marking equation: example



$M_0 = [\,4 \quad 0 \quad 1 \quad 0 \quad 0\,]$ $\sigma = t_3 t_5 t_3 t_4 t_2$ $\vec{\sigma} = [\,0 \quad 1 \quad 2 \quad 1 \quad 1\,]$

$$\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Marking equation lemma: consequences

The marking reached by any occurrence sequence only depends on the number of occurrences of each transition

It does not depend on the order in which transitions occur

Every fireable permutation of the same transitions leads to the same marking

Monotonicity lemma (1)

Lemma: If $M \xrightarrow{\sigma} M'$ then $M + L \xrightarrow{\sigma} M' + L$ for any L

The proof is by induction on the length of σ

base $(\sigma = \epsilon)$**:** the empty sequence is always enabled, at any marking

induction $(\sigma = \sigma' t$ **for some sequence** σ' **and transition** $t)$**:**

Let $M \xrightarrow{\sigma'} M'' \xrightarrow{t} M'$.

By the marking equation lemma: $M' = M'' + \mathbf{N} \cdot \vec{t}$

By the induction hypothesis $M + L \xrightarrow{\sigma'} M'' + L$

Moreover, $M'' + L \xrightarrow{t}$ because $M'' \xrightarrow{t}$.

By the marking equation lemma: $M'' + L \xrightarrow{t} M'' + L + \mathbf{N} \cdot \vec{t} = M' + L$

Monotonicity lemma (2)

Lemma: If $M \xrightarrow{\sigma}$ then $M + L \xrightarrow{\sigma}$ for any L

If σ is finite then the thesis follows from monotonicity lemma 1

If σ is infinite, then it suffices to prove that:
$M + L \xrightarrow{\sigma'}$ for any finite prefix σ' of σ

Take any such prefix σ'. Then, $M \xrightarrow{\sigma'}$ (because $M \xrightarrow{\sigma}$)
By the marking equation lemma, $M \xrightarrow{\sigma'} M + \mathbf{N} \cdot \vec{\sigma'}$.
By monotonicity lemma 1, $M + L \xrightarrow{\sigma'} M + \mathbf{N} \cdot \vec{\sigma'} + L$
Hence $M + L \xrightarrow{\sigma'}$