Business Processes Modelling

MPB (6 cfu, 295AA)

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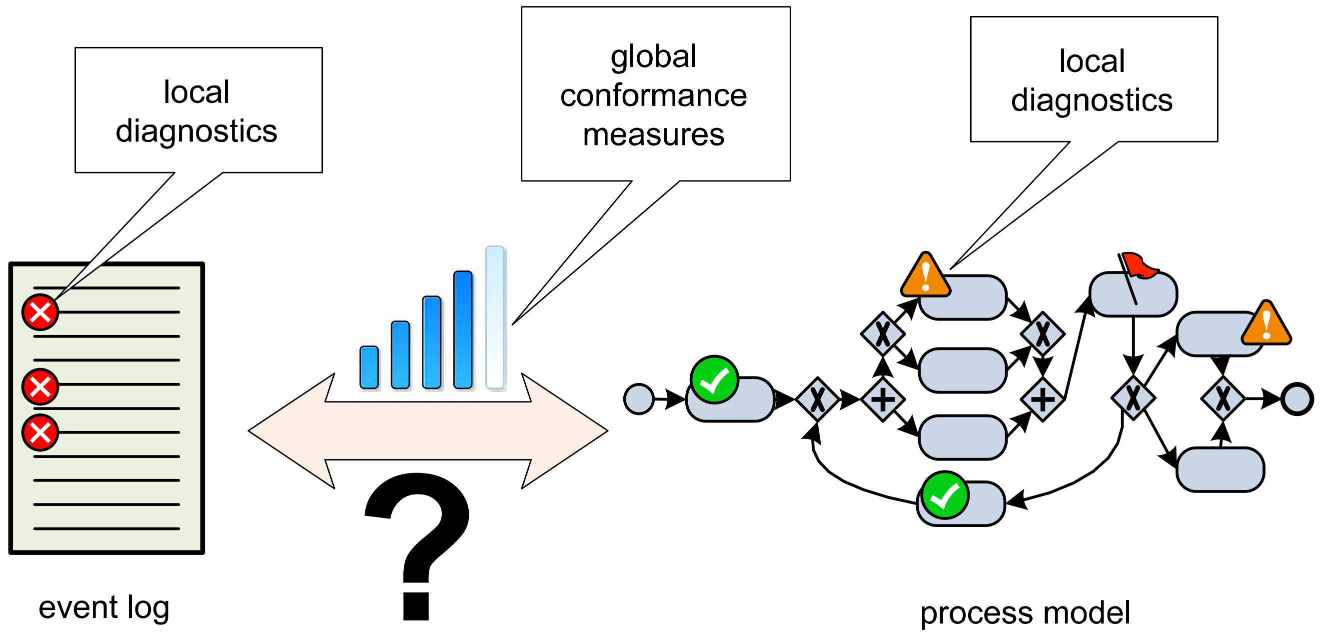
05 - Process Mining

Object



We overview the key principles of process mining

Chapters 1, 6. Process Mining. W. van der Aalst

Process Mining

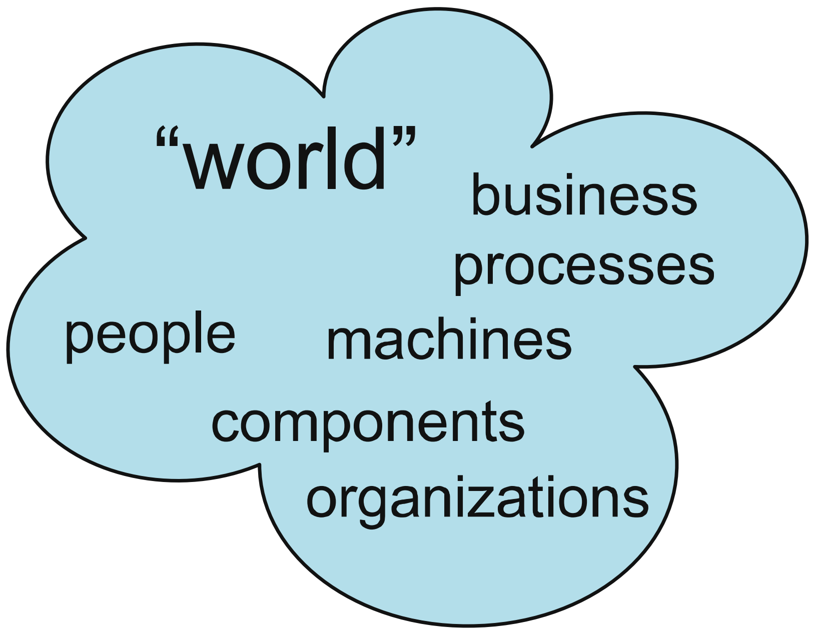
Process mining is a relative young research discipline
that sits between

machine learning and data mining on the one hand

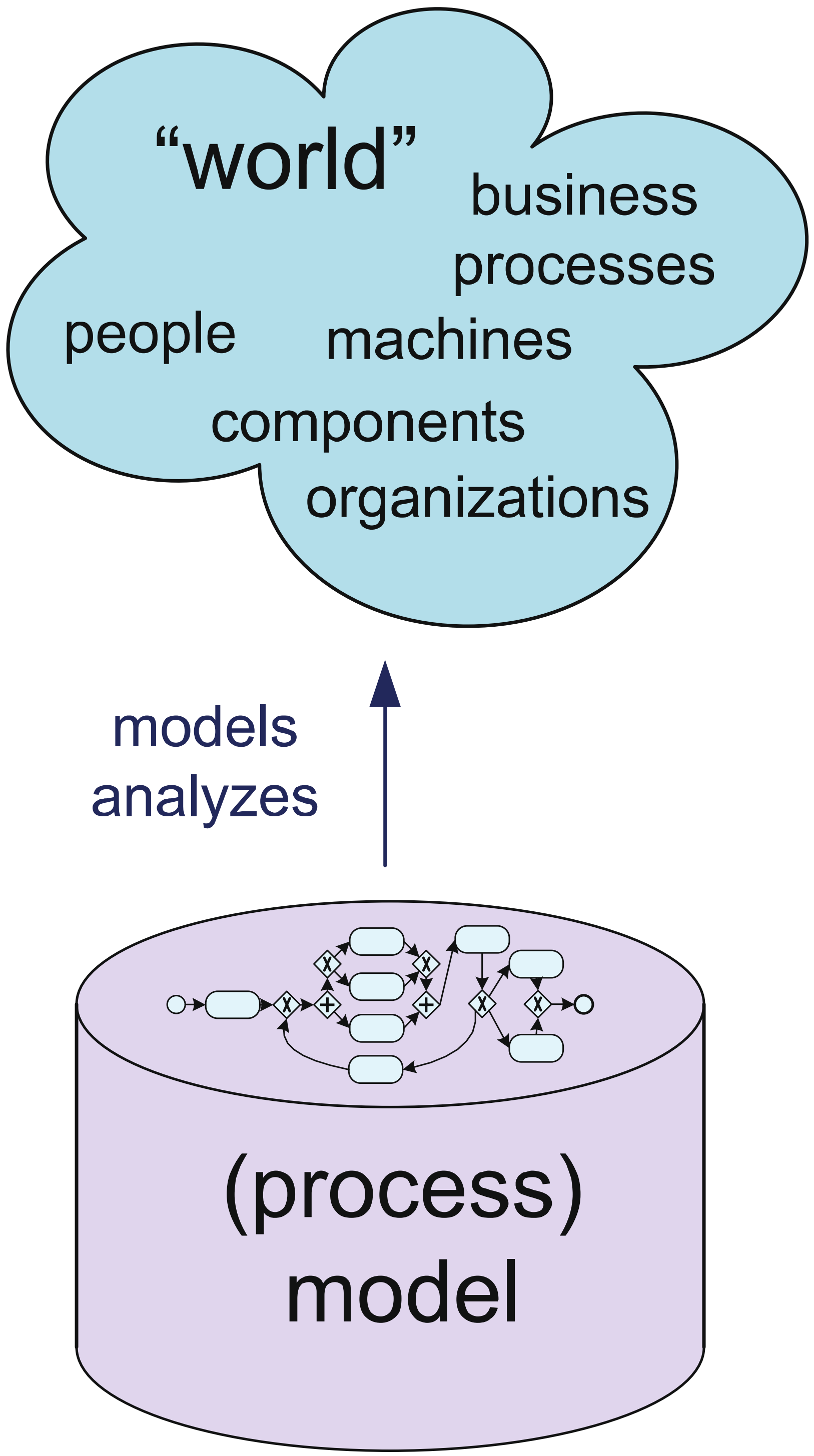
and process modeling and analysis on the other hand.

The idea is to discover, monitor and improve real processes
by **extracting knowledge from event logs**
readily available in today's systems.

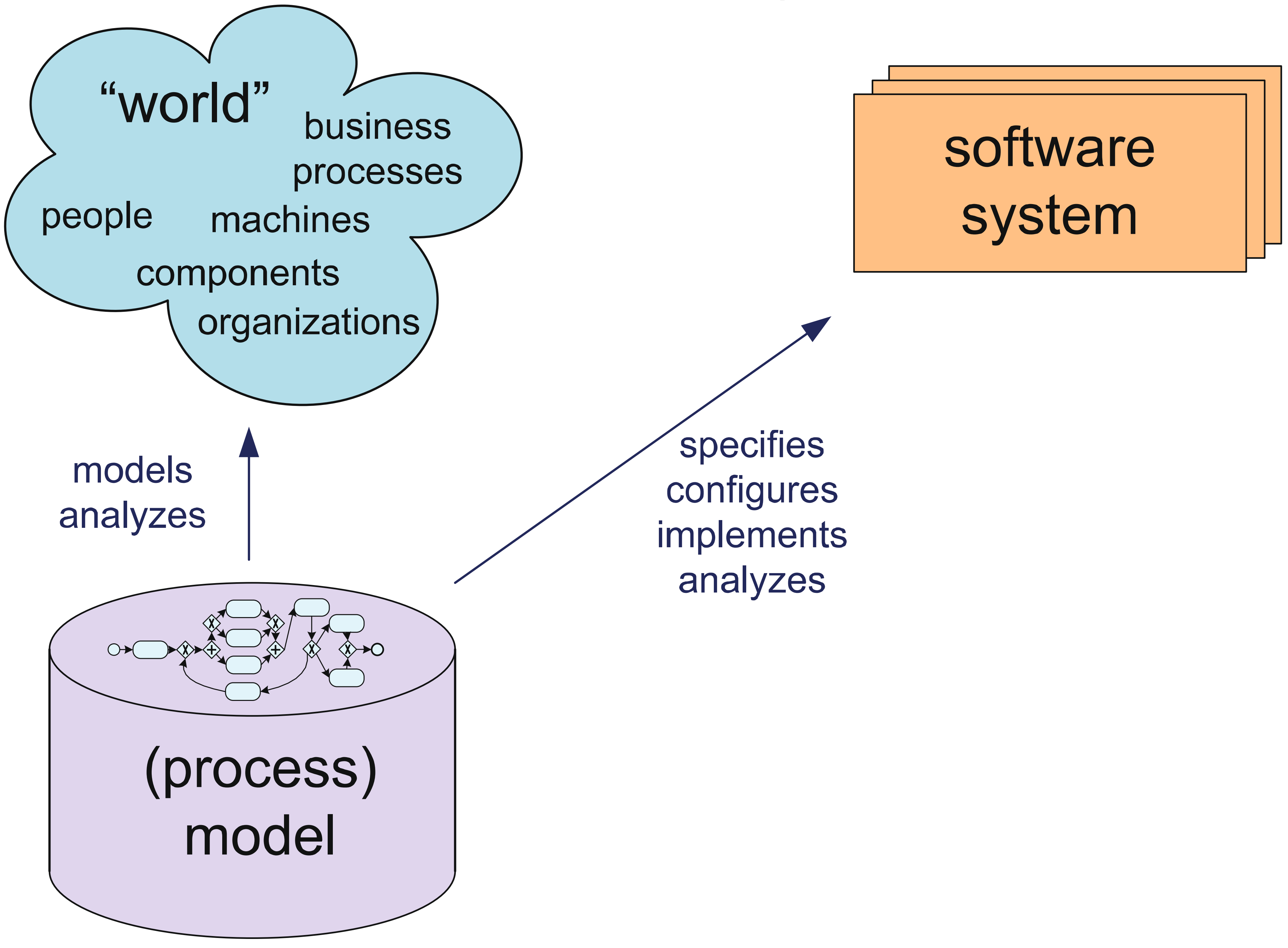
Process Mining Scheme



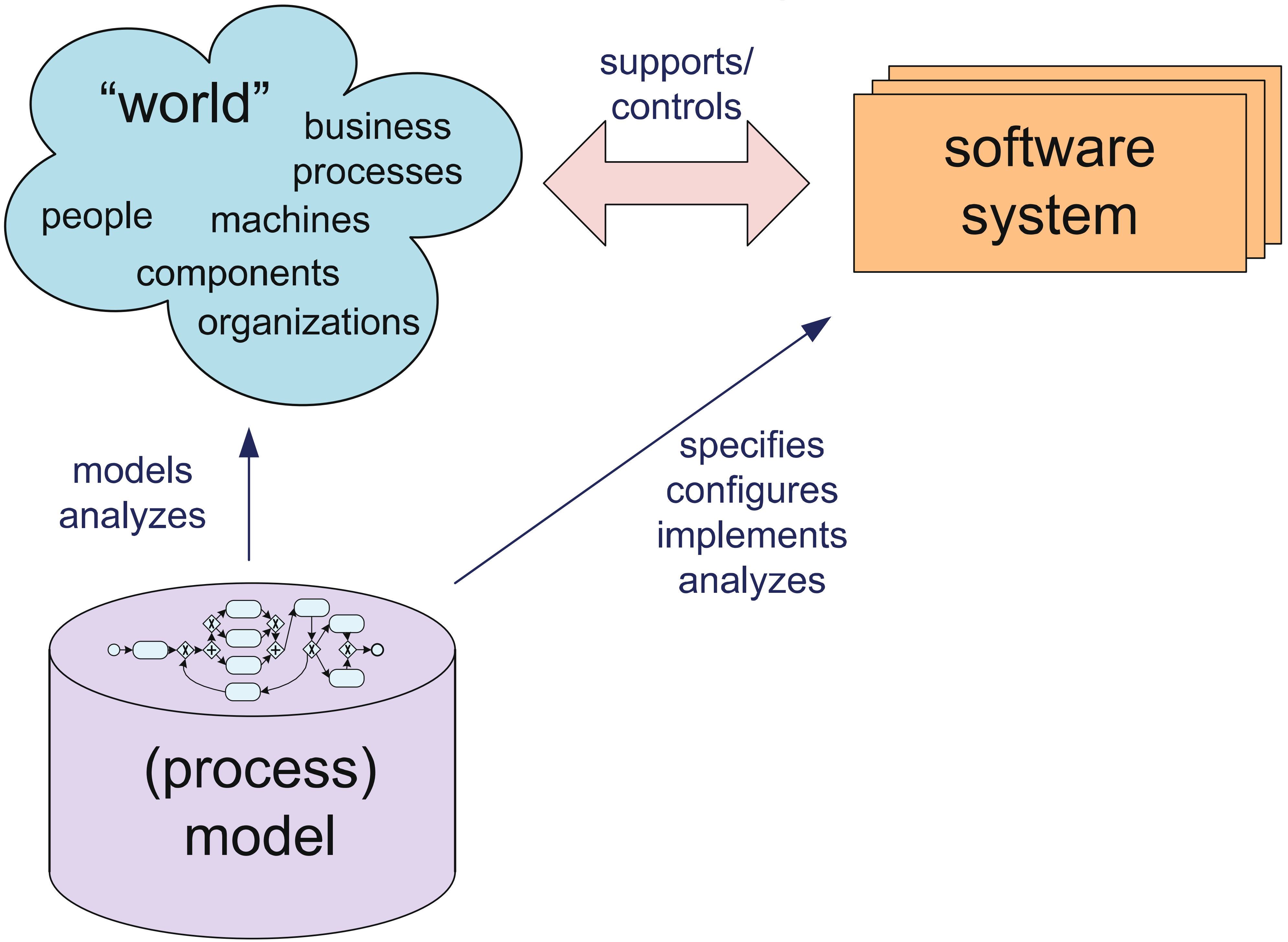
Process Mining Scheme



Process Mining Scheme



Process Mining Scheme



Processes, Cases, Events, Attributes

Each process defines a set of cases.

Each case consists of a series of events.

Each event is a unique instance of a task in the process and it relates to exactly one case

Events are ordered in time through their timestamps.

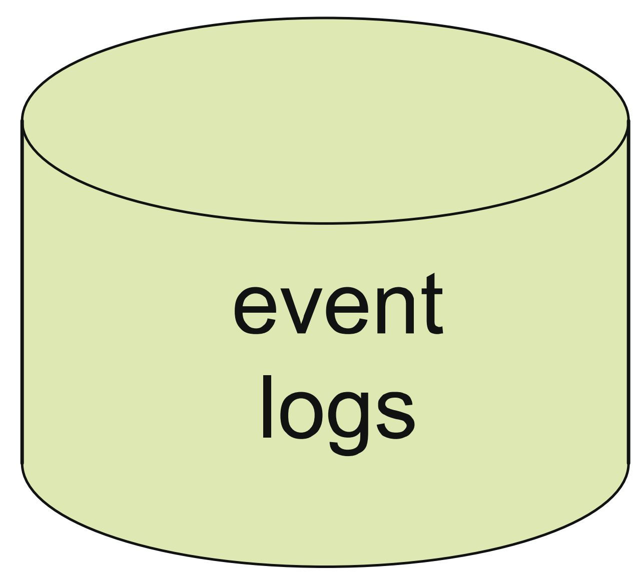
Events can have attributes, like
case id, activity, timestamp, duration, cost, and resource.

Event Logs

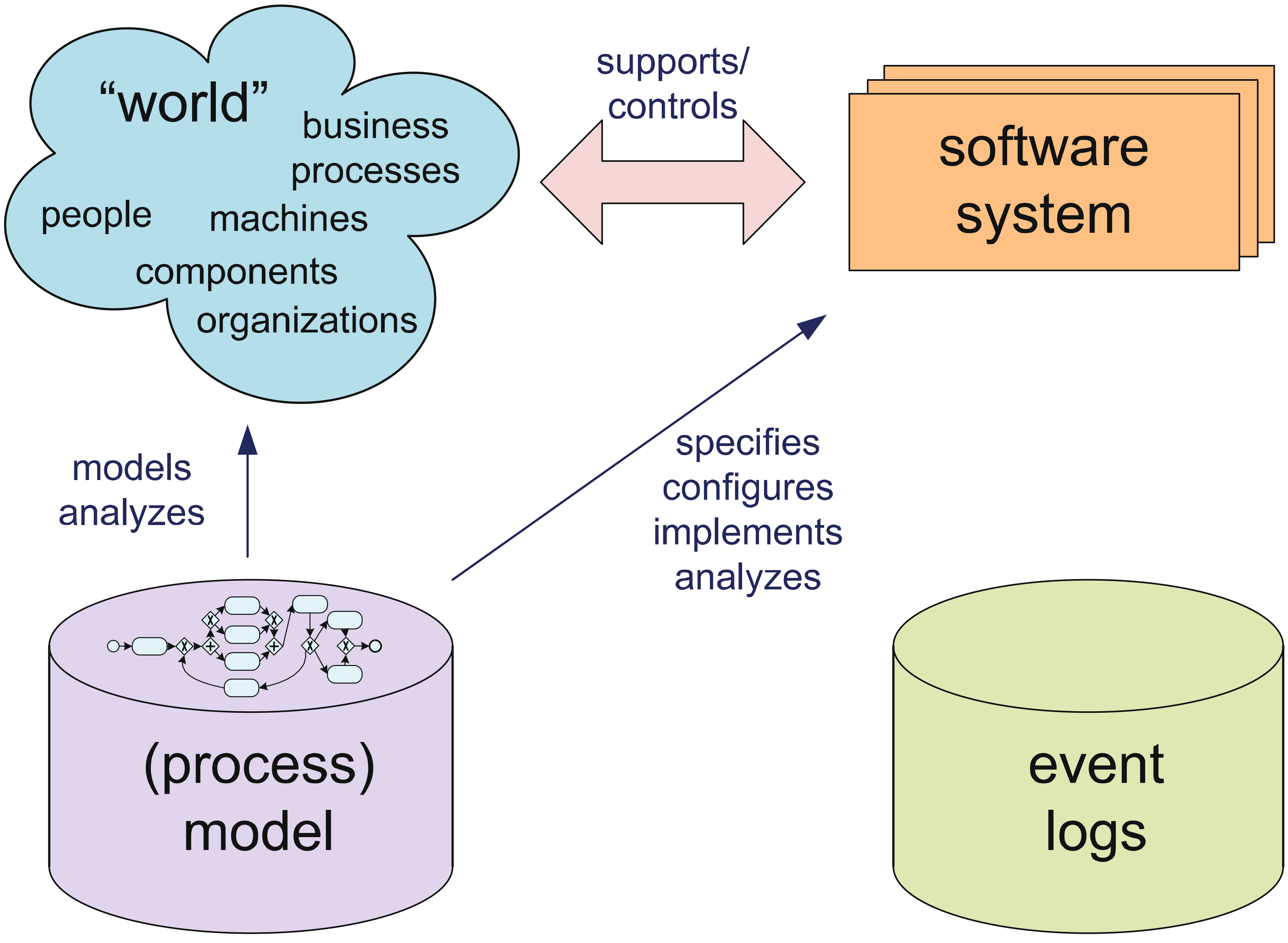
Let us assume that some software system
can record the information about all events:

each event is related by "case id" to a particular case
(i.e., a process instance).

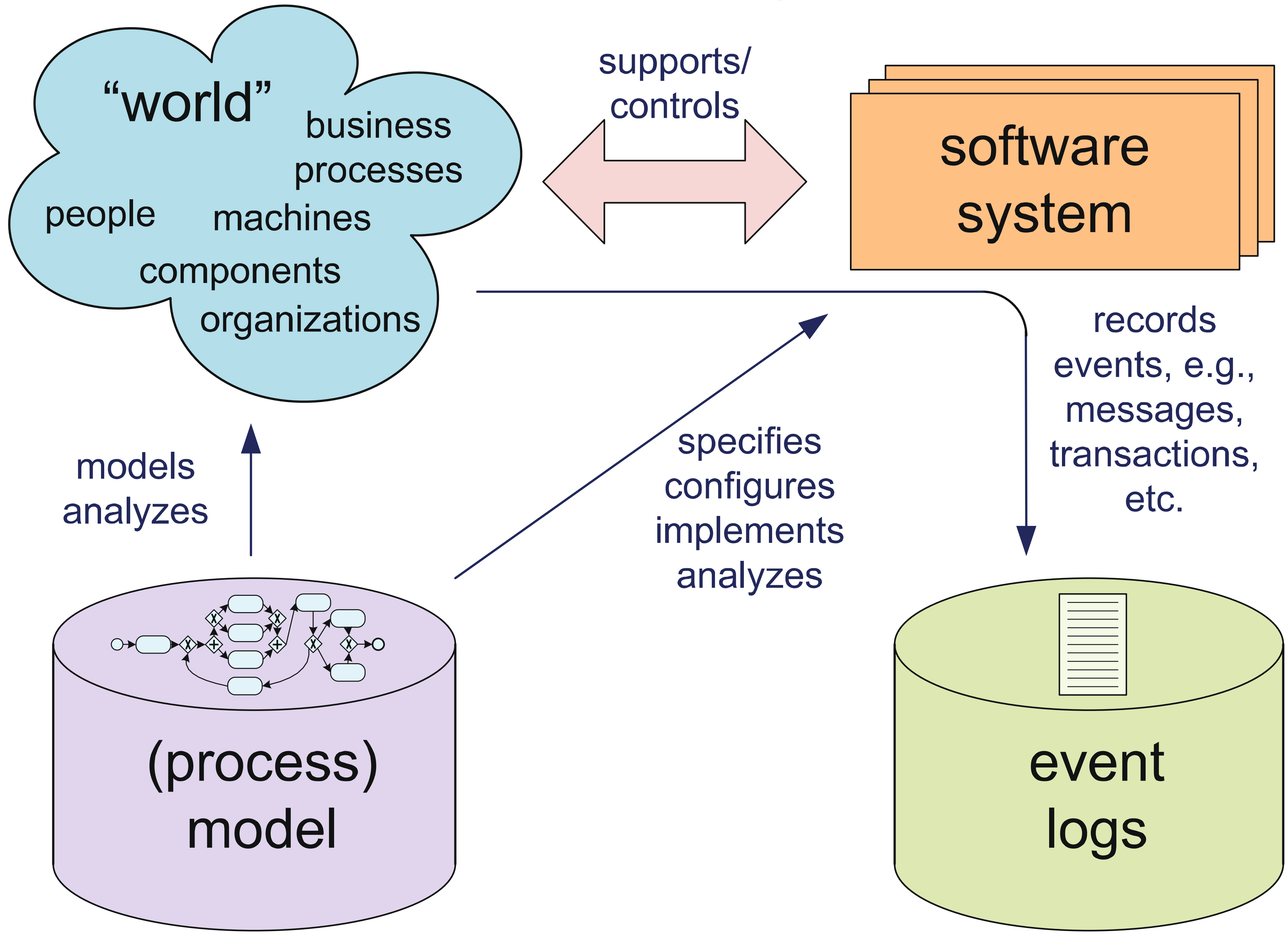
each event refers a particular "activity"
(i.e., a well-defined step in the process)



Process Mining Scheme



Process Mining Scheme



Event Log Example

Case id	Event id	Properties				
		Timestamp	Activity	Resource	Cost	...
1	35654423	30-12-2010:11.02	Register request	Pete	50	...
2	35654483	30-12-2010:11.32	Register request	Mike	50	...
2	35654485	30-12-2010:12.12	Check ticket	Mike	100	...
2	35654487	30-12-2010:14.16	Examine casually	Pete	400	...
1	35654424	31-12-2010:10.06	Examine thoroughly	Sue	400	...
2	35654488	05-01-2011:11.22	Decide	Sara	200	...
1	35654425	05-01-2011:15.12	Check ticket	Mike	100	...
1	35654426	06-01-2011:11.18	Decide	Sara	200	...
1	35654427	07-01-2011:14.24	Reject request	Pete	200	...
2	35654489	08-01-2011:12.05	Pay compensation	Ellen	200	...

Event Log Example

ordered by Timestamp

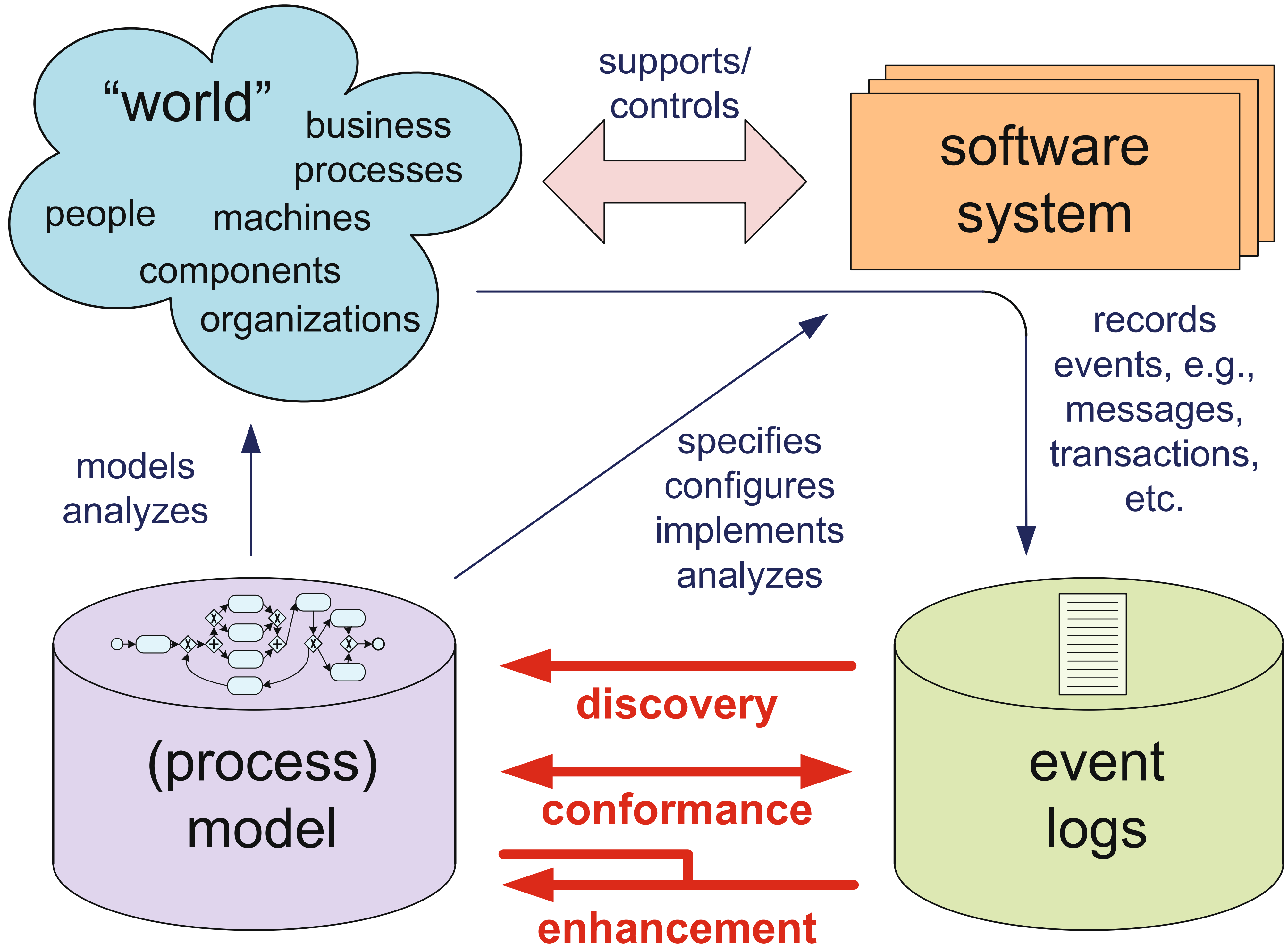
Case id	Event id	Properties				
		Timestamp	Activity	Resource	Cost	...
1	35654423	30-12-2010:11.02	Register request	Pete	50	...
2	35654483	30-12-2010:11.32	Register request	Mike	50	...
2	35654485	30-12-2010:12.12	Check ticket	Mike	100	...
2	35654487	30-12-2010:14.16	Examine casually	Pete	400	...
1	35654424	31-12-2010:10.06	Examine thoroughly	Sue	400	...
2	35654488	05-01-2011:11.22	Decide	Sara	200	...
1	35654425	05-01-2011:15.12	Check ticket	Mike	100	...
1	35654426	06-01-2011:11.18	Decide	Sara	200	...
1	35654427	07-01-2011:14.24	Reject request	Pete	200	...
2	35654489	08-01-2011:12.05	Pay compensation	Ellen	200	...

Event Log Example

grouped by Case id,
ordered by Timestamp

Case id	Event id	Properties				
		Timestamp	Activity	Resource	Cost	...
1	35654423	30-12-2010:11.02	Register request	Pete	50	...
	35654424	31-12-2010:10.06	Examine thoroughly	Sue	400	...
	35654425	05-01-2011:15.12	Check ticket	Mike	100	...
	35654426	06-01-2011:11.18	Decide	Sara	200	...
	35654427	07-01-2011:14.24	Reject request	Pete	200	...
2	35654483	30-12-2010:11.32	Register request	Mike	50	...
	35654485	30-12-2010:12.12	Check ticket	Mike	100	...
	35654487	30-12-2010:14.16	Examine casually	Pete	400	...
	35654488	05-01-2011:11.22	Decide	Sara	200	...
	35654489	08-01-2011:12.05	Pay compensation	Ellen	200	...

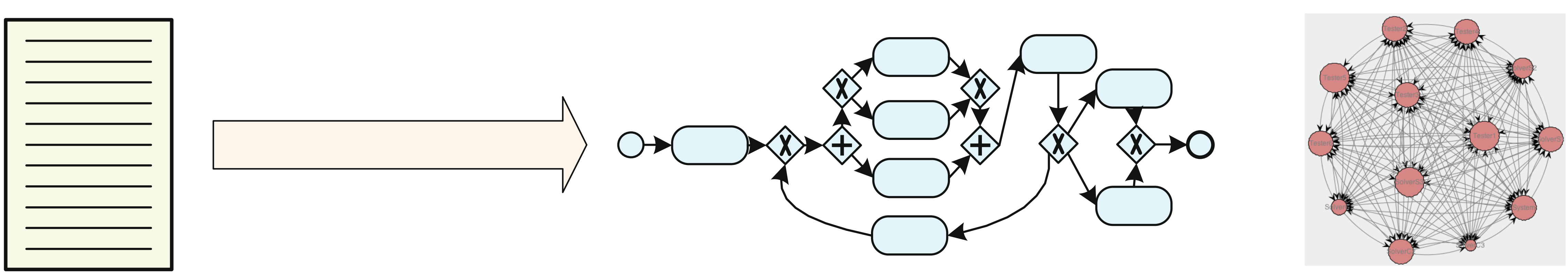
Process Mining Scheme



Discovery

A **discovery** technique
takes an event log and produces a model
(without using any a-priori information)

If the event log contains information about resources,
one can also discover resource-related models,
e.g., a social network
showing how people work together in an organization.



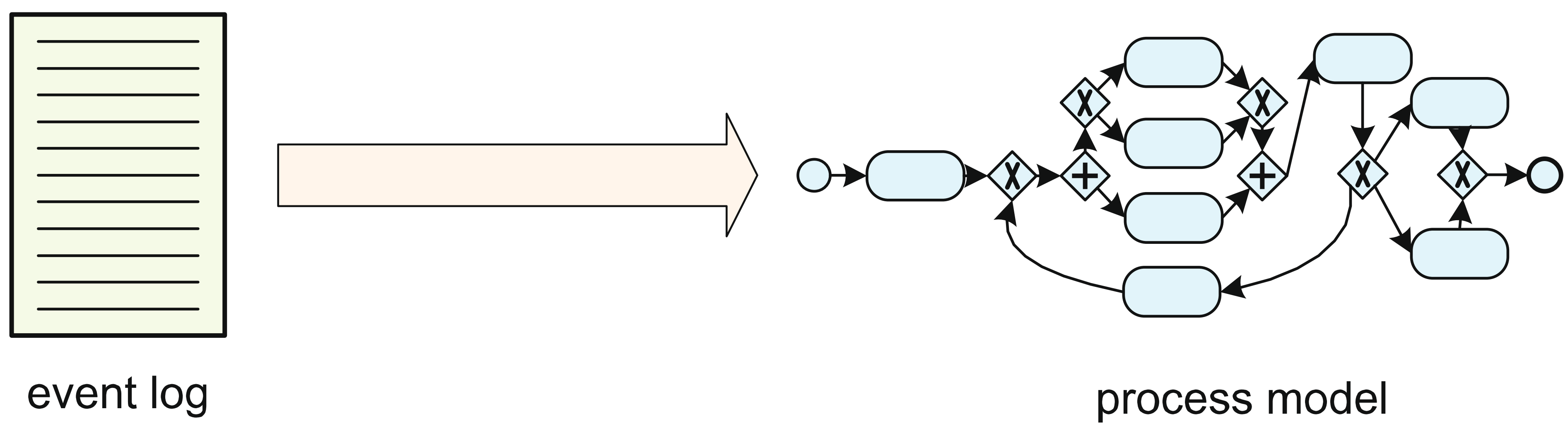
event log

process model

social network

Play-in

Play-In



Mining

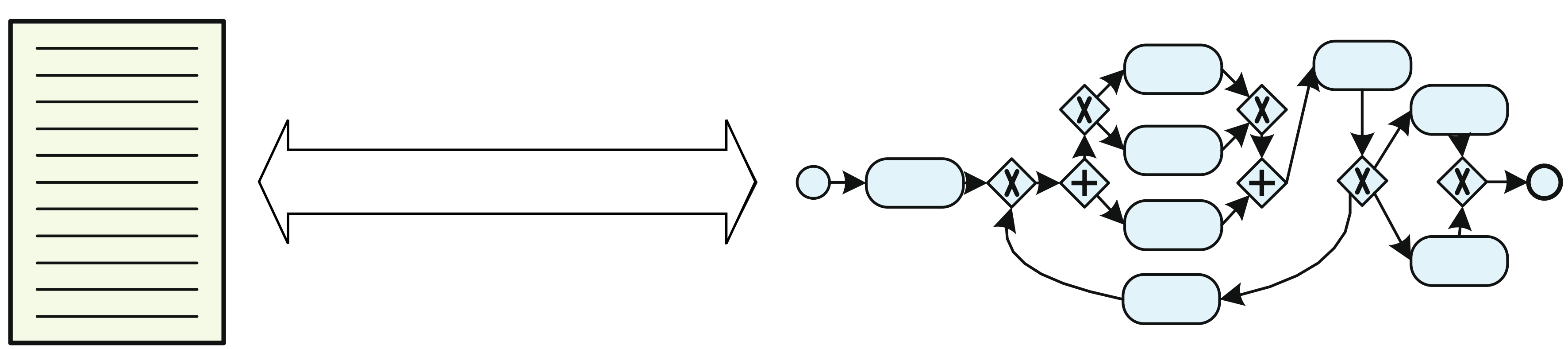
Discovery

Conformance

Conformance checking measures
how reality, as recorded in the log,
conforms to the process model, and vice versa.

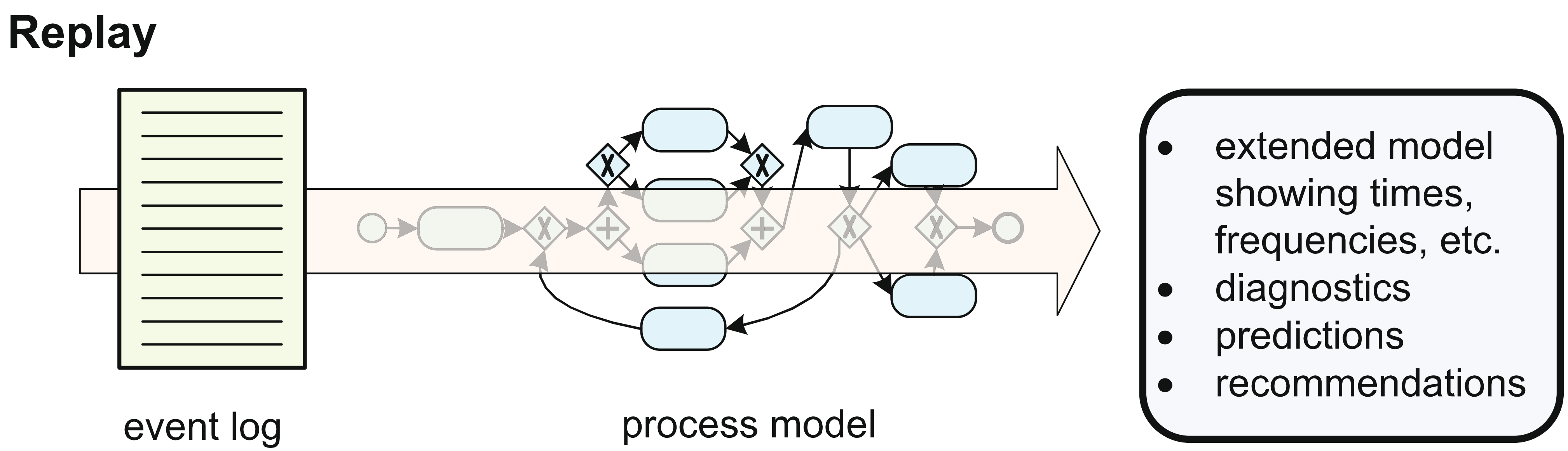
An existing process model is compared with an event log.

Conformance checking may be used
to detect, locate and explain deviations, and
to measure the severity of these deviations.



event log process model

Replay



Conformance checking

Performance analysis

Bottlenecks detection

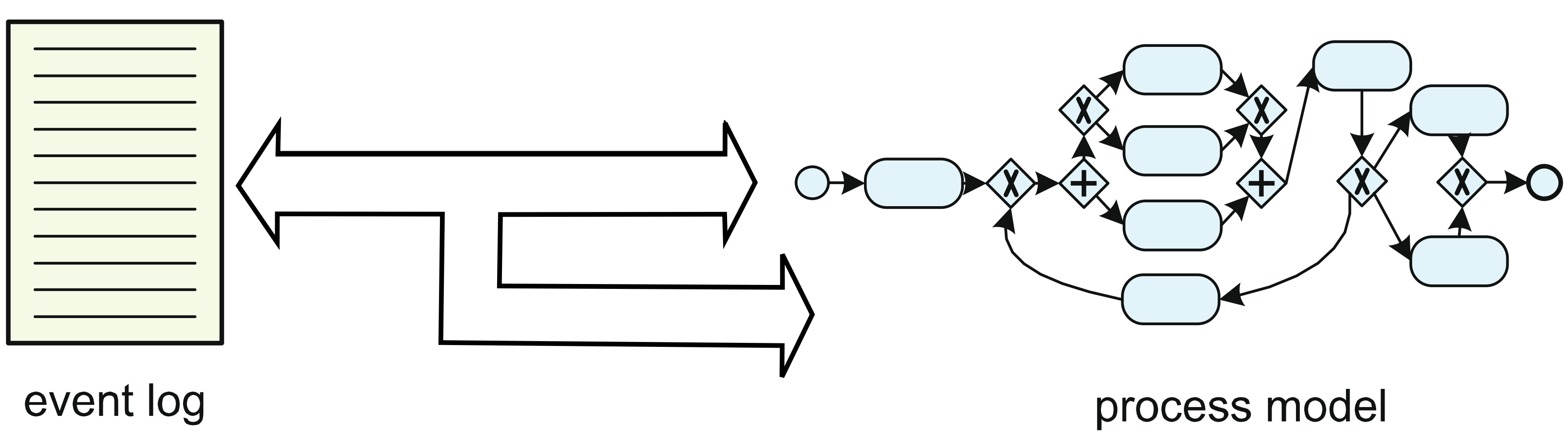
Predictive models (based on past)

Operational support (deviation detection)

Enhancement

Whereas conformance checking
measures the alignment between a model and reality

enhancement aims to
extend/improve existing models/systems
using information about the actual process
recorded in some event log.



Enhancement: Two angles

First viewpoint (the model is supposed to be **descriptive**):
the model does not capture the real behavior
("the model is wrong, how to improve it?")

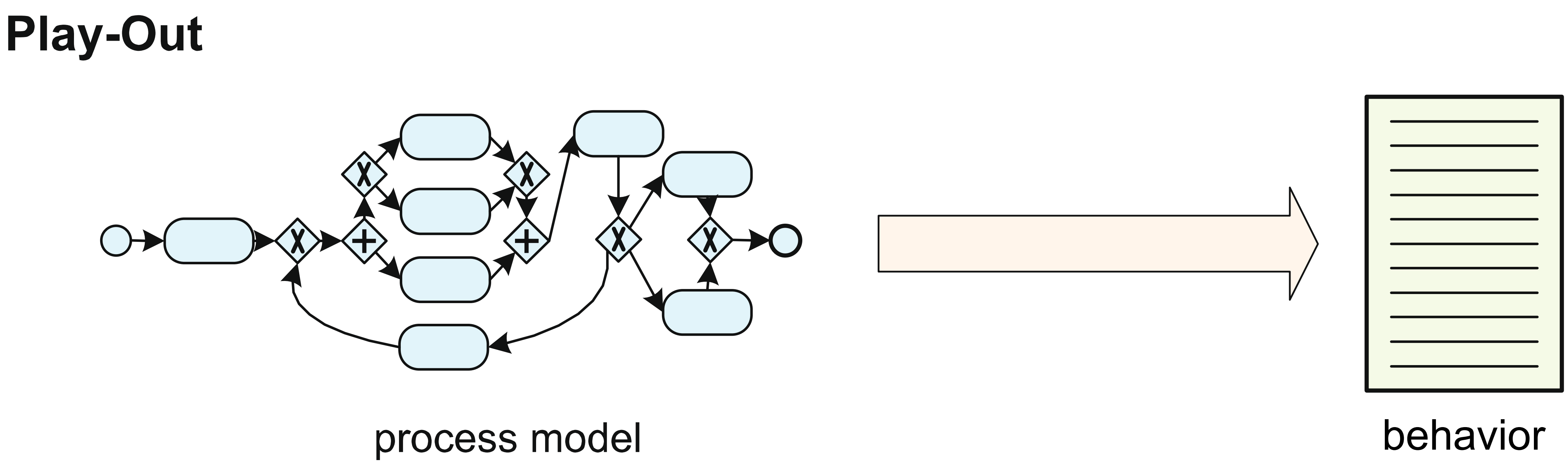
Second viewpoint (the model is **normative**)
reality deviates from the desired model
("the event log is wrong, how to control execution?").

Enhancement: Model Repair

One type of enhancement is **repair**,
i.e., modifying the model to better reflect reality.

For example, if two activities are modeled sequentially
but in reality can happen in any order,
then the model may be corrected to reflect this.

Play-out



Workflow engine
Simulation engine
Trace generation
Model checking

Discovery and conformance: an example

Event Log Fragment

Case id	Event id	Properties				
		Timestamp	Activity	Resource	Cost	...
1	35654423	30-12-2010:11.02	Register request	Pete	50	...
	35654424	31-12-2010:10.06	Examine thoroughly	Sue	400	...
	35654425	05-01-2011:15.12	Check ticket	Mike	100	...
	35654426	06-01-2011:11.18	Decide	Sara	200	...
	35654427	07-01-2011:14.24	Reject request	Pete	200	...
2	35654483	30-12-2010:11.32	Register request	Mike	50	...
	35654485	30-12-2010:12.12	Check ticket	Mike	100	...
	35654487	30-12-2010:14.16	Examine casually	Pete	400	...
	35654488	05-01-2011:11.22	Decide	Sara	200	...
	35654489	08-01-2011:12.05	Pay compensation	Ellen	200	...

Two cases Two traces Ten (ordered) events

Event Log Fragment

Case id	Event id	Properties				
		Timestamp	Activity	Resource	Cost	...
1			Register request			
			Examine thoroughly			
			Check ticket			
			Decide			
			Reject request			
2			Register request			
			Check ticket			
			Examine casually			
			Decide			
			Pay compensation			

Event Log Fragment

Case id	Event id	Properties				
		Timestamp	Activity	Resource	Cost	...
1			a Register request			
			b Examine thoroughly			
			d Check ticket			
			e Decide			
			h Reject request			
2			a Register request			
			d Check ticket			
			c Examine casually			
			e Decide			
			g Pay compensation			

Event Log Fragment

Case id	Event id	Properties				
		Timestamp	Activity	Resource	Cost	...
1			a			
			b			
			d			
			e			
			h			
2			a			
			d			
			c			
			e			
			g			

Event Log Fragment

Case id	Event id	Properties				
		Timestamp	Activity	Resource	Cost	...
1						
			$\langle$ a b d e h $\rangle$			
2						
			$\langle$ a d c e g $\rangle$			

Event Log Example

Case id	Event id	Properties				
		Timestamp	Activity	Resource	Cost	…
1	35654423	30-12-2010:11.02	Register request	Pete	50	…
	35654424	31-12-2010:10.06	Examine thoroughly	Sue	400	…
	35654425	05-01-2011:15.12	Check ticket	Mike	100	…
	35654426	06-01-2011:11.18	Decide	Sara	200	…
	35654427	07-01-2011:14.24	Reject request	Pete	200	…
2	35654483	30-12-2010:11.32	Register request	Mike	50	…
	35654485	30-12-2010:12.12	Check ticket	Mike	100	…
	35654487	30-12-2010:14.16	Examine casually	Pete	400	…
	35654488	05-01-2011:11.22	Decide	Sara	200	…
	35654489	08-01-2011:12.05	Pay compensation	Ellen	200	…
3	35654521	30-12-2010:14.32	Register request	Pete	50	…
	35654522	30-12-2010:15.06	Examine casually	Mike	400	…
	35654524	30-12-2010:16.34	Check ticket	Ellen	100	…
	35654525	06-01-2011:09.18	Decide	Sara	200	…
	35654526	06-01-2011:12.18	Reinitiate request	Sara	200	…
	35654527	06-01-2011:13.06	Examine thoroughly	Sean	400	…
	35654530	08-01-2011:11.43	Check ticket	Pete	100	…
	35654531	09-01-2011:09.55	Decide	Sara	200	…
	35654533	15-01-2011:10.45	Pay compensation	Ellen	200	…
4	35654641	06-01-2011:15.02	Register request	Pete	50	…
	35654643	07-01-2011:12.06	Check ticket	Mike	100	…
	35654644	08-01-2011:14.43	Examine thoroughly	Sean	400	…
	35654645	09-01-2011:12.02	Decide	Sara	200	…
	35654647	12-01-2011:15.44	Reject request	Ellen	200	…
5	35654711	06-01-2011:09.02	Register request	Ellen	50	…
	35654712	07-01-2011:10.16	Examine casually	Mike	400	…
	35654714	08-01-2011:11.22	Check ticket	Pete	100	…
	35654715	10-01-2011:13.28	Decide	Sara	200	…
	35654716	11-01-2011:16.18	Reinitiate request	Sara	200	…
	35654718	14-01-2011:14.33	Check ticket	Ellen	100	…
	35654719	16-01-2011:15.50	Examine casually	Mike	400	…
	35654720	19-01-2011:11.18	Decide	Sara	200	…
	35654721	20-01-2011:12.48	Reinitiate request	Sara	200	…
	35654722	21-01-2011:09.06	Examine casually	Sue	400	…
	35654724	21-01-2011:11.34	Check ticket	Pete	100	…
	35654725	23-01-2011:13.12	Decide	Sara	200	…
	35654726	24-01-2011:14.56	Reject request	Mike	200	…
6	35654871	06-01-2011:15.02	Register request	Mike	50	…
	35654873	06-01-2011:16.06	Examine casually	Ellen	400	…
	35654874	07-01-2011:16.22	Check ticket	Mike	100	…
	35654875	07-01-2011:16.52	Decide	Sara	200	…
	35654877	16-01-2011:11.47	Pay compensation	Mike	200	…
…	…	…	…	…	…	…

Table 1.2 A more compact representation of log shown in Table 1.1: a = *register request*, b = *examine thoroughly*, c = *examine casually*, d = *check ticket*, e = *decide*, f = *reinitiate request*, g = *pay compensation*, and h = *reject request*

Case id	Trace
1	$\langle a, b, d, e, h \rangle$
2	$\langle a, d, c, e, g \rangle$
3	$\langle a, c, d, e, f, b, d, e, g \rangle$
4	$\langle a, d, b, e, h \rangle$
5	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h \rangle$
6	$\langle a, c, d, e, g \rangle$
…	…

Discovery Example

Case id	Trace
1	$\langle a, b, d, e, h\rangle$
2	$\langle a, d, c, e, g\rangle$
3	$\langle a, c, d, e, f, b, d, e, g\rangle$
4	$\langle a, d, b, e, h\rangle$
5	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h\rangle$
6	$\langle a, c, d, e, g\rangle$
...	...

Discovery Example

All cases start with a

Case id	Trace
1	$\langle a, b, d, e, h\rangle$
2	$\langle a, d, c, e, g\rangle$
3	$\langle a, c, d, e, f, b, d, e, g\rangle$
4	$\langle a, d, b, e, h\rangle$
5	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h\rangle$
6	$\langle a, c, d, e, g\rangle$
...	...

Discovery Example

All cases start with a

Case id	Trace
1	$\langle a, b, d, e, h \rangle$
2	$\langle a, d, c, e, g \rangle$
3	$\langle a, c, d, e, f, b, d, e, g \rangle$
4	$\langle a, d, b, e, h \rangle$
5	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h \rangle$
6	$\langle a, c, d, e, g \rangle$
...	...

Discovery Example

All cases start with a and end with either g or h.

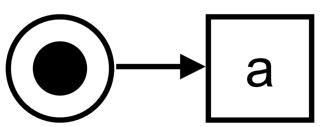
Case id	Trace
1	$\langle a, b, d, e, h \rangle$
2	$\langle a, d, c, e, g \rangle$
3	$\langle a, c, d, e, f, b, d, e, g \rangle$
4	$\langle a, d, b, e, h \rangle$
5	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h \rangle$
6	$\langle a, c, d, e, g \rangle$
...	...

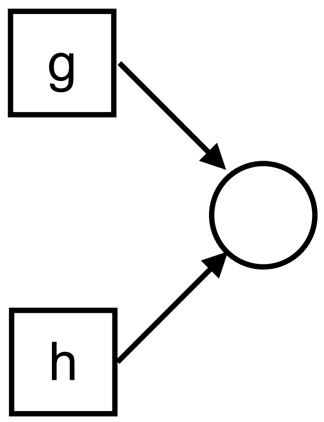
Discovery Example

All cases start with a
and end with either g or h.

Case id	Trace
1	$\langle a, b, d, e, h \rangle$
2	$\langle a, d, c, e, g \rangle$
3	$\langle a, c, d, e, f, b, d, e, g \rangle$
4	$\langle a, d, b, e, h \rangle$
5	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h \rangle$
6	$\langle a, c, d, e, g \rangle$
...	...

Discovery Example

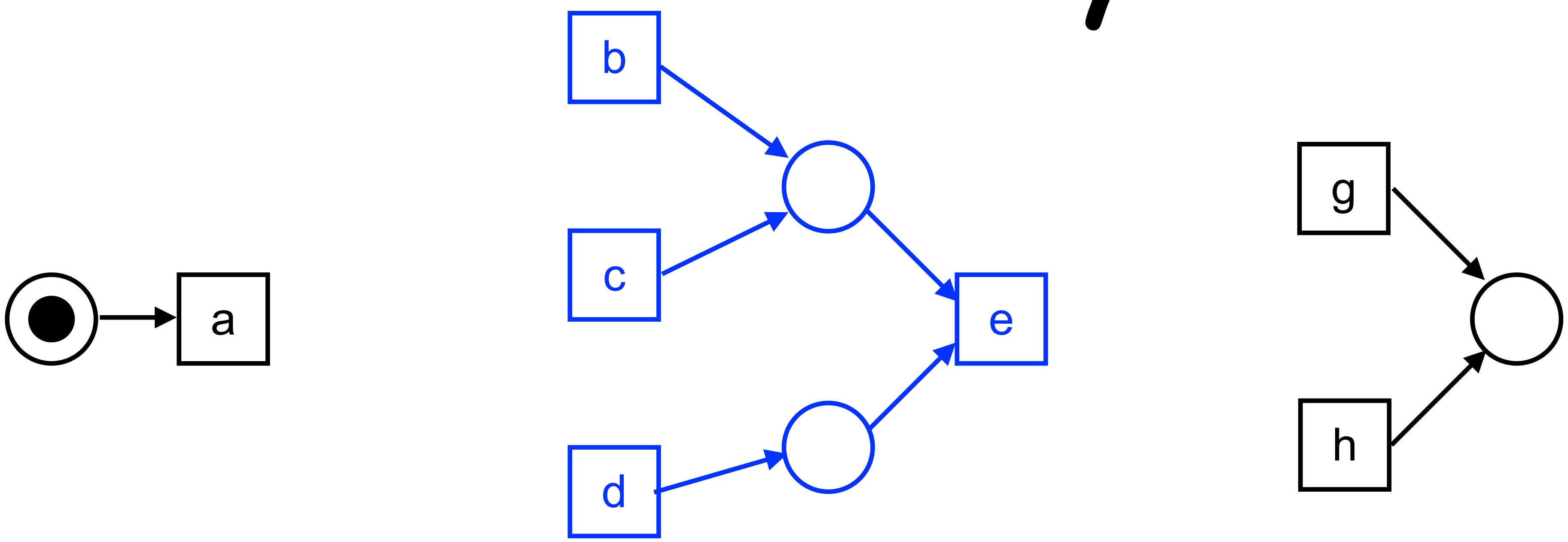




Every e is preceded by d and one of the examination activities (b or c).

Case id	Trace
1	$\langle a, b, \boxed{d, e}, h \rangle$
2	$\langle a, d, \boxed{c, e}, g \rangle$
3	$\langle a, c, \boxed{d, e}, f, b, \boxed{d, e}, g \rangle$
4	$\langle a, d, \boxed{b, e}, h \rangle$
5	$\langle a, c, \boxed{d, e}, f, d, \boxed{c, e}, f, c, \boxed{d, e}, h \rangle$
6	$\langle a, c, \boxed{d, e}, g \rangle$
...	...

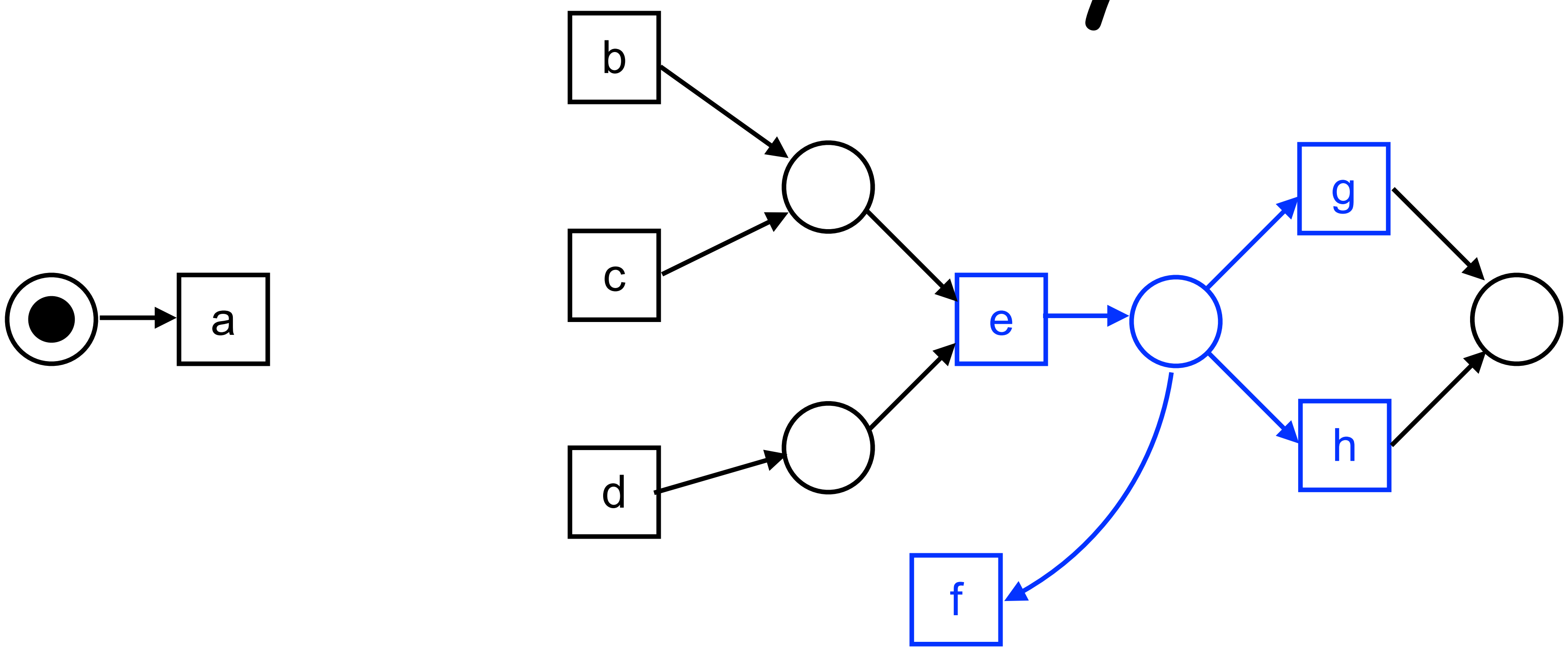
Discovery Example



Every e is preceded by d and one of the examination activities (b or c).

Case id	Trace
1	$\langle a, b, d, e, h\rangle$
2	$\langle a, d, c, e, g\rangle$
3	$\langle a, c, d, e, f, b, d, e, g\rangle$
4	$\langle a, d, b, e, h\rangle$
5	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h\rangle$
6	$\langle a, c, d, e, g\rangle$
...	...

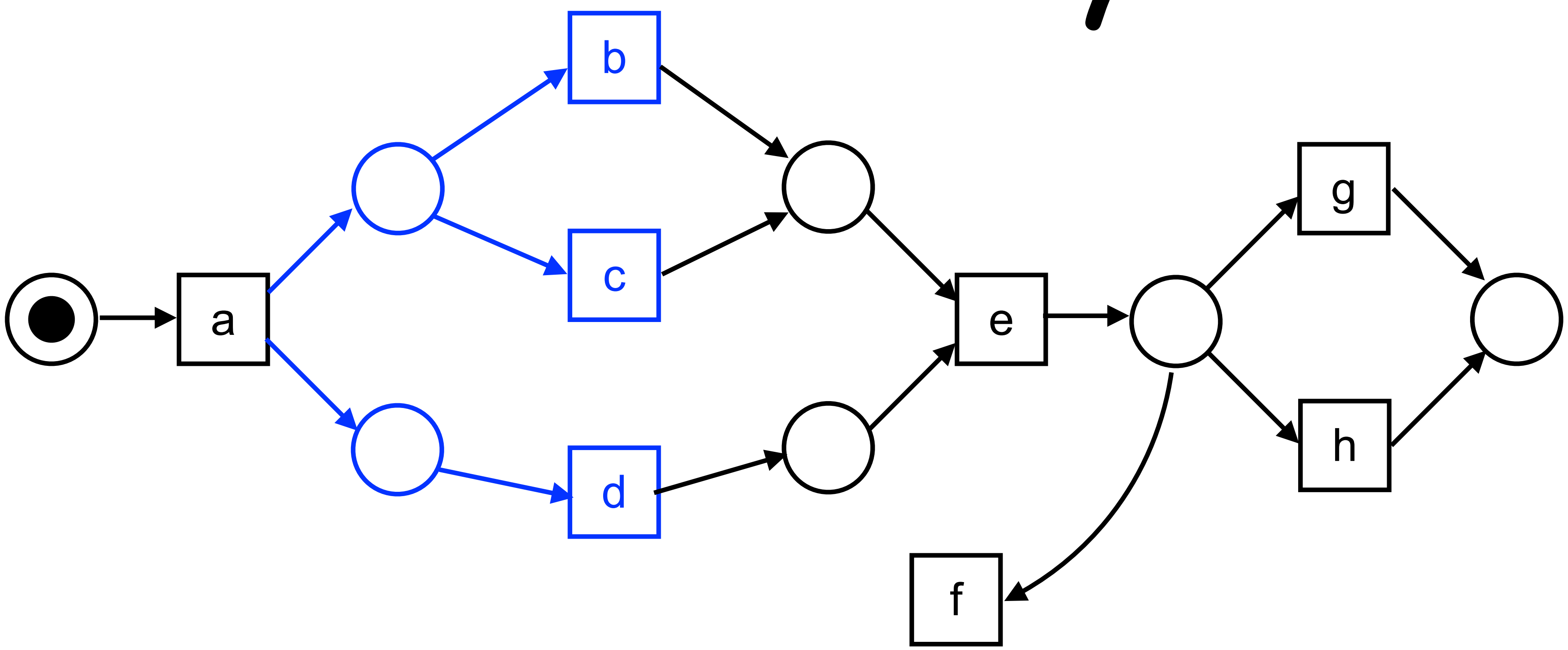
Discovery Example



Moreover, e is always followed by f, g, or h.

Case id	Trace
1	$\langle a, b, d, e, h\rangle$
2	$\langle a, d, c, e, g\rangle$
3	$\langle a, c, d, e, f, b, d, e, g\rangle$
4	$\langle a, d, b, e, h\rangle$
5	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h\rangle$
6	$\langle a, c, d, e, g\rangle$
...	...

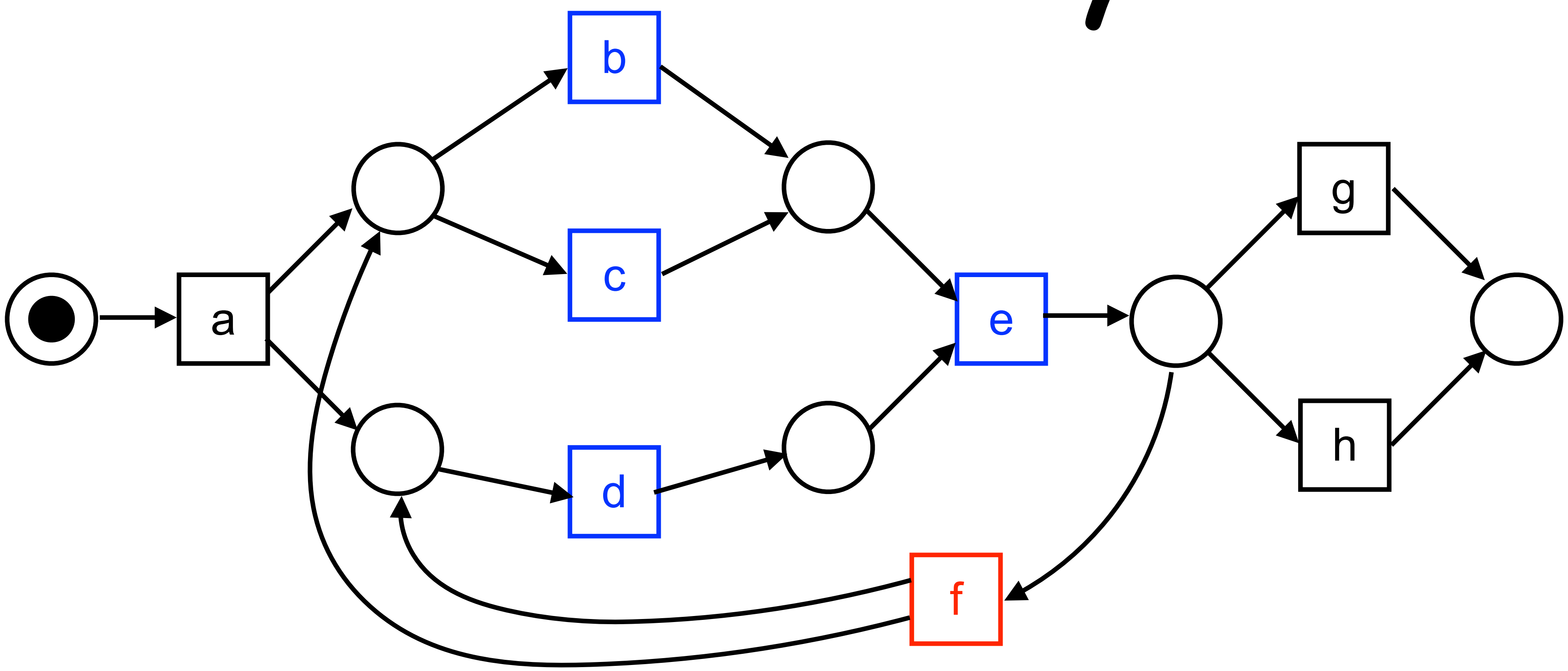
Discovery Example



b/c and d
are executed in any order
(bd,db,cd,dc)
which suggests they are
executed in parallel

Case id	Trace
1	$\langle a, b, d, e, h\rangle$
2	$\langle a, d, c, e, g\rangle$
3	$\langle a, c, d, e, f, b, d, e, g\rangle$
4	$\langle a, d, b, e, h\rangle$
5	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h\rangle$
6	$\langle a, c, d, e, g\rangle$
…	…

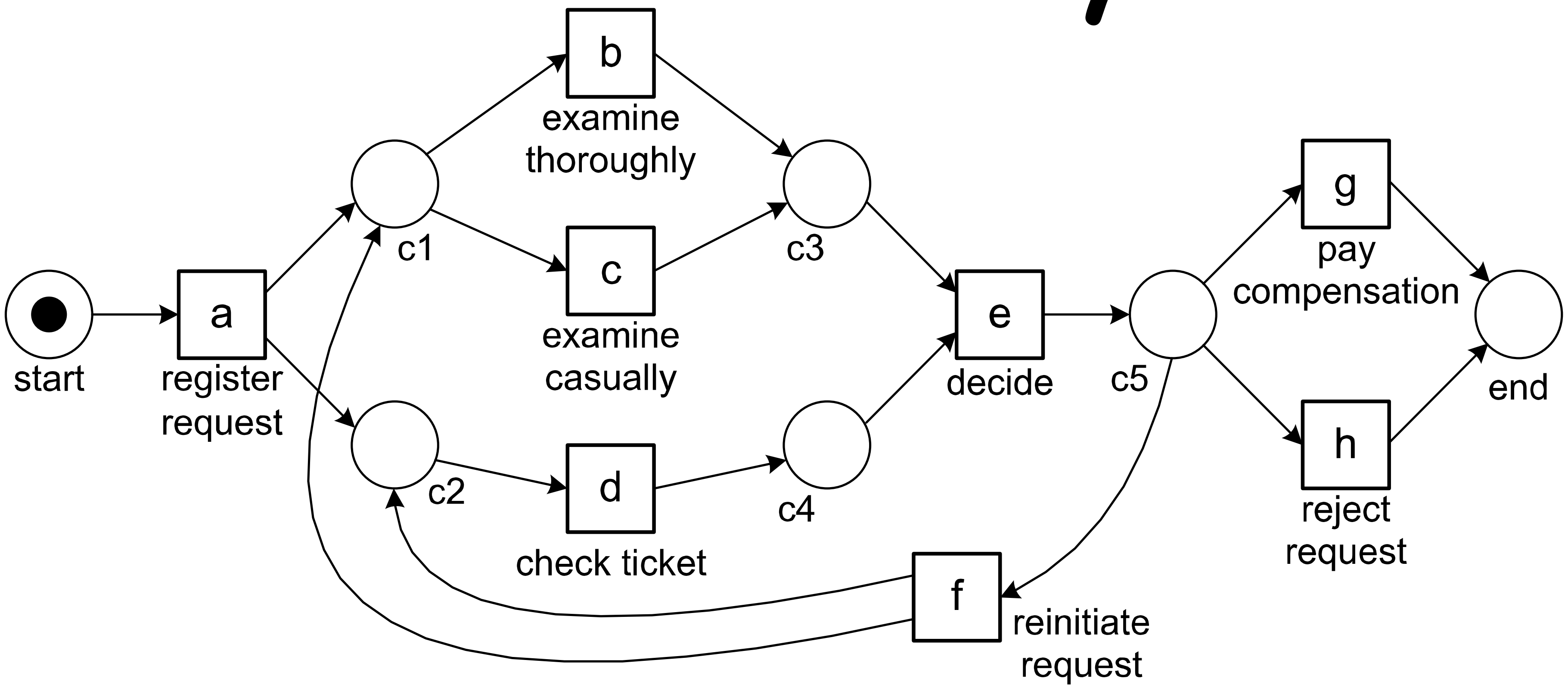
Discovery Example



The repeated execution of b/c, d, and e suggests the presence of a loop (over f).

Case id	Trace
1	$\langle a, b, d, e, h \rangle$
2	$\langle a, d, c, e, g \rangle$
3	$\langle a, c, d, e, f, b, d, e, g \rangle$
4	$\langle a, d, b, e, h \rangle$
5	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h \rangle$
6	$\langle a, c, d, e, g \rangle$
...	...

Discovery Example



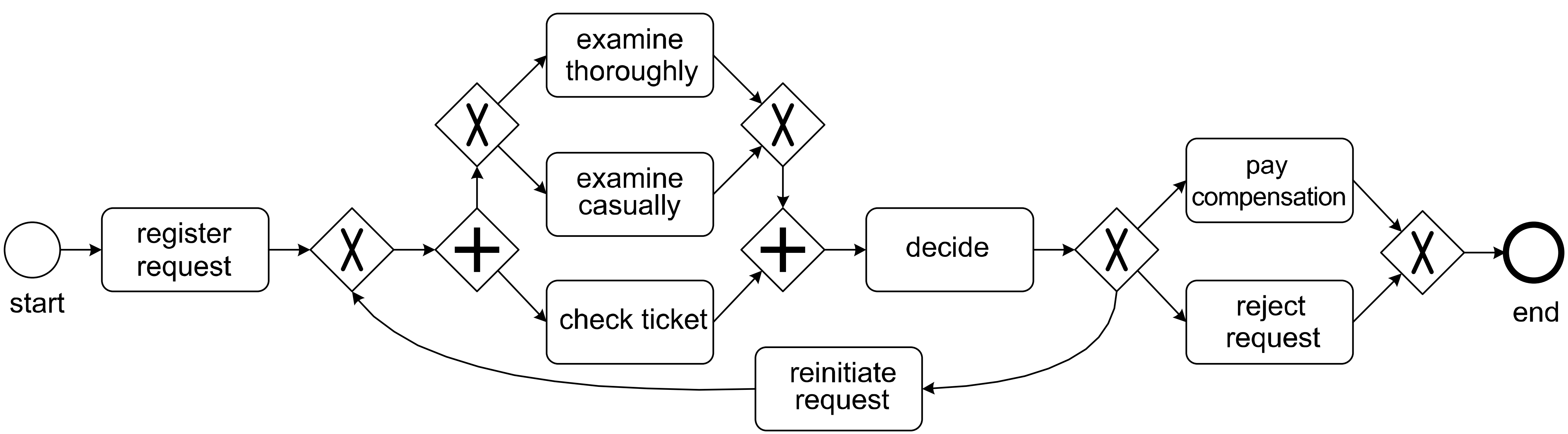
Case id		Trace
1	$L(N) \ni$	$\langle a, b, d, e, h \rangle$
2	$L(N) \ni$	$\langle a, d, c, e, g \rangle$
3	$L(N) \ni$	$\langle a, c, d, e, f, b, d, e, g \rangle$
4	$L(N) \ni$	$\langle a, d, b, e, h \rangle$
5	$L(N) \ni$	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h \rangle$
6	$L(N) \ni$	$\langle a, c, d, e, g \rangle$
...		...

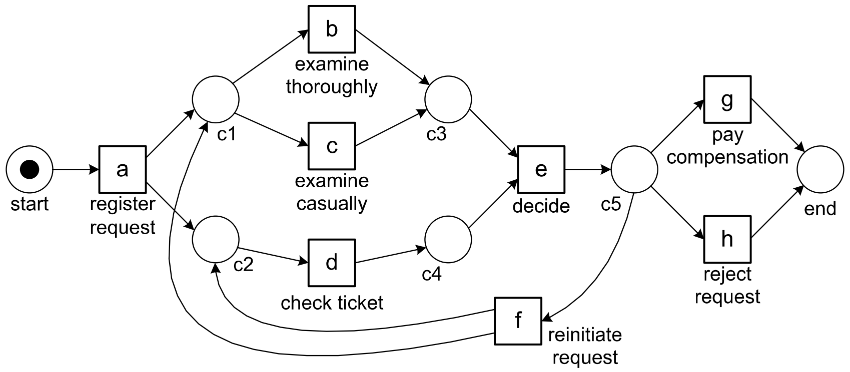
Replay:
log features are adequately captured by the net

Mining Other Models

We used Petri nets to represent the discovered process models, because Petri nets are a succinct way of representing processes and have unambiguous but intuitive semantics.

However, some mining techniques can apply to other representations as well.





Discussion

Case id	Trace
1	$\langle a, b, d, e, h \rangle$
2	$\langle a, d, c, e, g \rangle$
3	$\langle a, c, d, e, f, b, d, e, g \rangle$
4	$\langle a, d, b, e, h \rangle$
5	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h \rangle$
6	$\langle a, c, d, e, g \rangle$
...	...

The discovered net also allows for traces not in the log, e.g.

$\langle$ a, d, c, e, f, b, d, e, g $\rangle$

$\langle$ a, c, d, e, f, c, d, e, f, c, d, e, f, c, d, e, f, b, d, e, g $\rangle$

This is a desired phenomenon:
the goal of a discovery procedure is not to represent exactly the particular set of sample traces in the event log.

Process mining algorithms must generalize the behavior contained in the log to show the most likely underlying model that is not invalidated by the next set of observations

Overfitting and Underfitting

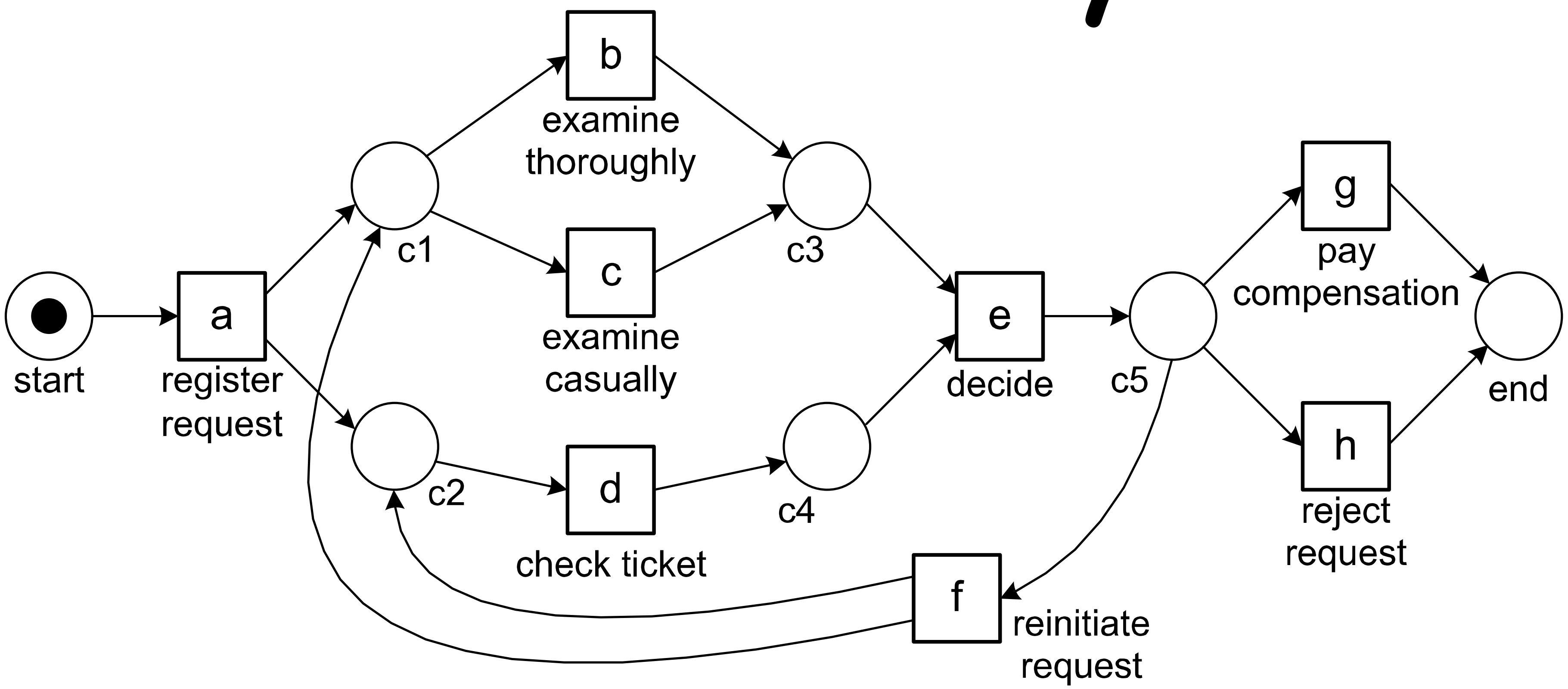
One of the challenges of process mining is
to balance between

overfitting:
the model is too specific
it only allows for the accidental behavior observed

and

underfitting:
the model is too general
it allows for behavior unrelated to the behavior observed

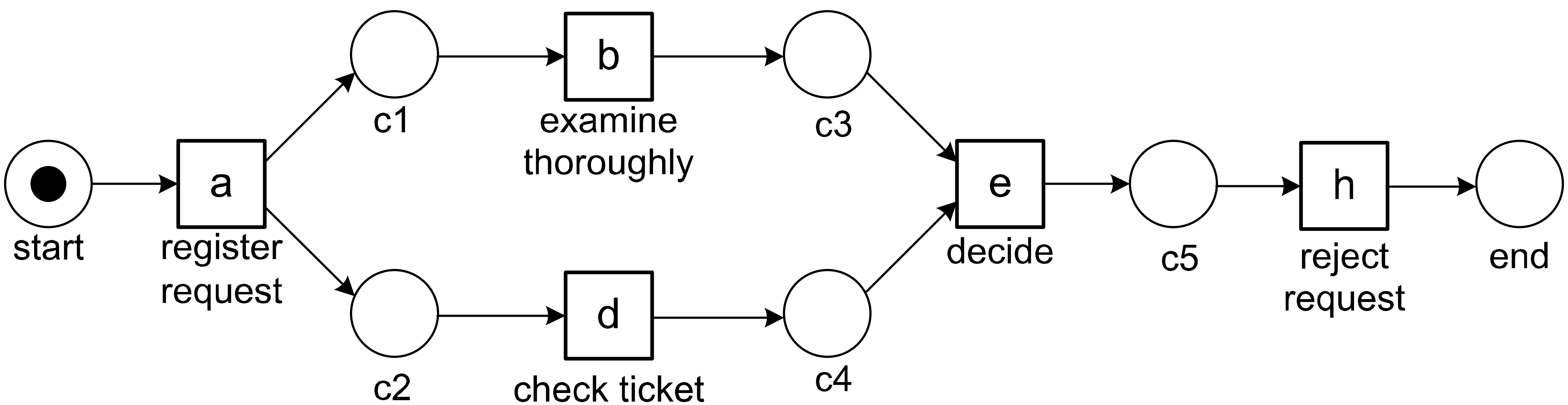
Discovery Example



When comparing the event log and the model, there seems to be a good balance between "overfitting" and "underfitting".

Case id	Trace
1	$\langle a, b, d, e, h \rangle$
2	$\langle a, d, c, e, g \rangle$
3	$\langle a, c, d, e, f, b, d, e, g \rangle$
4	$\langle a, d, b, e, h \rangle$
5	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h \rangle$
6	$\langle a, c, d, e, g \rangle$
...	...

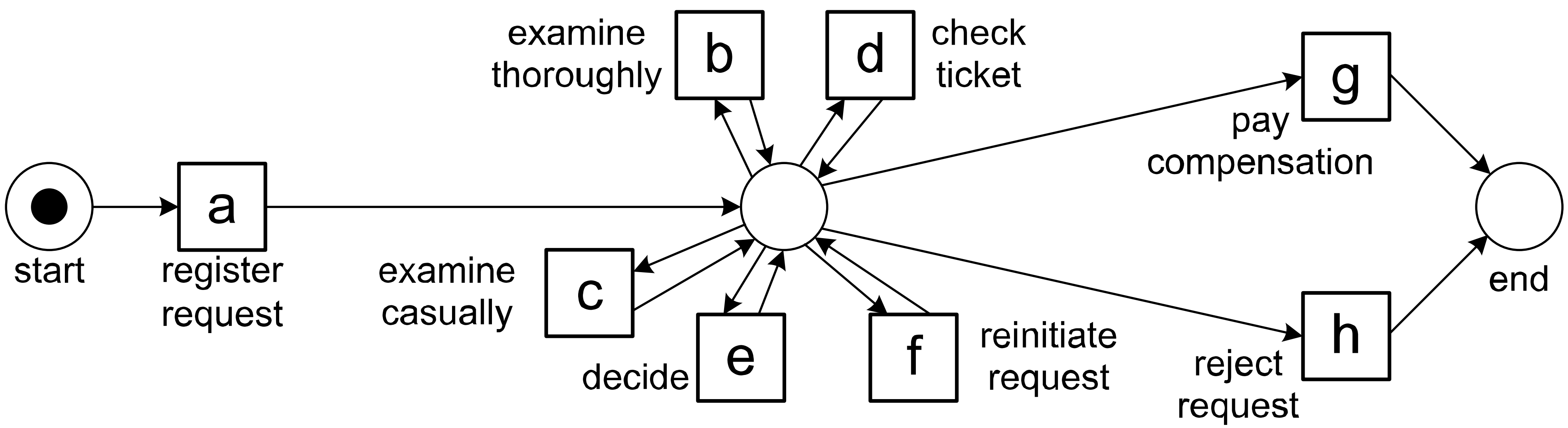
Another Discovery Example



Another net could fail to replay some traces

Case id	Trace
1	$\langle a, b, d, e, h \rangle$
~~2~~	~~$\langle a, d, c, e, g \rangle$~~
~~3~~	~~$\langle a, c, d, e, f, b, d, e, g \rangle$~~
4	$\langle a, d, b, e, h \rangle$
~~5~~	~~$\langle a, c, d, e, f, d, c, e, f, c, d, e, h \rangle$~~
~~6~~	~~$\langle a, c, d, e, g \rangle$~~
...	...

Another Discovery Example

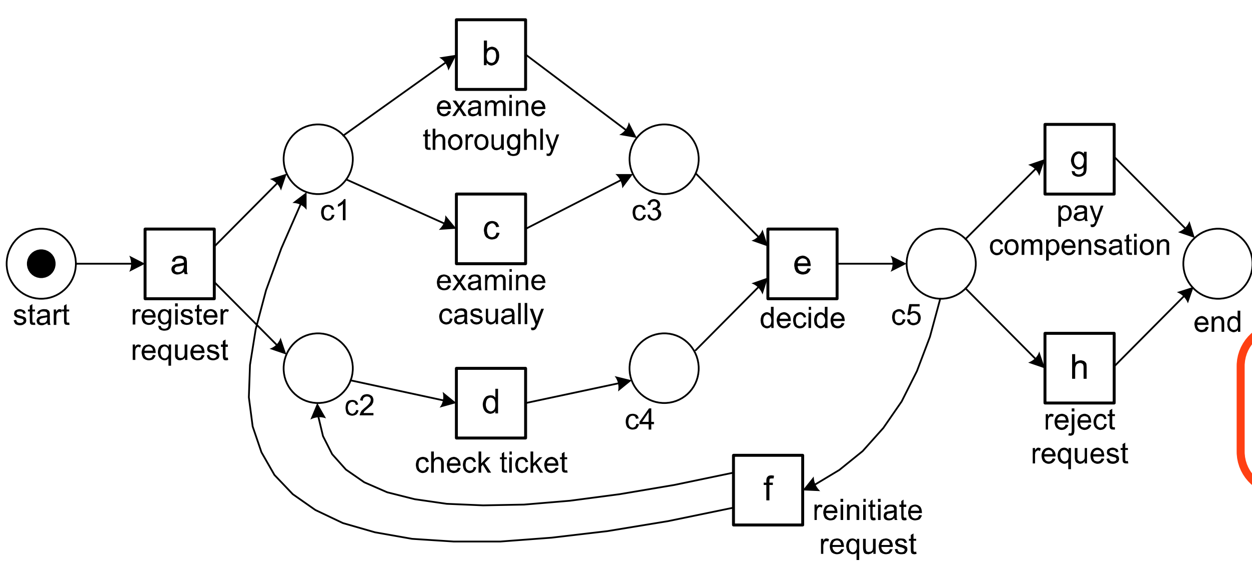


Another net could allow for too many other traces (nets of this kind are called **flower nets**) and deliver little information about the underlying process

Case id	Trace
1	$\langle a, b, d, e, h \rangle$
2	$\langle a, d, c, e, g \rangle$
3	$\langle a, c, d, e, f, b, d, e, g \rangle$
4	$\langle a, d, b, e, h \rangle$
5	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h \rangle$
6	$\langle a, c, d, e, g \rangle$
...	...

Conformance Example

We would like to measure the ``conformance'' between a net and en event log (how well they pair together)

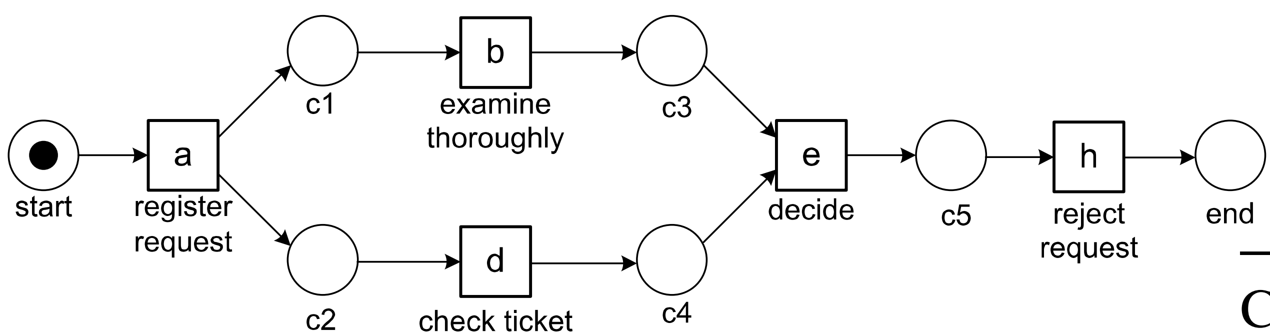


7 ok out of 10

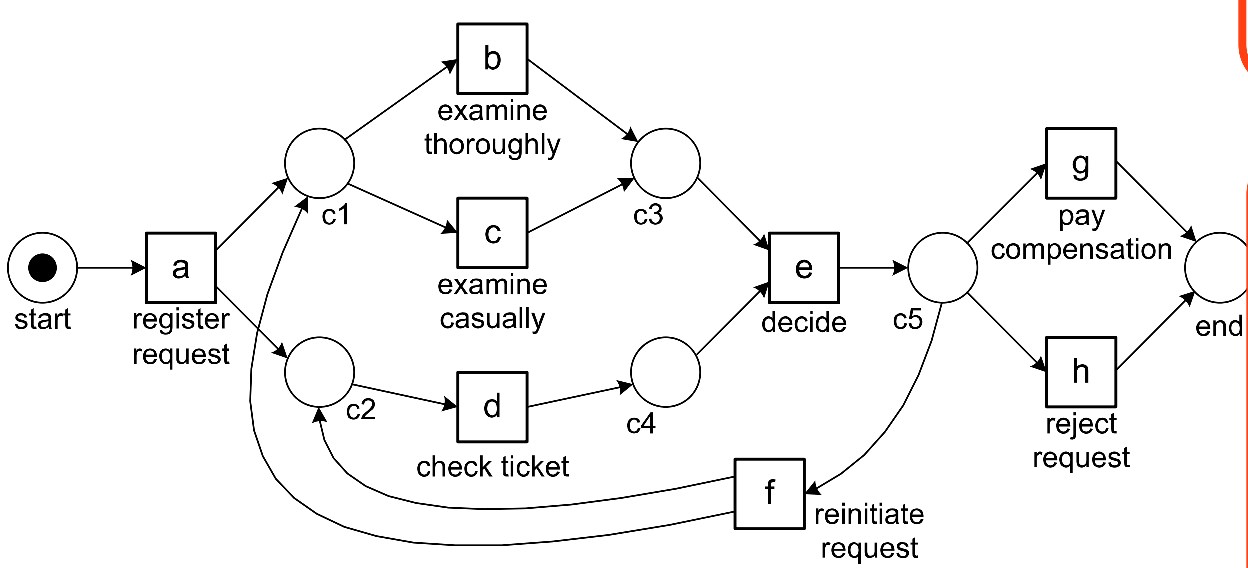
Case id		Trace
1	√	$\langle a, b, d, e, h \rangle$
2	√	$\langle a, d, c, e, g \rangle$
3	√	$\langle a, c, d, e, f, b, d, e, g \rangle$
4	√	$\langle a, d, b, e, h \rangle$
5	√	$\langle a, c, d, e, f, d, c, e, f, c, d, e, h \rangle$
6	√	$\langle a, c, d, e, g \rangle$
7	✗	$\langle \mathbf{a}, \mathbf{b}, \mathbf{e}, \mathbf{g} \rangle$
8	✗	$\langle \mathbf{a}, \mathbf{b}, \mathbf{d}, \mathbf{e} \rangle$
9	√	$\langle a, d, c, e, f, d, c, e, f, b, d, e, h \rangle$
10	✗	$\langle \mathbf{a}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{b}, \mathbf{d}, \mathbf{g} \rangle$

Conformance Example

We would like to measure the ``conformance'' between a net and en event log (how well they pair together)



2 ok out of 10



7 ok out of 10

Case id		Trace
1	√	$\langle a, b, d, e, h \rangle$
2		$\langle a, d, c, e, g \rangle$
3		$\langle a, c, d, e, f, b, d, e, g \rangle$
4	√	$\langle a, d, b, e, h \rangle$
5		$\langle a, c, d, e, f, d, c, e, f, c, d, e, h \rangle$
6		$\langle a, c, d, e, g \rangle$
7		$\langle \mathbf{a, b, e, g} \rangle$
8		$\langle \mathbf{a, b, d, e} \rangle$
9		$\langle a, d, c, e, f, d, c, e, f, b, d, e, h \rangle$
10		$\langle \mathbf{a, c, d, e, f, b, d, g} \rangle$

Quality Criteria

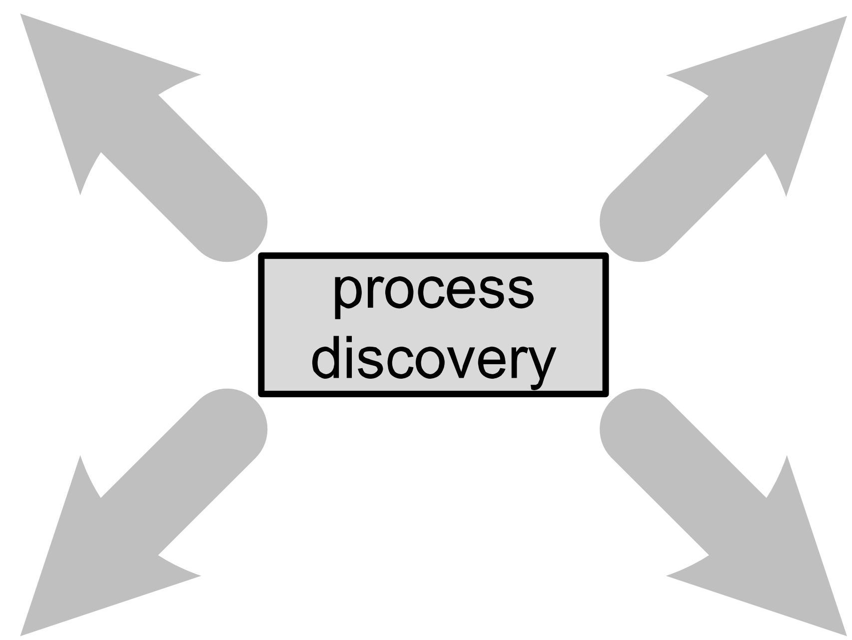
Simple structure

“Occam’s razor”

simplicity

“able to replay event log”

fitness



generalization

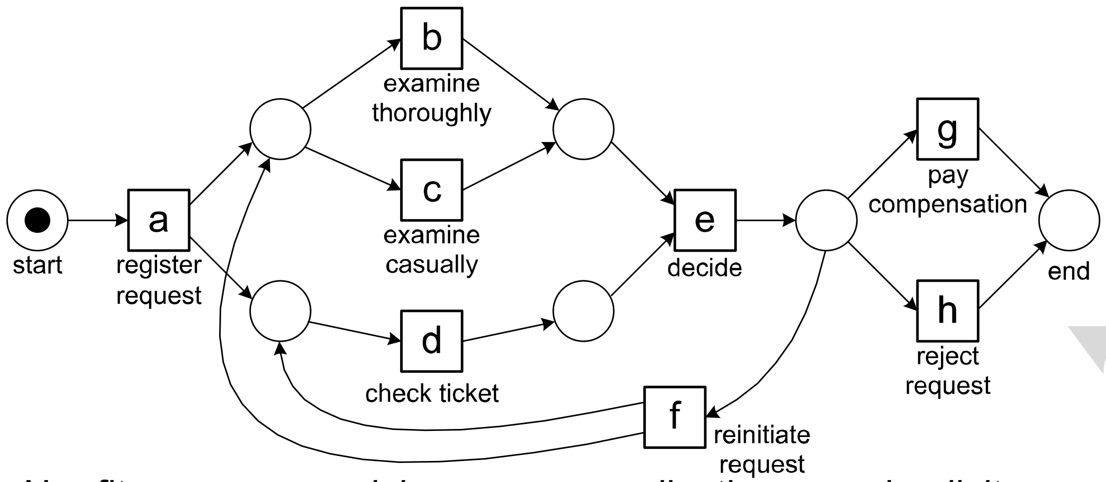
“not overfitting the log”

Other behaviors allowed

precision

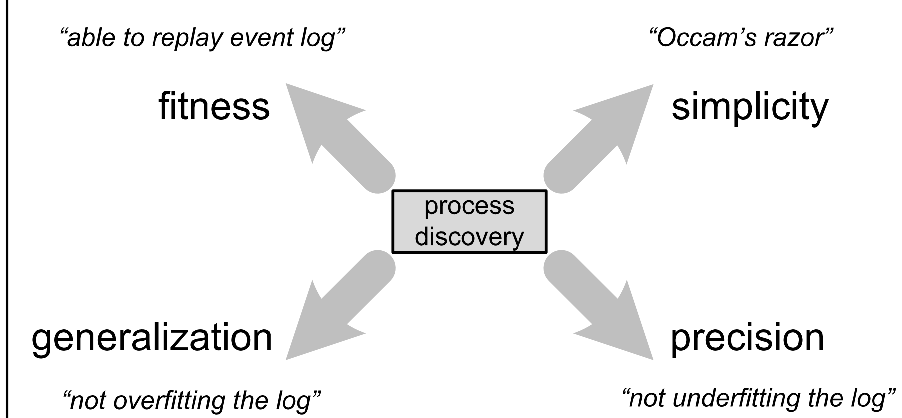
“not underfitting the log”

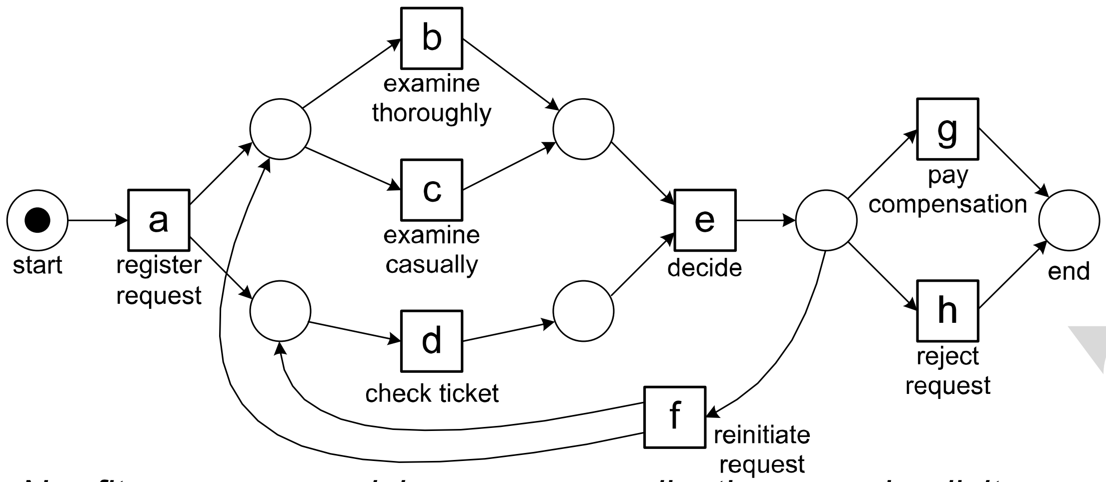
No completely unrelated behavior



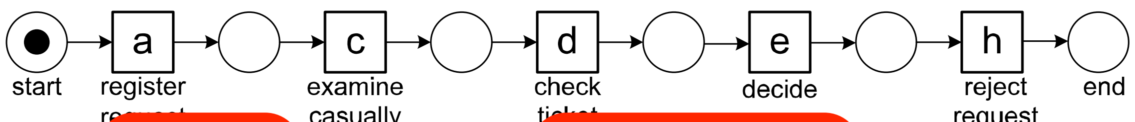
N_1 : fitness = +, precision = +, generalization = +, simplicity = +

#	trace
455	acdeh
191	abdeg
177	adceh
144	abdeh
111	acdeg
82	adceg
56	adbeh
47	acdefdbeh
38	adbeg
33	acdefbdeh
14	acdefbdeg
11	acdefdbeg
9	adcefcdeh
8	adcefdbeh
5	adcefbdeg
3	acdefbdefdbeg
2	adcefdbeg
2	adcefbdefbdeg
1	adcefdbefbdeh
1	adbefbdefdbeg
1	adcefdbefcdefdbeg
1391	



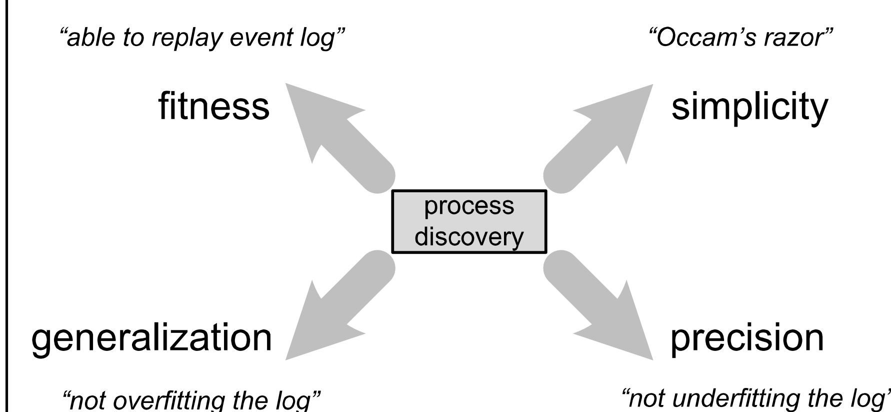


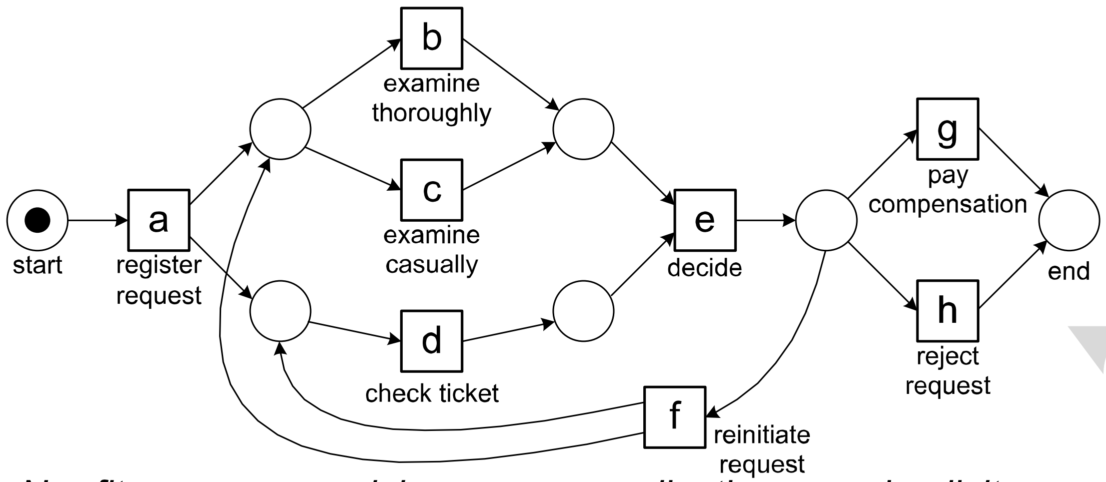
N_1 : fitness = +, precision = +, generalization = +, simplicity = +



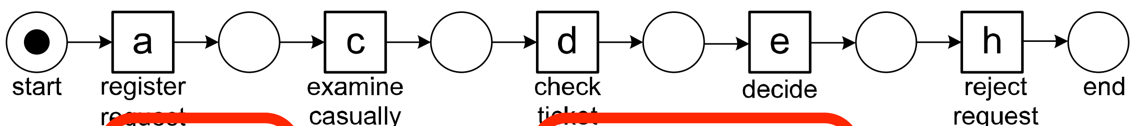
N_2 fitness = -, precision = +, generalization = -, simplicity = +

#	trace
455	acdeh
191	abdeg
177	adceh
144	abdeh
111	acdeg
82	adceg
56	adbeh
47	acdefdbeh
38	adbeg
33	acdefbdeh
14	acdefbdeg
11	acdefdbeg
9	adcefcdeh
8	adcefdbeh
5	adcefbdeg
3	acdefbdefdbeg
2	adcefdbeg
2	adcefbdefbdeg
1	adcefdbefbdeh
1	adbefbdefdbeg
1	adcefdbefcdefdbeg
1391	

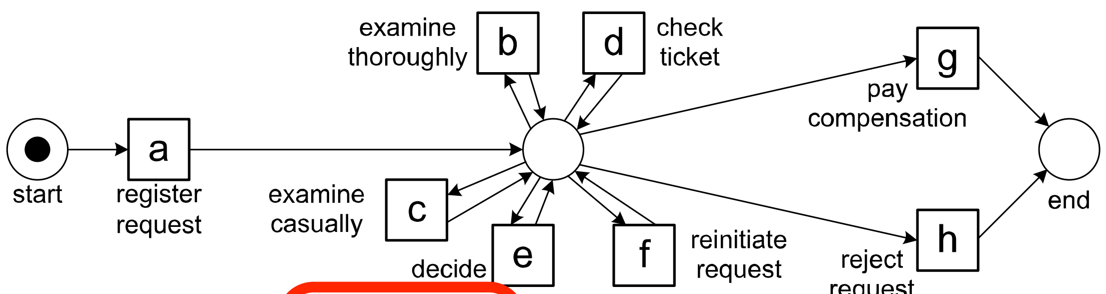




N_1 : *fitness* = +, *precision* = +, *generalization* = +, *simplicity* = +

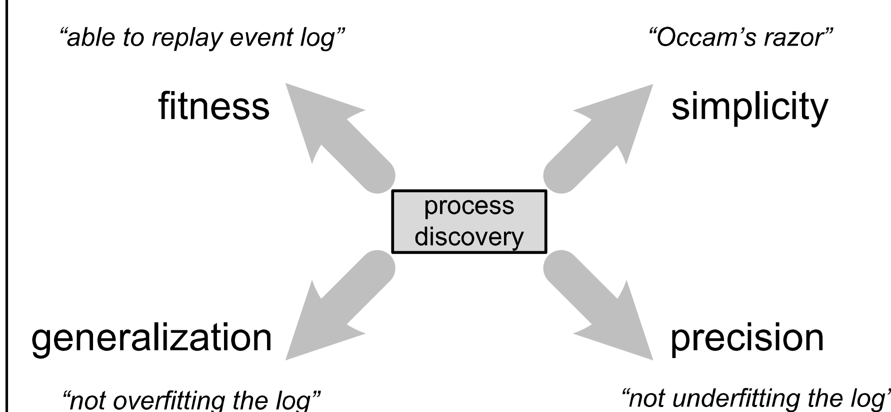


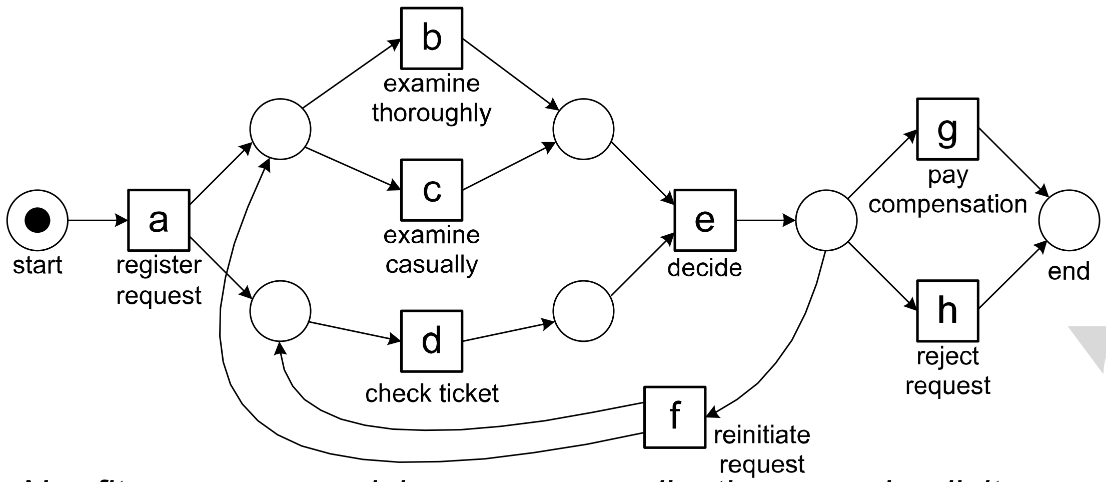
N_2 : *fitness* = -, *precision* = +, *generalization* = -, *simplicity* = +



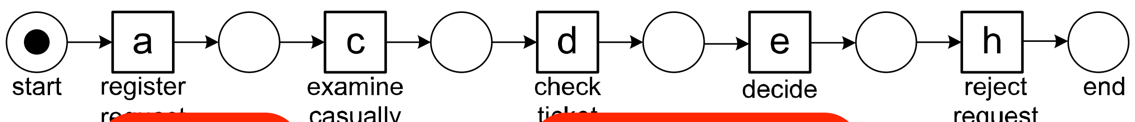
N_3 : *fitness* = +, *precision* = -, *generalization* = +, *simplicity* = +

#	trace
455	acdeh
191	abdeg
177	adceh
144	abdeh
111	acdeg
82	adceg
56	adbeh
47	acdefdbeh
38	adbeg
33	acdefbdeh
14	acdefbdeg
11	acdefdbeg
9	adcefcdeh
8	adcefdbeh
5	adcefbdeg
3	acdefbdefdbeg
2	adcefdbeg
2	adcefbdefbdeg
1	adcefdbefbdeh
1	adbefbdefdbeg
1	adcefdbefcdefdbeg
1391	

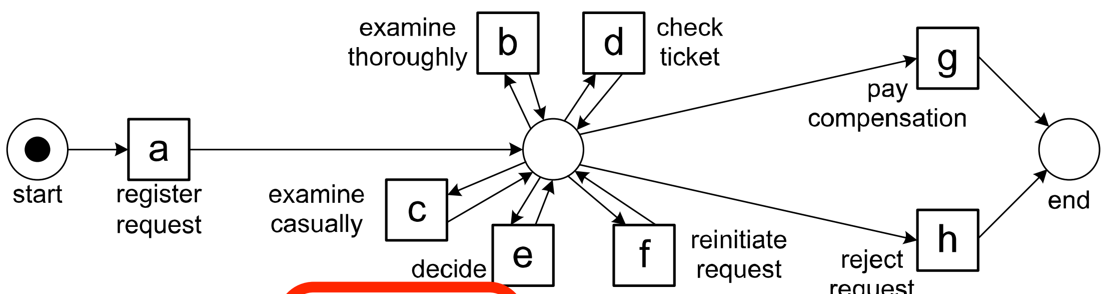




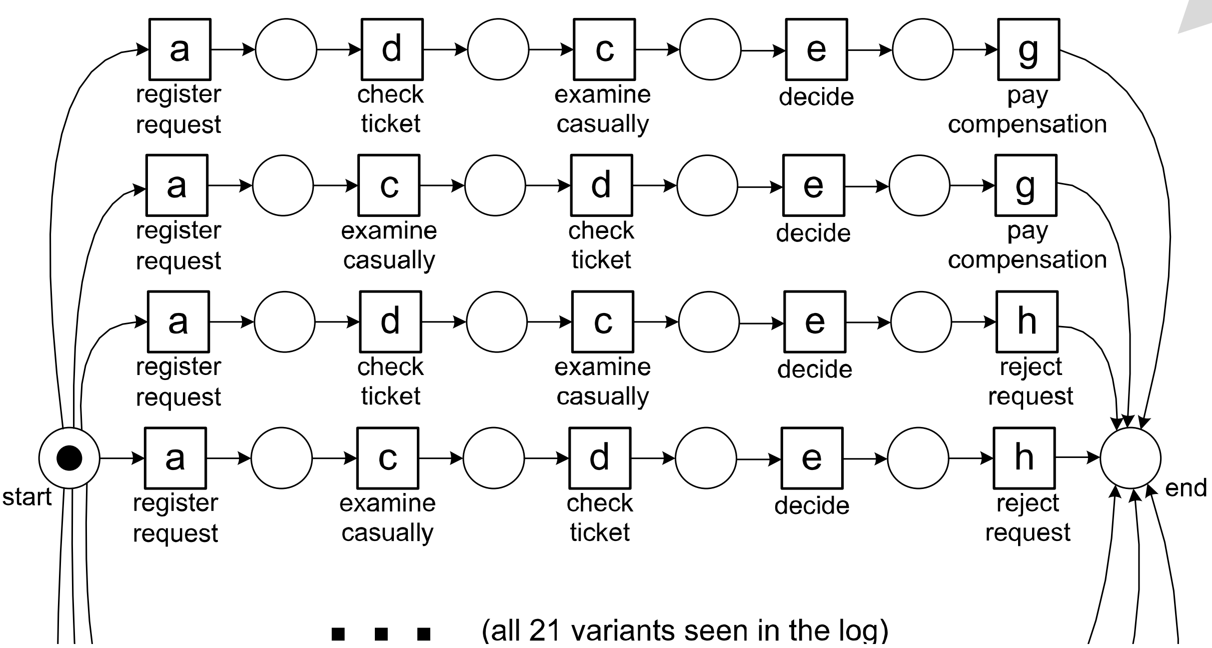
N_1 : fitness = +, precision = +, generalization = +, simplicity = +



N_2 : fitness = -, precision = +, generalization = -, simplicity = +



N_3 : fitness = +, precision = -, generalization = +, simplicity = +



N_4 : fitness = +, precision = +, generalization = -, simplicity = -

#	trace
455	acdeh
191	abdeg
177	adceh
144	abdeh
111	acdeg
82	adceg
56	adbeh
47	acdefdbeh
38	adbeg
33	acdefbdeh
14	acdefbdeg
11	acdefdbeg
9	adcefcdeh
8	adcefdbeh
5	adcefbdeg
3	acdefbdefdbeg
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2	adcefbdefbdeg
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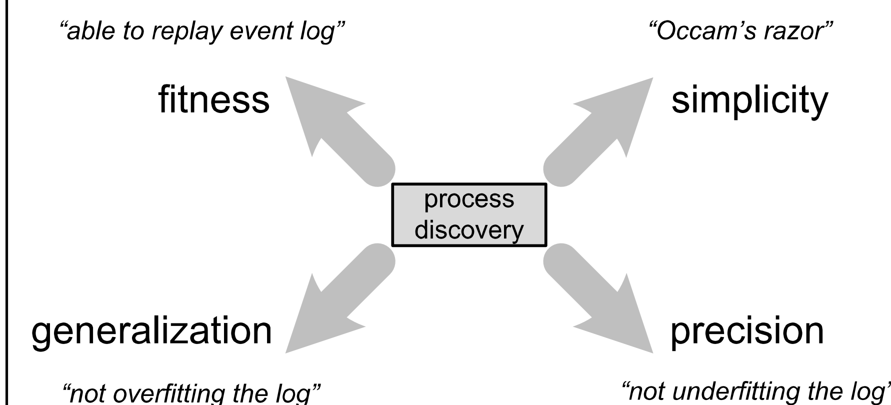
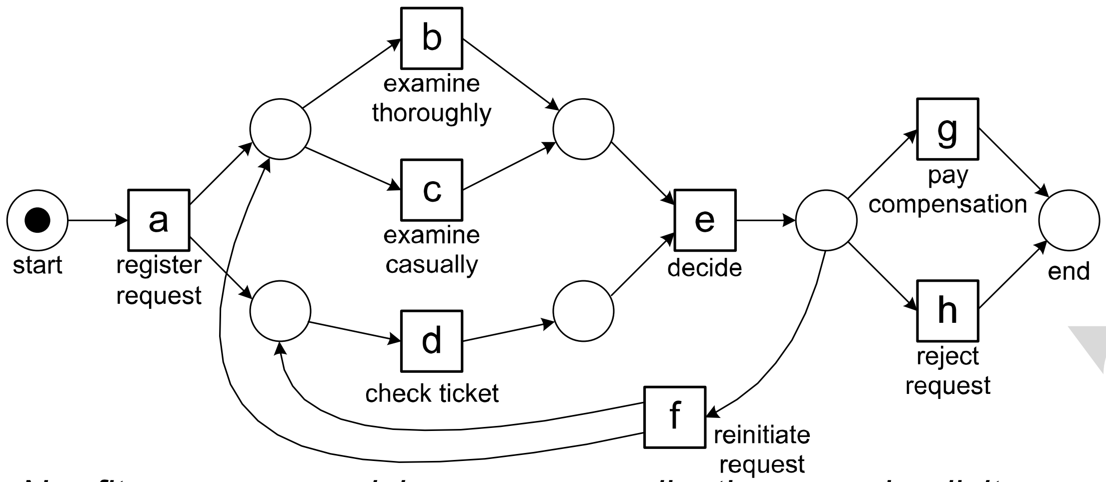
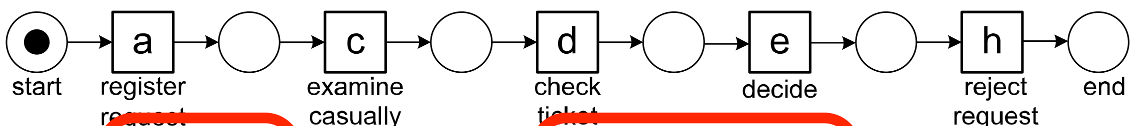
1	adcefdbefcdefdbeg
1391	



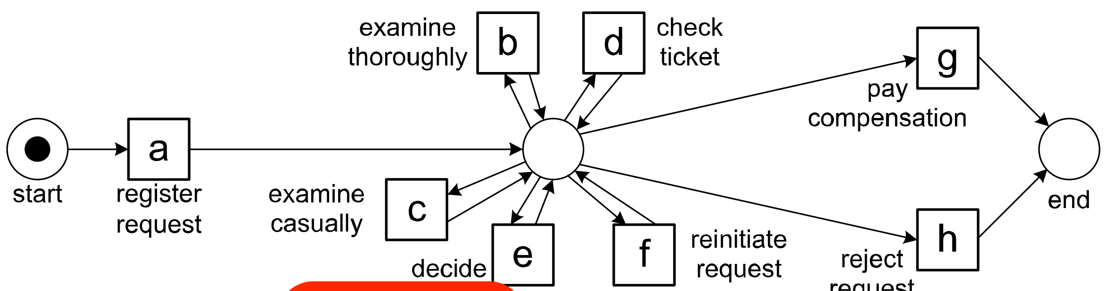


N_1 : *fitness = +, precision = +, generalization = +, simplicity = +*

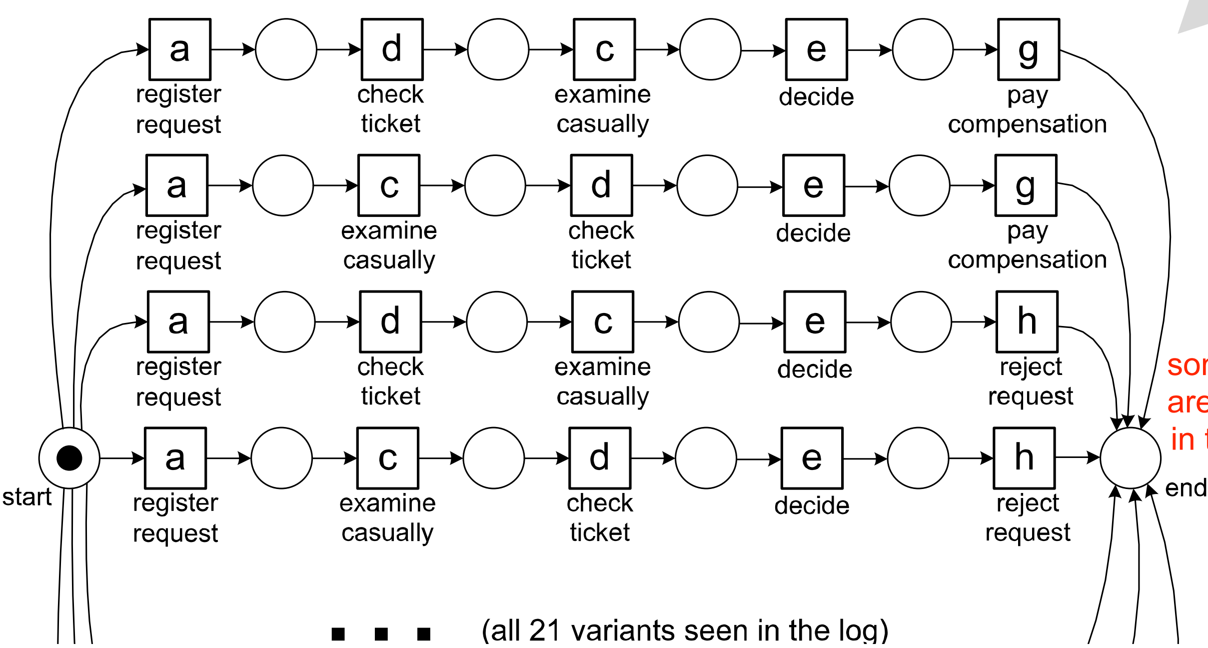


some events are missing in the model

N_2 : *fitness = -, precision = +, generalization = -, simplicity = +*



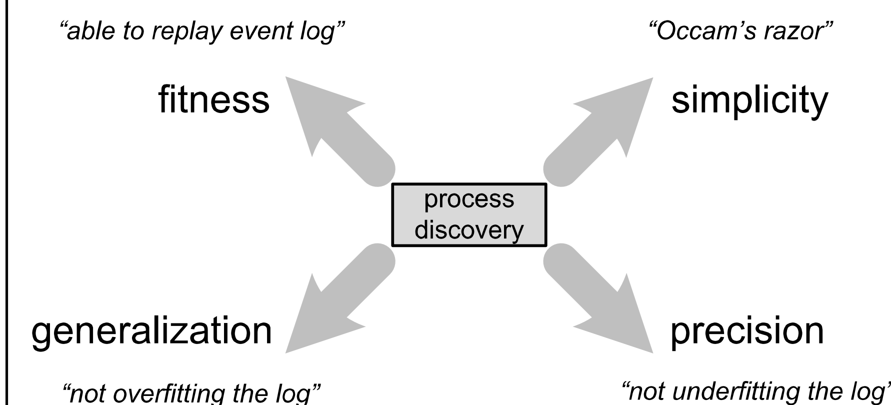
N_3 : *fitness = +, precision = -, generalization = +, simplicity = +*



some events are repeated in the model

N_4 : *fitness = +, precision = +, generalization = -, simplicity = -*

#	trace
455	acdeh
191	abdeg
177	adceh
144	abdeh
111	acdeg
82	adceg
56	adbeh
47	acdefdbeh
38	adbeg
33	acdefbdeh
14	acdefbdeg
11	acdefdbeg
9	adcefcdeh
8	adcefdbeh
5	adcefbdeg
3	acdefbdefdbeg
2	adcefdbeg
2	adcefbdefbdeg
1	adcefdbefbdeh
1	adbefbdefdbeg
1	adcefdbefcdefdbeg
1391	



Quality Measures

We have seen four quality criteria:
fitness, precision, generalization, and simplicity.

In the example, for each of these models, a subjective judgment is given with respect to the four quality criteria. As the models are rather extreme, the scores +/- for the various quality criteria are easy to assign.

However, in a more realistic setting it would be much more difficult to judge the quality of a model.

We will discuss how the notion of **fitness** can be quantified.

Question time

Suppose you are given a log with:

#6 traces of the form $\langle$ a , c , d $\rangle$

#3 traces of the form $\langle$ b , c , e $\rangle$

Which model (i.e., Petri net) would you infer?

The Petri net you derive must have exactly
five transitions named a, b, c, d, e
(and the places / arcs you like)

Question time

can start with a or b

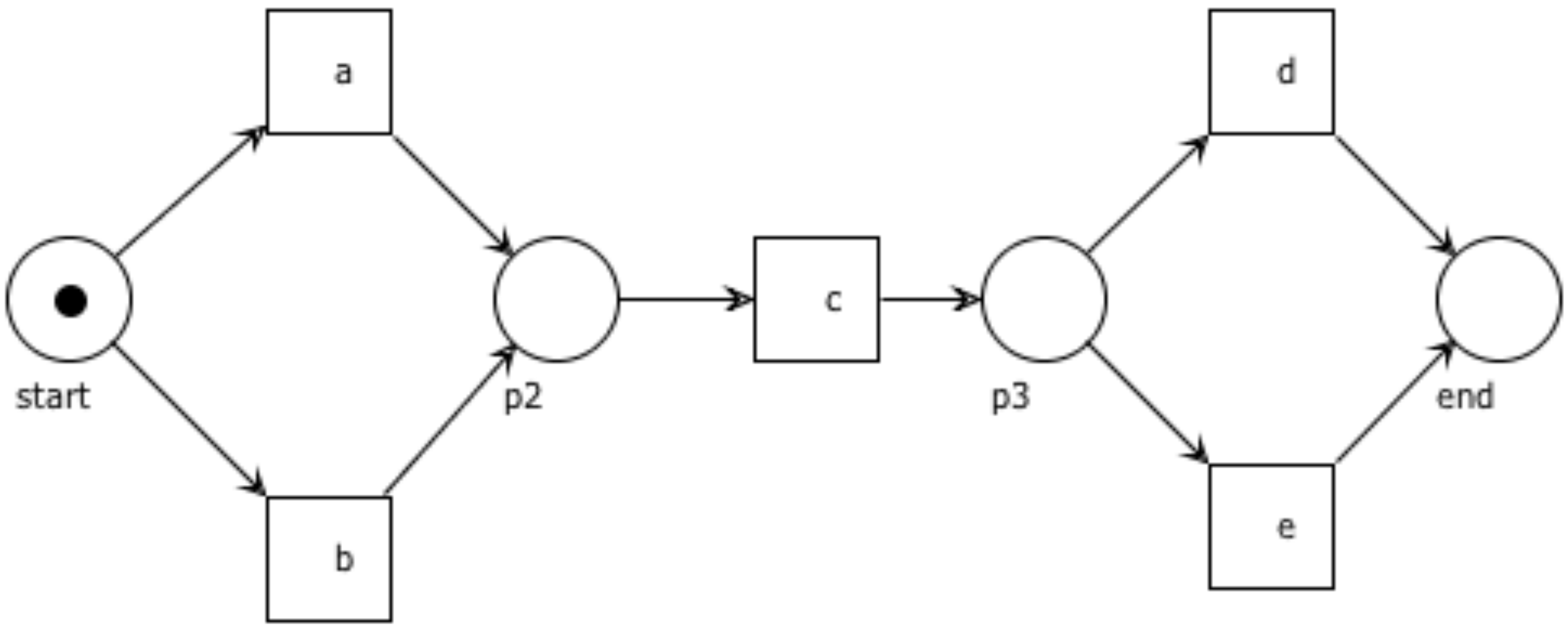
can end with d or e

c is always executed in between

$\langle$ a , c , d $\rangle$

$\langle$ b , c , e $\rangle$

Question time

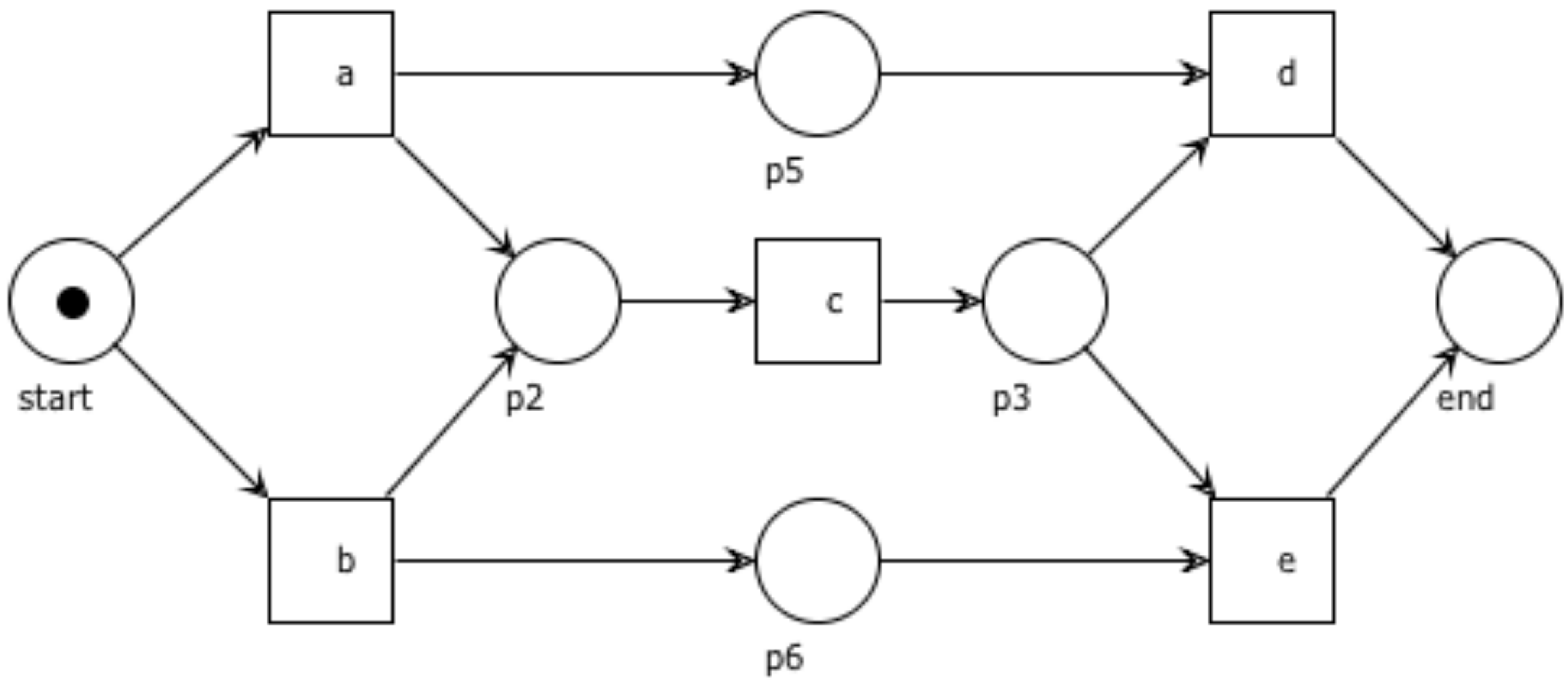


also allowed:

$\langle$ a , c , d $\rangle$
$\langle$ b , c , e $\rangle$

$\langle$ a , c , e $\rangle$
$\langle$ b , c , d $\rangle$

Question time



⟨ a , c , d ⟩

⟨ b , c , e ⟩

nothing else allowed!

Question time

Suppose you are given a log with:

#3 traces of the form $\langle$ a , b , c , d $\rangle$

#1 traces of the form $\langle$ a , e , d $\rangle$

#2 traces of the form $\langle$ a , c , b , d $\rangle$

Which model (i.e., Petri net) would you infer?

The Petri net you derive must have exactly five transitions named a, b, c, d, e (and the places / arcs you like)

Question time

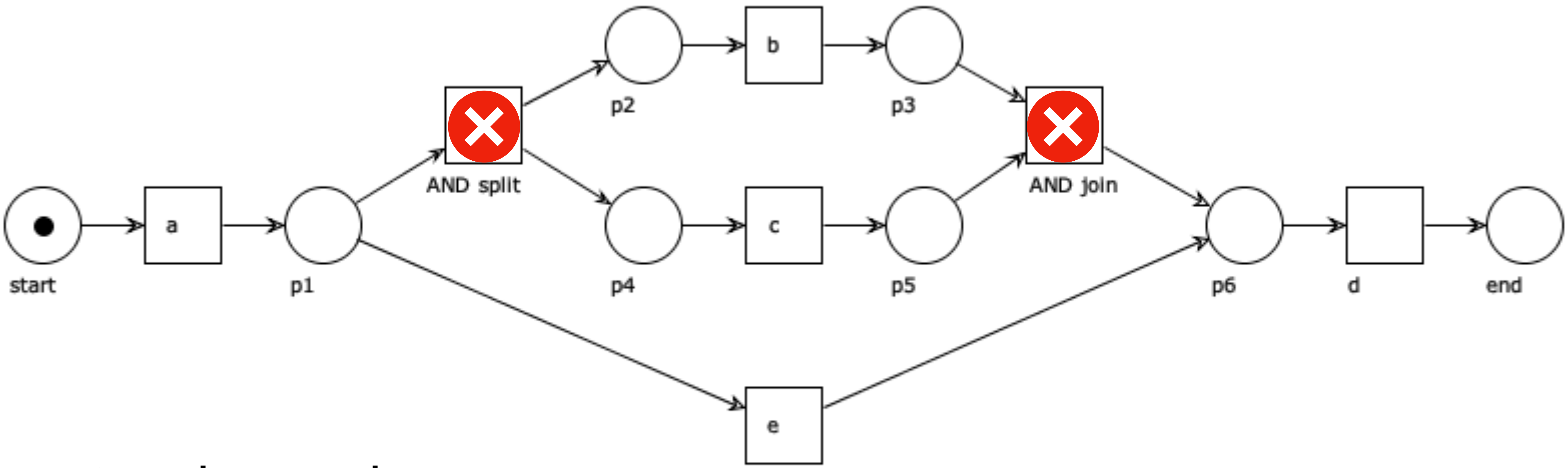
must start with a

must end with d

b/c in any order OR just e

$\langle$ a , b , c , d $\rangle$
$\langle$ a , e , d $\rangle$
$\langle$ a , c , b , d $\rangle$

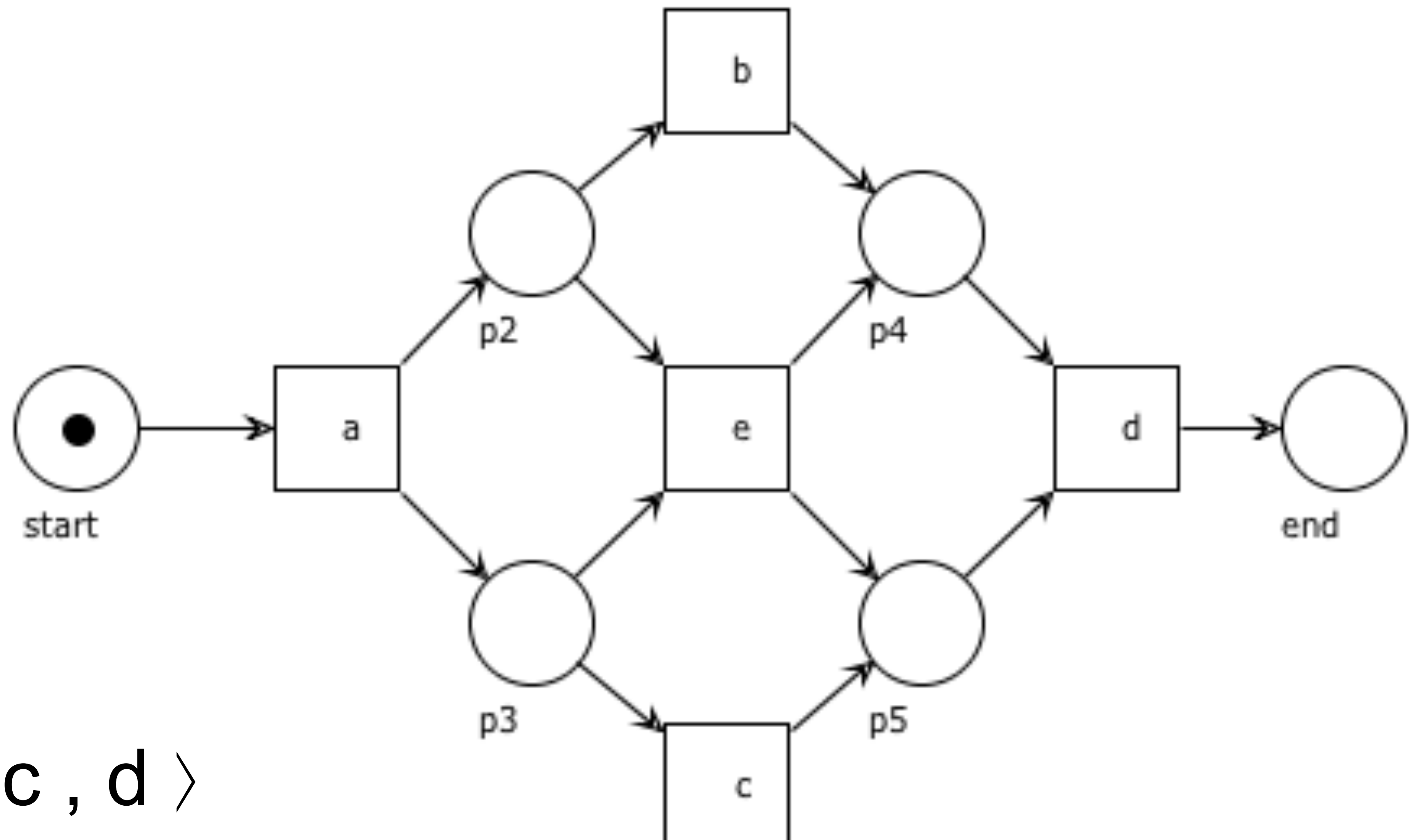
Question time



⟨ a , b , c , d ⟩
⟨ a , e , d ⟩
⟨ a , c , b , d ⟩

contains events that are not present in the log

Question time



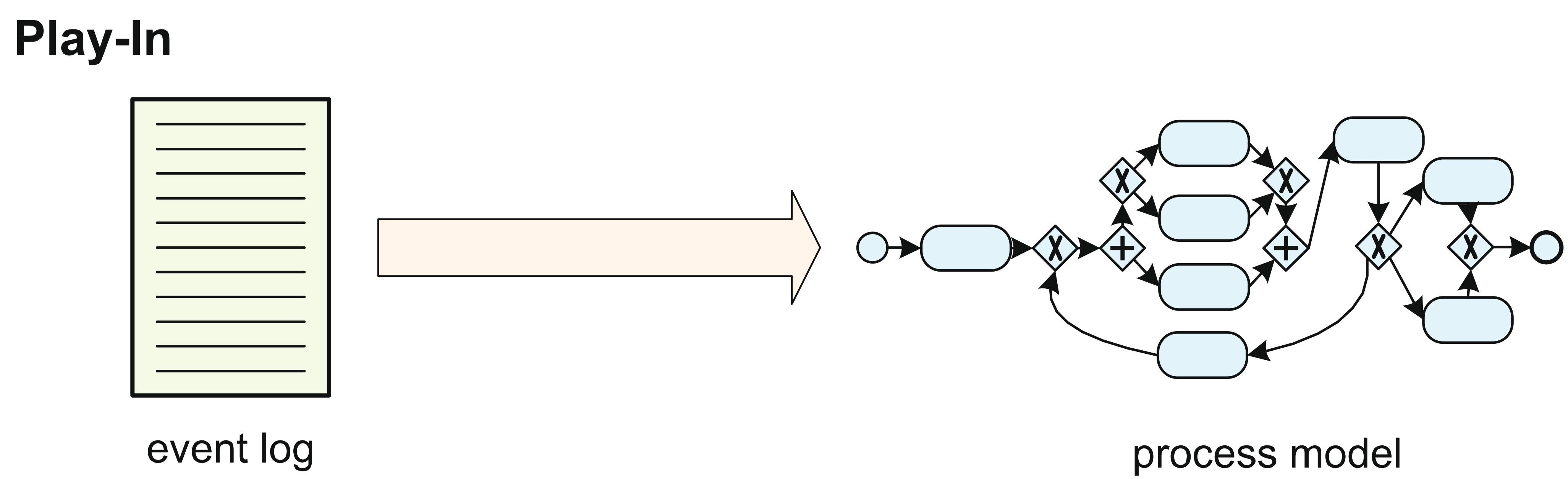
$\langle$ a , b , c , d $\rangle$
$\langle$ a , e , d $\rangle$
$\langle$ a , c , b , d $\rangle$

Process Discovery: α-Algorithm

Process Discovery

A **process discovery algorithm** is a function that maps an event log L onto a process model M such that the model M is "representative" for the behaviour seen in the event log L.

We focus on *simple event logs* and Petri net models (possibly sound workflow nets).



Simple Event Log

Let A be a set of activities.

A **simple trace** σ over A is a finite sequence of activities.

A **simple event log** L over A is a multiset of traces.

trace multiplicity multiplicity 1

$$L_1 = \left[\langle a, b, c, d\rangle^3, \langle a, c, b, d\rangle^2, \langle a, e, d\rangle\right]$$

$$L_2 = \left[\langle a, b, c, d\rangle^3, \langle a, c, b, d\rangle^4, \langle a, b, c, e, f, b, c, d\rangle^2, \langle a, b, c, e, f, c, b, d\rangle, \langle a, c, b, e, f, b, c, d\rangle^2, \langle a, c, b, e, f, b, c, e, f, c, b, d\rangle\right]$$

α-Algorithm

The α-algorithm was one of the first process discovery algorithms that could adequately deal with concurrency.

It has several limitations,
but it provides a good introduction into the topic:
The α-algorithm is simple and many of its ideas have been embedded in more complex and robust techniques.

The α-algorithm uses the **play-in** strategy
to scan the event log for particular patterns,
called **log-based ordering relations,**
to create a **footprint** matrix of the log.

Log-based Ordering Relations

a is (sometimes) immediately followed by b

$a >_L b$ if and only if there is a trace $\sigma = \langle t_1, t_2, t_3, \ldots, t_n \rangle$ and $i \in \{1, \ldots, n-1\}$ such that $\sigma \in L$ and $t_i = a$ and $t_{i+1} = b$

Example: L = $\{ \langle a, c, d \rangle, \langle b, c, e \rangle \}$

$a >_L c$ $b >_L c$

$c >_L d$ $c >_L e$

$a >_L d$ No!

Directly Follow Graphs

A Directly Follows Graph (DFG) is a graph that represents the observed behaviour in an event log, by showing how activities can immediately follow other activities

One node per each activity il the log

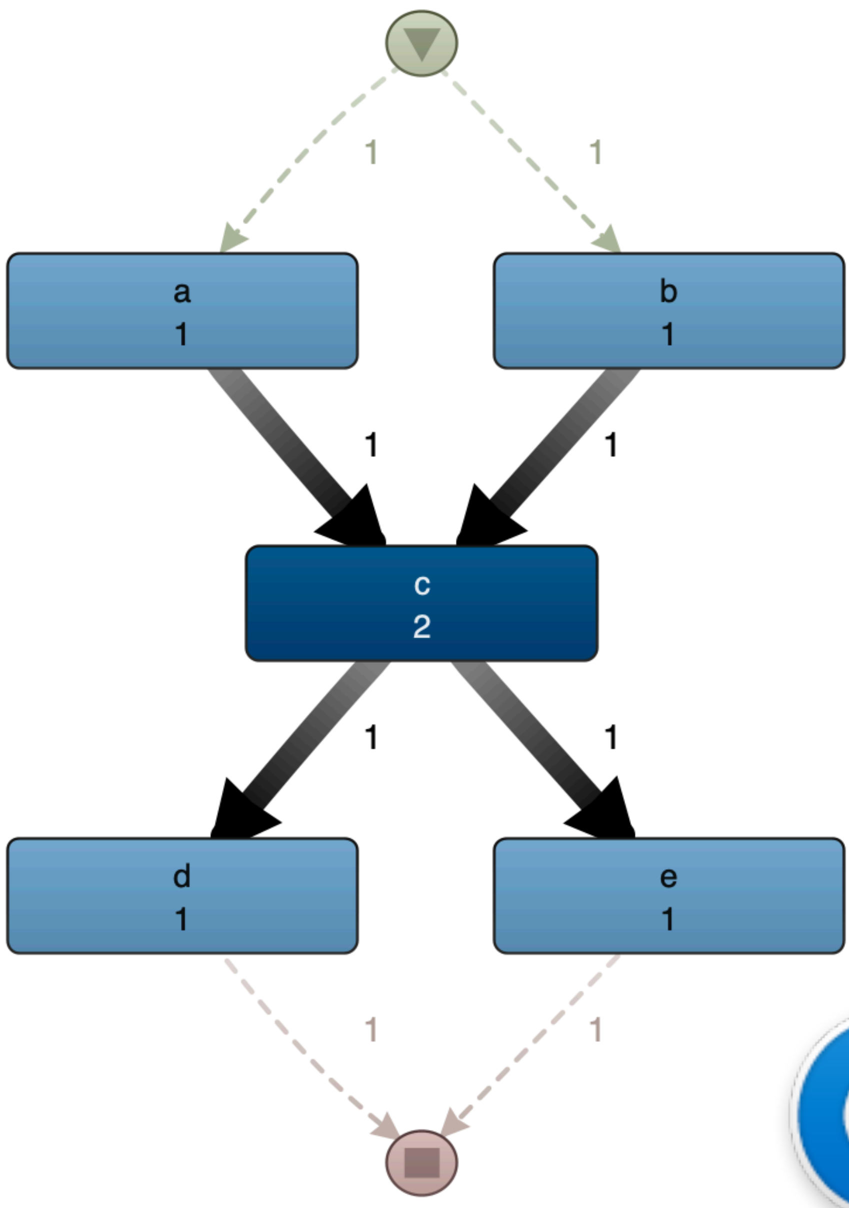
One edge from node a to node b whenever $a >_L b$
(optionally: arcs are weighted by frequencies,
i.e., how often a is immediately followed by b)

Relation $>_L$ as DFG

Example: $L = \{ \langle a, c, d \rangle, \langle b, c, e \rangle \}$

$a >_L c \quad c >_L d$

$b >_L c \quad c >_L e$



Log-based Ordering Relations

a is (sometimes) immediately followed by b

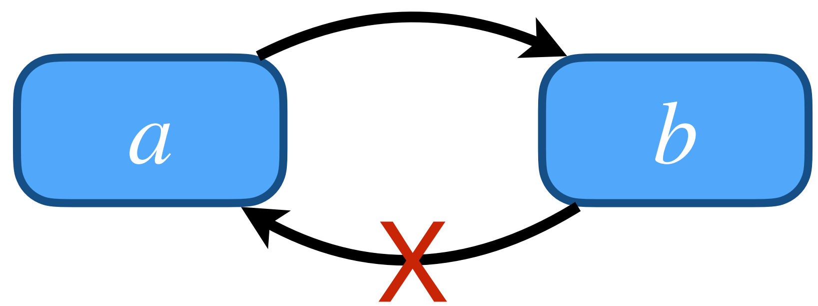
$\boxed{a >_L b}$ if and only if there is a trace $\sigma = \langle t_1, t_2, t_3, \ldots, t_n \rangle$ and $i \in \{1, \ldots, n-1\}$ such that $\sigma \in L$ and $t_i = a$ and $t_{i+1} = b$

- $\boxed{a \rightarrow_L b}$ if and only if $a >_L b$ and $b \not>_L a$ (causality)
- $\boxed{a \#_L b}$ if and only if $a \not>_L b$ and $b \not>_L a$ (mutual exclusion)
- $\boxed{a \parallel_L b}$ if and only if $a >_L b$ and $b >_L a$ (concurrency)

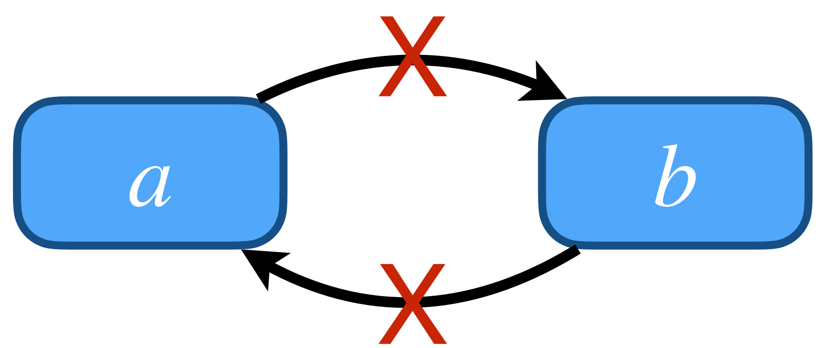
$$x \rightarrow_L y,\ y \rightarrow_L x,\ x \#_L y, \text{ or } x \parallel_L y$$

Log-based Ordering Relations

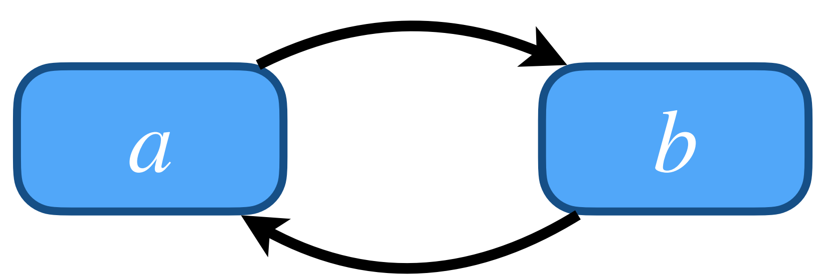
$a \rightarrow_L b$
$b \leftarrow_L a$



$a \;\#_L\; b$



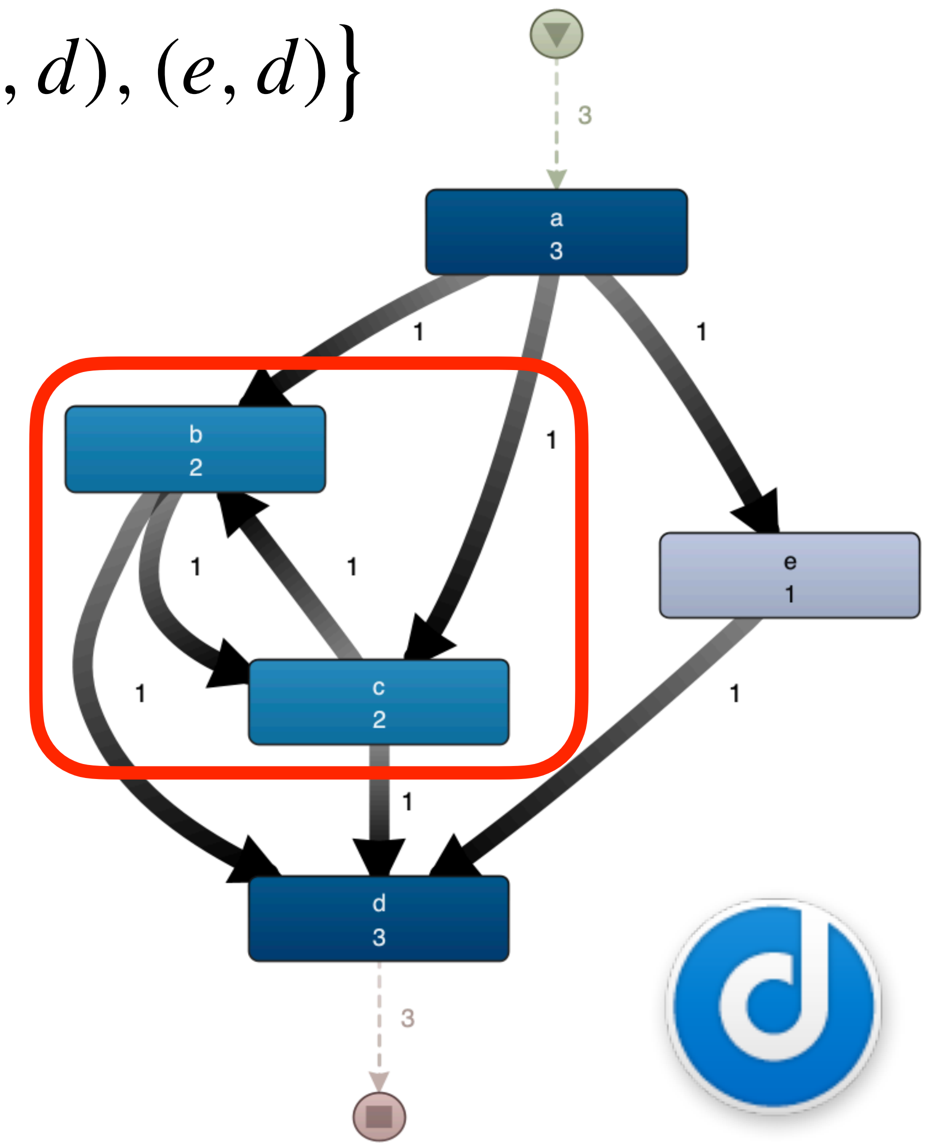
$a \;\|_L\; b$



Log-based Ordering Relations

$$L_1 = [\langle a,b,c,d\rangle^3, \langle a,c,b,d\rangle^2, \langle a,e,d\rangle]$$

$$>_{L_1} = \{(a,b),(a,c),(a,e),(b,c),(c,b),(b,d),(c,d),(e,d)\}$$



$$\|_{L_1} = \{(b,c),(c,b)\}$$

$$\rightarrow_{L_1} = \{(a,b),(a,c),(a,e),(b,d),(c,d),(e,d)\}$$

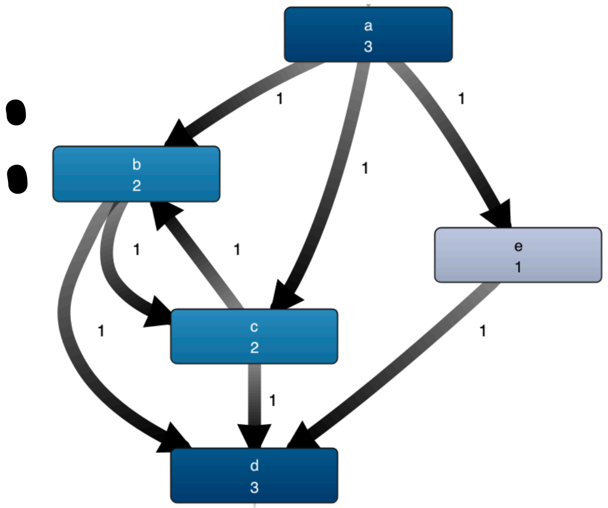
$$\#_{L_1} = \{(a,a),(a,d),(b,b),(b,e),(c,c),(c,e),(d,a),(d,d),(e,b),(e,c),(e,e)\}$$

Footprint Matrix

We can record all information about
log-based ordering relations
in a concise way as a matrix:

one row for each event
one column for each event
the entry in row a and column b tells us their relation

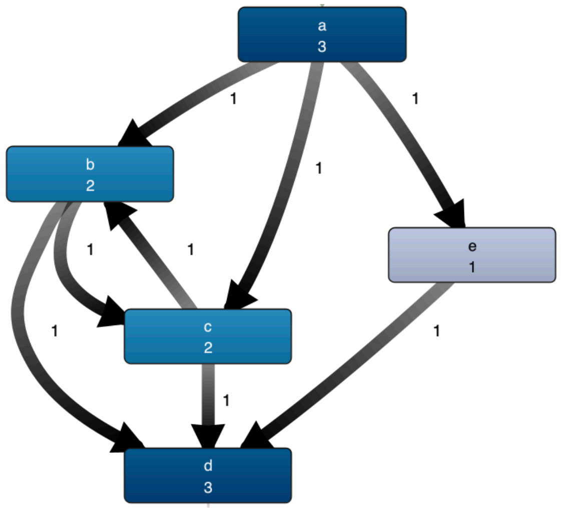
Footprint Matrix: Example



$$L_1 = [\langle a, b, c, d\rangle^3, \langle a, c, b, d\rangle^2, \langle a, e, d\rangle]$$

	a	b	c	d	e
a					
b			$\|_{L_1}$		
c		$\|_{L_1}$			
d					
e					

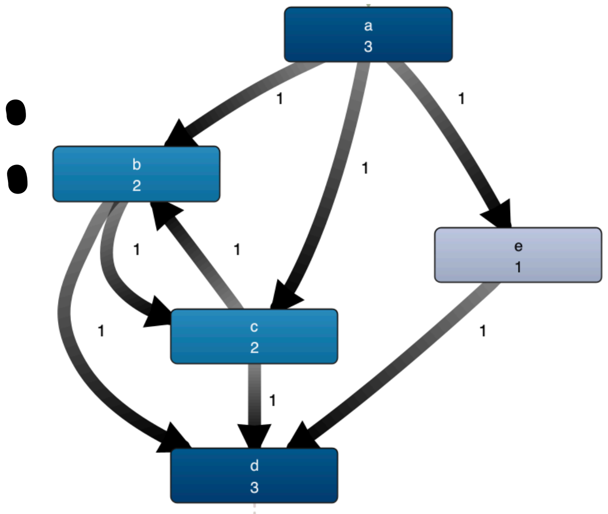
Footprint Matrix: Example



$$L_1 = [\langle a, b, c, d\rangle^3, \langle a, c, b, d\rangle^2, \langle a, e, d\rangle]$$

	a	b	c	d	e
a		$\rightarrow_{L_1}$	$\rightarrow_{L_1}$		$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$		$\|_{L_1}$		
c	$\leftarrow_{L_1}$	$\|_{L_1}$			
d					
e	$\leftarrow_{L_1}$				

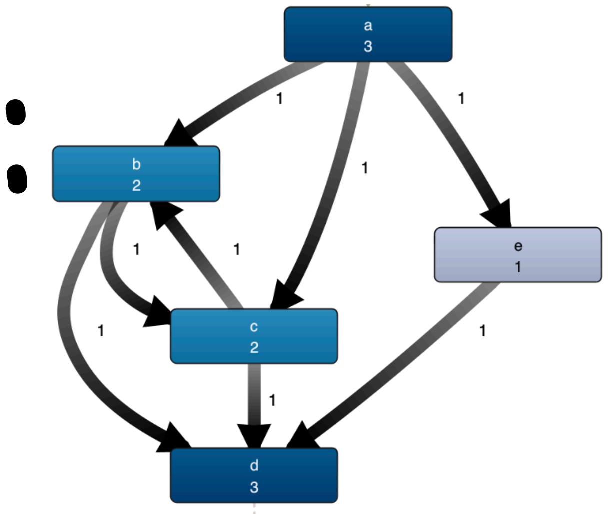
Footprint Matrix: Example



$$L_1 = [\langle a, b, c, d\rangle^3, \langle a, c, b, d\rangle^2, \langle a, e, d\rangle]$$

	a	b	c	d	e
a		$\rightarrow_{L_1}$	$\rightarrow_{L_1}$		$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$		$\Vert_{L_1}$	$\rightarrow_{L_1}$	
c	$\leftarrow_{L_1}$	$\Vert_{L_1}$		$\rightarrow_{L_1}$	
d		$\leftarrow_{L_1}$	$\leftarrow_{L_1}$		$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$			$\rightarrow_{L_1}$	

Footprint Matrix: Example



$$L_1 = [\langle a,b,c,d\rangle^3, \langle a,c,b,d\rangle^2, \langle a,e,d\rangle]$$

	a	b	c	d	e
a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\Vert_{L_1}$	$\rightarrow_{L_1}$	
c	$\leftarrow_{L_1}$	$\Vert_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

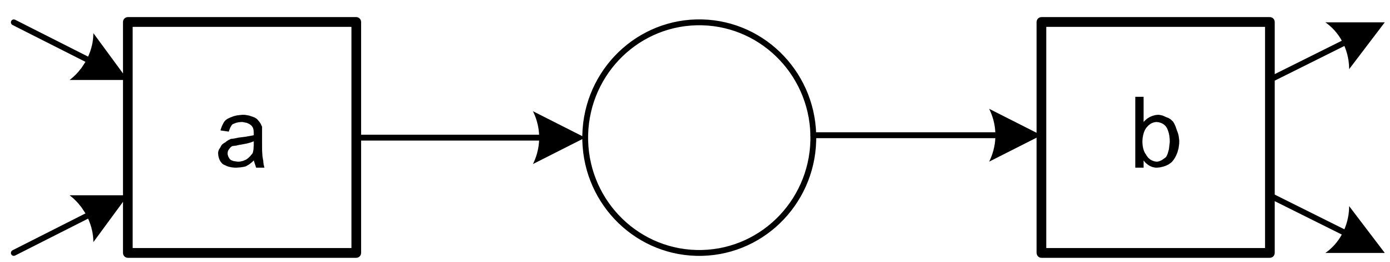
Footprint Matrix: Example

Note the symmetry w.r.t. the diagonal

	a	b	c	d	e
a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\boxed{\#_{L_1}}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\boxed{\Vert_{L_1}}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\boxed{\Vert_{L_1}}$	$\#_{L_1}$	$\boxed{\rightarrow_{L_1}}$	$\#_{L_1}$
d	$\boxed{\#_{L_1}}$	$\leftarrow_{L_1}$	$\boxed{\leftarrow_{L_1}}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

Patterns

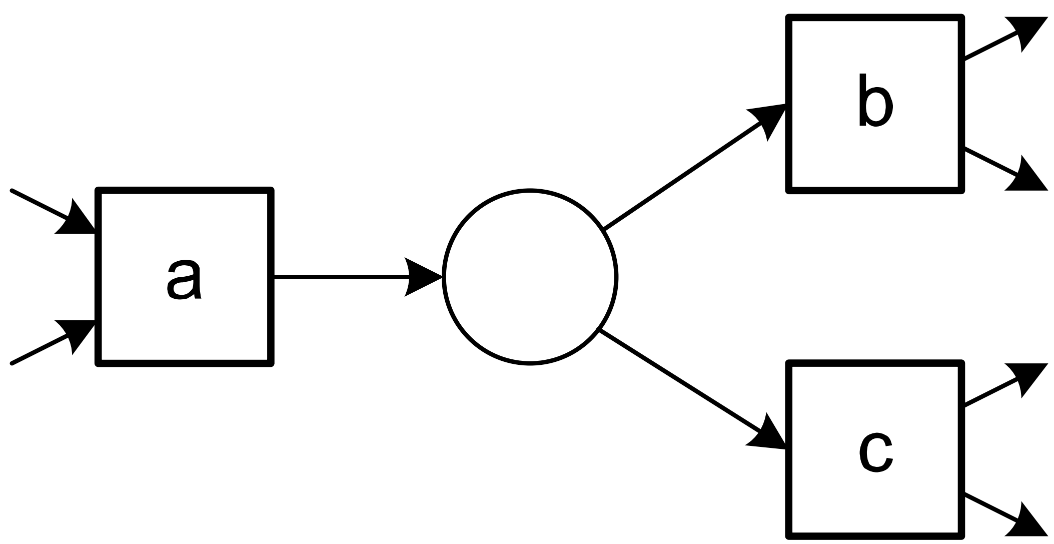
Footprints are useful to discover typical patterns of activities in the corresponding process model



(a) sequence pattern: a→b

Patterns

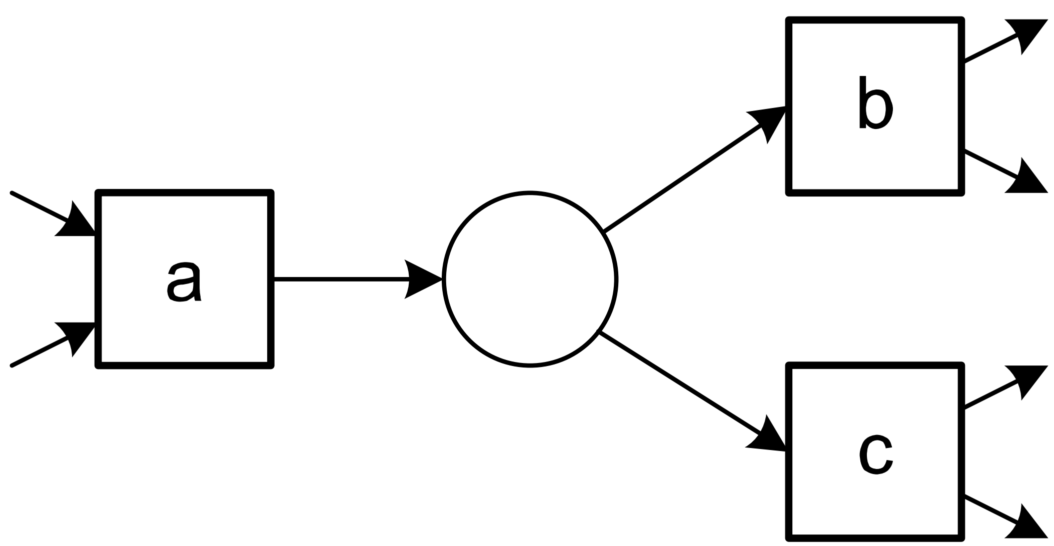
Footprints are useful to discover typical patterns of activities in the corresponding process model



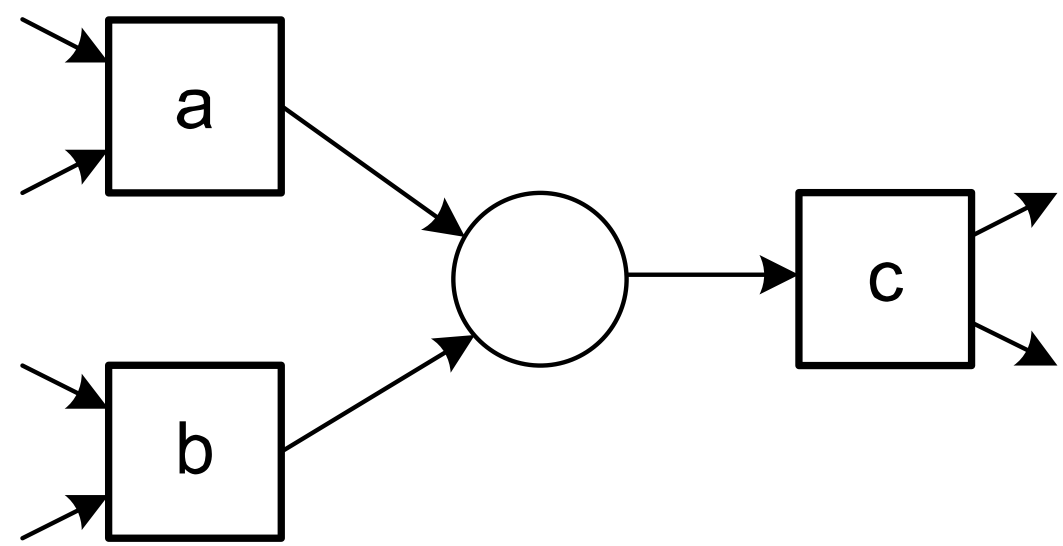
(b) XOR-split pattern:
a→b, a→c, and b#c

Patterns

Footprints are useful to discover typical patterns of activities in the corresponding process model



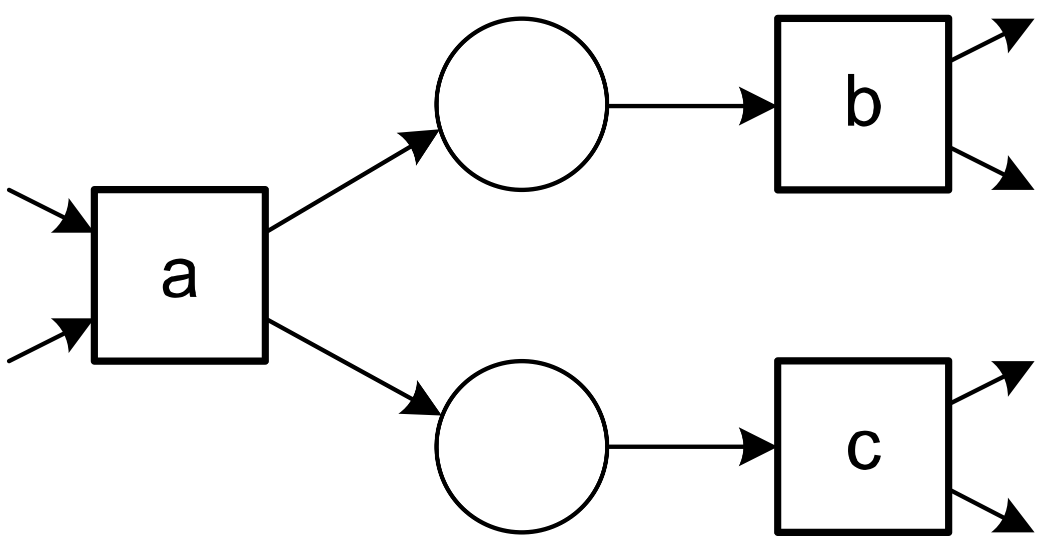
(b) XOR-split pattern:
a→b, a→c, and b#c



(c) XOR-join pattern:
a→c, b→c, and a#b

Patterns

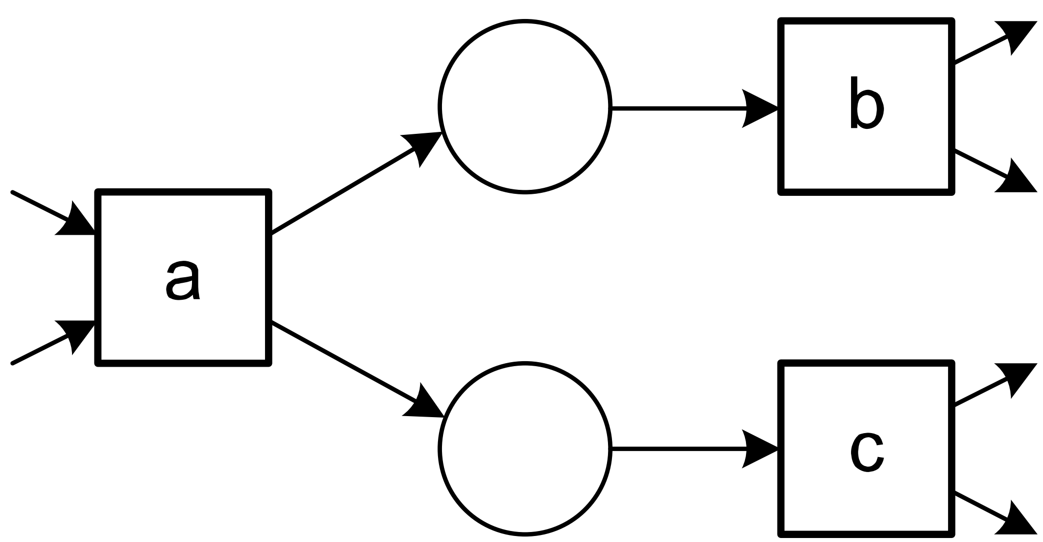
Footprints are useful to discover typical patterns of activities in the corresponding process model



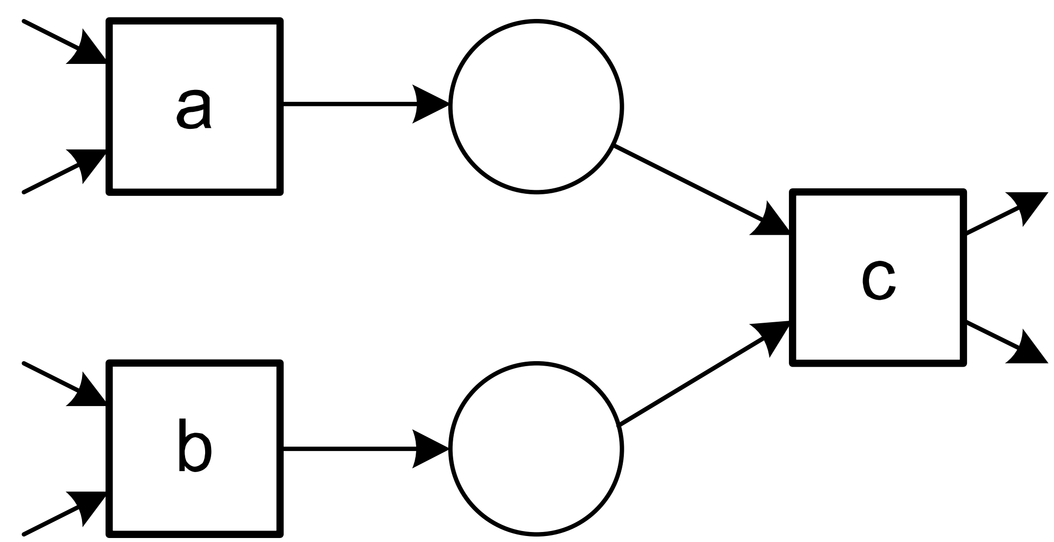
(d) AND-split pattern:
$a \rightarrow b$, $a \rightarrow c$, and $b || c$

Patterns

Footprints are useful to discover typical patterns of activities in the corresponding process model



(d) AND-split pattern:
a→b, a→c, and b||c



(e) AND-join pattern:
a→c, b→c, and a||b

The α-Algorithm

1. $T_L = \{\ t \in T \ \mid\ \exists_{\sigma \in L}\ t \in \sigma\ \}$ transitions
2. $T_I = \{\ t \in T \ \mid\ \exists_{\sigma \in L}\ t = \mathit{first}(\sigma)\ \}$ start events
3. $T_O = \{\ t \in T \ \mid\ \exists_{\sigma \in L}\ t = \mathit{last}(\sigma)\ \}$ end events
4. $X_L = \left\{ (A,B) \ \middle|\ \begin{array}{lll} A,B \subseteq T_L & \wedge \quad A,B \neq \emptyset & \wedge \\ \forall_{a \in A} \forall_{b \in B} & a \rightarrow_L b & \wedge \\ \forall_{a_1,a_2 \in A} & a_1 \#_L a_2 & \wedge \\ \forall_{b_1,b_2 \in B} & b_1 \#_L b_2 & \end{array} \right\}$ decision points
5. $Y_L = \left\{ (A,B) \in X_L \ \middle|\ \forall_{(A',B') \in X_L} \begin{array}{c} A \subseteq A' \ \wedge\ B \subseteq B' \\ \Rightarrow \\ (A',B') = (A,B) \end{array} \right\}$ max. dec. points
6. $P_L = \{\ p_{(A,B)} \ \mid\ (A,B) \in Y_L\ \} \cup \{\ i_L, o_L\ \}$ places
7. $F_L = \{\ (a, p_{(A,B)}) \ \mid\ (A,B) \in Y_L \ \wedge\ a \in A\ \} \cup$
   $\{\ (p_{(A,B)}, b) \ \mid\ (A,B) \in Y_L \ \wedge\ b \in B\ \} \cup$
   $\{\ (i_L, t) \ \mid\ t \in T_I\ \} \cup$
   $\{\ (t, o_L) \ \mid\ t \in T_O\ \}$ arcs
8. $\alpha(L) = (P_L, T_L, F_L, i_L)$ net

The α-Algorithm

one transition for each event in the log

1. $T_L = \{\ t \in T \mid \exists_{\sigma \in L}\ t \in \sigma\ \}$ transitions

2. $T_I = \{\ t \in T \mid \exists_{\sigma \in L}\ t = \mathit{first}(\sigma)\ \}$ start events

3. $T_O = \{\ t \in T \mid \exists_{\sigma \in L}\ t = \mathit{last}(\sigma)\ \}$ end events

transitions that start/end at least one trace

Steps 1-3: Example

$$L_1 = [\langle a, b, c, d\rangle^3, \langle a, c, b, d\rangle^2, \langle a, e, d\rangle]$$

1. $T_L = \{ \, t \in T \mid \exists_{\sigma \in L} \, t \in \sigma \, \}$ transitions
2. $T_I = \{ \, t \in T \mid \exists_{\sigma \in L} \, t = \mathit{first}(\sigma) \, \}$ start events
3. $T_O = \{ \, t \in T \mid \exists_{\sigma \in L} \, t = \mathit{last}(\sigma) \, \}$ end events

$$T_L = \{a, b, c, d, e\}$$

$$T_I = \{a\}$$

$$T_O = \{d\}$$

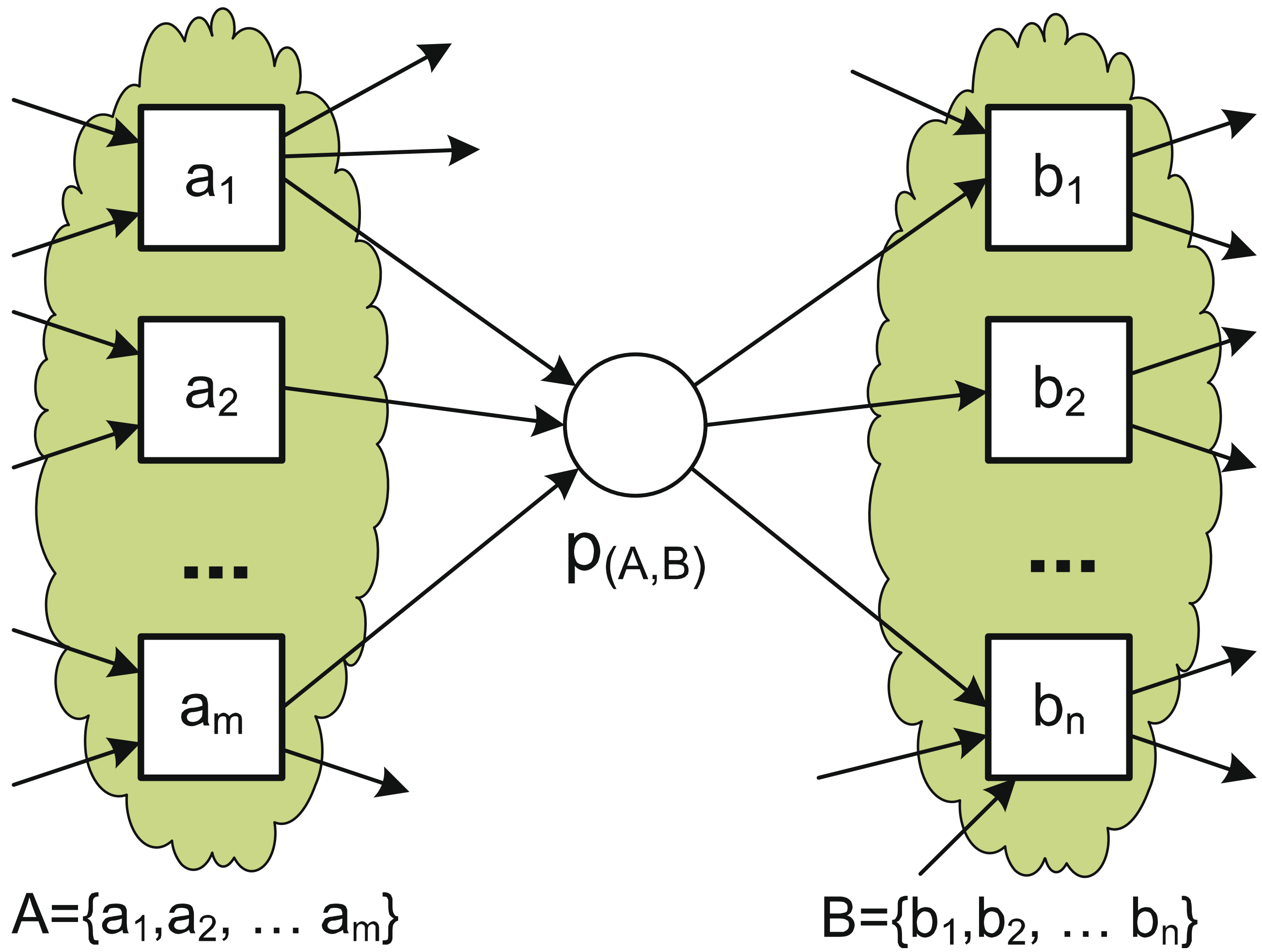
The α-Algorithm

we collect pairs of sets of events with certain features

4. $$X_L = \left\{ (A,B) \;\middle|\; \begin{array}{lll} A,B \subseteq T_L & \wedge \quad A,B \neq \emptyset & \wedge \\ \forall_{a \in A} \forall_{b \in B} & a \rightarrow_L b & \wedge \\ \forall_{a_1,a_2 \in A} & a_1 \#_L a_2 & \wedge \\ \forall_{b_1,b_2 \in B} & b_1 \#_L b_2 & \end{array} \right\}$$ decision points

each event in A causes all events in B
all events in A are mutually exclusive
all events in B are mutually exclusive

The Core of the α-Algorithm: Steps 4, 5



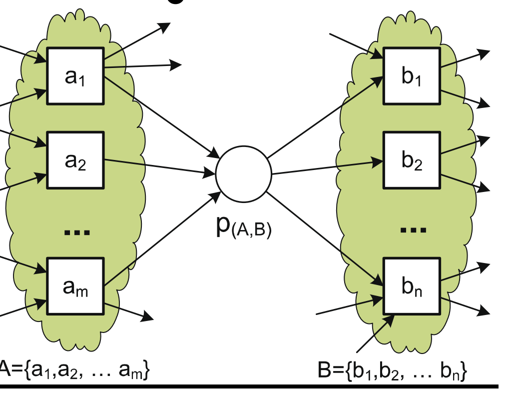
we are going to insert a place for each pair (A,B) to represent some sort of decision point

The Core of the α-Algorithm: Steps 4, 5

$\forall_{a \in A} \forall_{b \in B} \quad a \rightarrow_L b$

$\forall_{a_1, a_2 \in A} \quad a_1 \#_L a_2$

$\forall_{b_1, b_2 \in B} \quad b_1 \#_L b_2$



		a_1	a_2	$\dots$	a_m	b_1	b_2	$\dots$	b_n
A	a_1	#	#	$\dots$	#	$\rightarrow$	$\rightarrow$	$\dots$	$\rightarrow$
	a_2	#	#	$\dots$	#	$\rightarrow$	$\rightarrow$	$\dots$	$\rightarrow$
	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
	a_m	#	#	$\dots$	#	$\rightarrow$	$\rightarrow$	$\dots$	$\rightarrow$
B	b_1	$\leftarrow$	$\leftarrow$	$\dots$	$\leftarrow$	#	#	$\dots$	#
	b_2	$\leftarrow$	$\leftarrow$	$\dots$	$\leftarrow$	#	#	$\dots$	#
	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
	b_n	$\leftarrow$	$\leftarrow$	$\dots$	$\leftarrow$	#	#	$\dots$	#

The Core of the α-Algorithm: Step 5

$\forall_{a \in A} \forall_{b \in B} \quad a \rightarrow_L b$

$\forall_{a_1, a_2 \in A} \quad a_1 \#_L a_2$

$\forall_{b_1, b_2 \in B} \quad b_1 \#_L b_2$

If (A, B) forms a decision point any pair (A', B') with $A' \subseteq A$, $B' \subseteq B$ is still a decision point

	a_1	a_2	$\ldots$	a_m	b_1	b_2	$\ldots$	b_n
a_1	#	#	$\ldots$	#	$\rightarrow$	$\rightarrow$	$\ldots$	$\rightarrow$
a_2	#	#	$\ldots$	#	$\rightarrow$	$\rightarrow$	$\ldots$	$\rightarrow$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
a_m	#	#	$\ldots$	#	$\rightarrow$	$\rightarrow$	$\ldots$	$\rightarrow$
b_1	$\leftarrow$	$\leftarrow$	$\ldots$	$\leftarrow$	#	#	$\ldots$	#
b_2	$\leftarrow$	$\leftarrow$	$\ldots$	$\leftarrow$	#	#	$\ldots$	#
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
b_n	$\leftarrow$	$\leftarrow$	$\ldots$	$\leftarrow$	#	#	$\ldots$	#

The Core of the α-Algorithm: Step 5

$\forall_{a \in A} \forall_{b \in B} \quad a \rightarrow_L b$

$\forall_{a_1, a_2 \in A} \quad a_1 \#_L a_2$

$\forall_{b_1, b_2 \in B} \quad b_1 \#_L b_2$

If (A, B) forms a decision point any pair (A', B') with $A' \subseteq A$, $B' \subseteq B$ is still a decision point

	a_1	a_2	$\ldots$	a_m	b_1	b_2	$\ldots$	b_n
a_1	#	#	$\ldots$	#	$\rightarrow$	$\rightarrow$	$\ldots$	$\rightarrow$
a_2	#	#	$\ldots$	#	$\rightarrow$	$\rightarrow$	$\ldots$	$\rightarrow$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
a_m	#	#	$\ldots$	#	$\rightarrow$	$\rightarrow$	$\ldots$	$\rightarrow$
b_1	$\leftarrow$	$\leftarrow$	$\ldots$	$\leftarrow$	#	#	$\ldots$	#
b_2	$\leftarrow$	$\leftarrow$	$\ldots$	$\leftarrow$	#	#	$\ldots$	#
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
b_n	$\leftarrow$	$\leftarrow$	$\ldots$	$\leftarrow$	#	#	$\ldots$	#

The Core of the α-Algorithm: Step 5

$\forall_{a \in A} \forall_{b \in B} \quad a \rightarrow_L b$

$\forall_{a_1, a_2 \in A} \quad a_1 \#_L a_2$

$\forall_{b_1, b_2 \in B} \quad b_1 \#_L b_2$

If (A, B) forms a decision point any pair (A', B') with $A' \subseteq A$, $B' \subseteq B$ is still a decision point

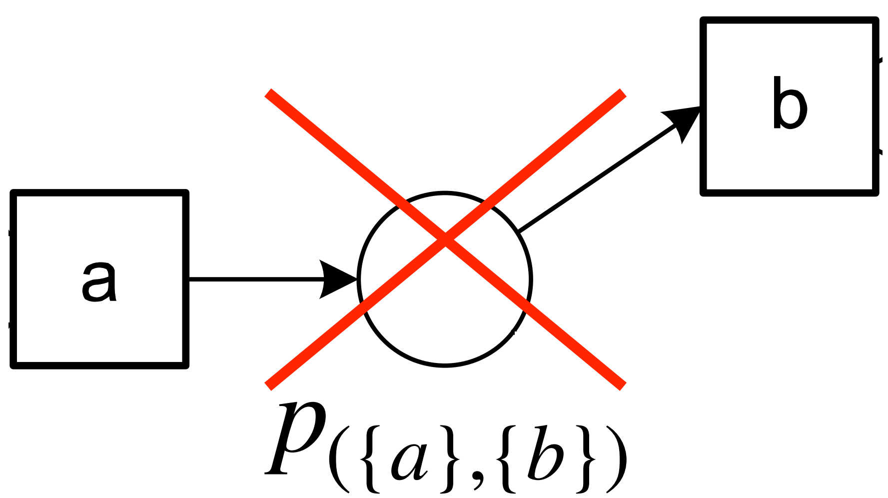
	a_1	a_2	$\dots$	a_m	b_1	b_2	$\dots$	b_n
a_1	#	#	$\dots$	#	$\rightarrow$	$\rightarrow$	$\dots$	$\rightarrow$
a_2	#	#	$\dots$	#	$\rightarrow$	$\rightarrow$	$\dots$	$\rightarrow$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
a_m	#	#	$\dots$	#	$\rightarrow$	$\rightarrow$	$\dots$	$\rightarrow$
b_1	$\leftarrow$	$\leftarrow$	$\dots$	$\leftarrow$	#	#	$\dots$	#
b_2	$\leftarrow$	$\leftarrow$	$\dots$	$\leftarrow$	#	#	$\dots$	#
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
b_n	$\leftarrow$	$\leftarrow$	$\dots$	$\leftarrow$	#	#	$\dots$	#

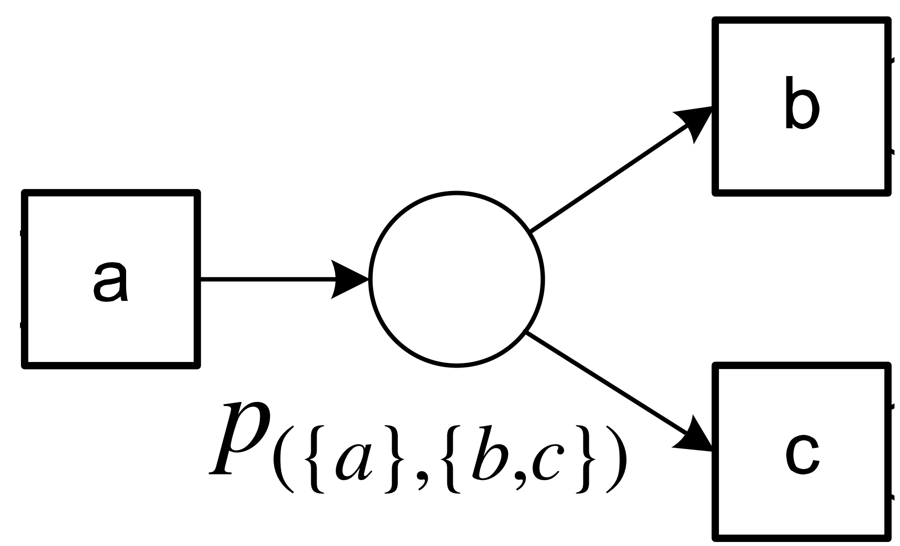
The α-Algorithm

We take only the **largest** pairs (A,B)

5. $Y_L = \left\{ (A,B) \in X_L \mid \forall_{(A',B') \in X_L} \begin{array}{c} A \subseteq A' \ \wedge \ B \subseteq B' \\ \Rightarrow \\ (A',B') = (A,B) \end{array} \right\}$ max. dec. points

$\boxed{Y_L \text{ contains all pairs in } X_L \text{ that are not dominated by other pairs}}$





Steps 4-5: Example

	a	b	c	d	e
a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\|_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\|_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

$$X_{L_1} = \big\{\big(\{a\},\{b\}\big),\big(\{a\},\{c\}\big),\big(\{a\},\{e\}\big),\big(\{a\},\{b,e\}\big),\big(\{a\},\{c,e\}\big),\\ \big(\{b\},\{d\}\big),\big(\{c\},\{d\}\big),\big(\{e\},\{d\}\big),\big(\{b,e\},\{d\}\big),\big(\{c,e\},\{d\}\big)\big\}$$

Steps 4-5: Example

	a	b	c	d	e
a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\|_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\|_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

$$X_{L_1} = \Big\{\boxed{\big(\{a\},\{b\}\big)}, \big(\{a\},\{c\}\big), \big(\{a\},\{e\}\big), \big(\{a\},\{b,e\}\big), \big(\{a\},\{c,e\}\big),$$
$$\big(\{b\},\{d\}\big), \big(\{c\},\{d\}\big), \big(\{e\},\{d\}\big), \big(\{b,e\},\{d\}\big), \big(\{c,e\},\{d\}\big)\Big\}$$

Steps 4-5: Example

	a	b	c	d	e
a	$\boxed{\#_{L_1}}$	$\rightarrow_{L_1}$	$\boxed{\rightarrow_{L_1}}$	$\#_{L_1}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\Vert_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\boxed{\leftarrow_{L_1}}$	$\Vert_{L_1}$	$\boxed{\#_{L_1}}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

$$X_{L_1} = \Big\{\big(\{a\},\{b\}\big), \boxed{\big(\{a\},\{c\}\big)}, \big(\{a\},\{e\}\big), \big(\{a\},\{b,e\}\big), \big(\{a\},\{c,e\}\big),$$
$$\big(\{b\},\{d\}\big), \big(\{c\},\{d\}\big), \big(\{e\},\{d\}\big), \big(\{b,e\},\{d\}\big), \big(\{c,e\},\{d\}\big)\Big\}$$

Steps 4-5: Example

	a	b	c	d	e
a	$\boxed{\#_{L_1}}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\boxed{\rightarrow_{L_1}}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\|_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\|_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\boxed{\leftarrow_{L_1}}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\boxed{\#_{L_1}}$

$$X_{L_1} = \Big\{\big(\{a\},\{b\}\big), \big(\{a\},\{c\}\big), \boxed{\big(\{a\},\{e\}\big)}\ \big(\{a\},\{b,e\}\big), \big(\{a\},\{c,e\}\big),$$
$$\big(\{b\},\{d\}\big), \big(\{c\},\{d\}\big), \big(\{e\},\{d\}\big), \big(\{b,e\},\{d\}\big), \big(\{c,e\},\{d\}\big)\Big\}$$

Steps 4-5: Example

	a	b	c	d	e
a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\parallel_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\parallel_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

$$X_{L_1} = \{(\{a\},\{b\}),(\{a\},\{c\}),(\{a\},\{e\}),(\boxed{\{a\},\{b,e\}}),(\{a\},\{c,e\}),$$
$$(\{b\},\{d\}),(\{c\},\{d\}),(\{e\},\{d\}),(\{b,e\},\{d\}),(\{c,e\},\{d\})\}$$

Steps 4-5: Example

	a	b	c	d	e
a	$\boxed{\#_{L_1}}$	$\rightarrow_{L_1}$	$\boxed{\rightarrow_{L_1}}$	$\#_{L_1}$	$\boxed{\rightarrow_{L_1}}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\Vert_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\boxed{\leftarrow_{L_1}}$	$\Vert_{L_1}$	$\boxed{\#_{L_1}}$	$\rightarrow_{L_1}$	$\boxed{\#_{L_1}}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\boxed{\leftarrow_{L_1}}$	$\#_{L_1}$	$\boxed{\#_{L_1}}$	$\rightarrow_{L_1}$	$\boxed{\#_{L_1}}$

$$X_{L_1} = \big\{ \big(\{a\},\{b\}\big), \big(\{a\},\{c\}\big), \big(\{a\},\{e\}\big), \big(\{a\},\{b,e\}\big), \big(\boxed{\{a\},\{c,e\}}\big), \big(\{b\},\{d\}\big), \big(\{c\},\{d\}\big), \big(\{e\},\{d\}\big), \big(\{b,e\},\{d\}\big), \big(\{c,e\},\{d\}\big) \big\}$$

Steps 4-5: Example

	a	b	c	d	e
a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\|_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\|_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

$$X_{L_1} = \Big\{\big(\{a\},\{b\}\big),\big(\{a\},\{c\}\big),\big(\{a\},\{e\}\big),\big(\{a\},\{b,e\}\big),\big(\{a\},\{c,e\}\big),\\ \big(\{b\},\{d\}\big),\big(\{c\},\{d\}\big),\big(\{e\},\{d\}\big),\big(\{b,e\},\{d\}\big),\big(\{c,e\},\{d\}\big)\Big\}$$

and so on for the other pairs

Steps 4-5: Example

	a	b	c	d	e
a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\\|_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\\|_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

$$X_{L_1} = \{\cancel{(\{a\},\{b\})}, (\{a\},\{c\}), \cancel{(\{a\},\{e\})}, \boxed{(\{a\},\{b,e\})}, (\{a\},\{c,e\}), (\{b\},\{d\}), (\{c\},\{d\}), (\{e\},\{d\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$$

$$Y_{L_1} = \{\boxed{(\{a\},\{b,e\})}, (\{a\},\{c,e\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$$

We take only the largest pairs

Steps 4-5: Example

	a	b	c	d	e
a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\Vert_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\Vert_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

$$X_{L_1} = \Big\{ \cancel{\big(\{a\},\{b\}\big)}, \boxed{\cancel{\big(\{a\},\{c\}\big)}}, \boxed{\cancel{\big(\{a\},\{e\}\big)}}, \big(\{a\},\{b,e\}\big), \boxed{\big(\{a\},\{c,e\}\big)}, \big(\{b\},\{d\}\big), \big(\{c\},\{d\}\big), \big(\{e\},\{d\}\big), \big(\{b,e\},\{d\}\big), \big(\{c,e\},\{d\}\big) \Big\}$$

$$Y_{L_1} = \Big\{ \big(\{a\},\{b,e\}\big), \boxed{\big(\{a\},\{c,e\}\big)}, \big(\{b,e\},\{d\}\big), \big(\{c,e\},\{d\}\big) \Big\}$$

We take only the largest pairs

Steps 4-5: Example

	a	b	c	d	e
a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\Vert_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\Vert_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

$$X_{L_1} = \Big\{\big(\cancel{\{a\},\{b\}}\big), \big(\cancel{\{a\},\{c\}}\big), \big(\cancel{\{a\},\{e\}}\big), \big(\{a\},\{b,e\}\big), \big(\{a\},\{c,e\}\big), \boxed{\big(\cancel{\{b\},\{d\}}\big)}, \big(\{c\},\{d\}\big), \boxed{\big(\cancel{\{e\},\{d\}}\big)}, \boxed{\big(\{b,e\},\{d\}\big)}, \big(\{c,e\},\{d\}\big)\Big\}$$

$$Y_{L_1} = \Big\{\big(\{a\},\{b,e\}\big), \big(\{a\},\{c,e\}\big), \boxed{\big(\{b,e\},\{d\}\big)}, \big(\{c,e\},\{d\}\big)\Big\}$$

We take only the largest pairs

Steps 4-5: Example

	a	b	c	d	e
a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\Vert_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\Vert_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

$$X_{L_1} = \Big\{\cancel{(\{a\},\{b\})}, \cancel{(\{a\},\{c\})}, \cancel{(\{a\},\{e\})}, \big(\{a\},\{b,e\}\big), \big(\{a\},\{c,e\}\big), \cancel{(\{b\},\{d\})}, \boxed{\cancel{(\{c\},\{d\})}}, \boxed{\cancel{(\{e\},\{d\})}}, \big(\{b,e\},\{d\}\big), \boxed{\big(\{c,e\},\{d\}\big)}\Big\}$$

$$Y_{L_1} = \Big\{\big(\{a\},\{b,e\}\big), \big(\{a\},\{c,e\}\big), \big(\{b,e\},\{d\}\big), \boxed{\big(\{c,e\},\{d\}\big)}\Big\}$$

We take only the largest pairs

The α-Algorithm

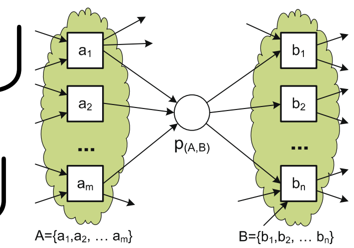
One place for for each pair

Initial Final

6. $P_L = \{\, p_{(A,B)} \mid (A,B) \in Y_L \,\} \cup \{\, i_L, o_L \,\}$ places

Connect decision points

7. $F_L = \{\, (a, p_{(A,B)}) \mid (A,B) \in Y_L \ \wedge \ a \in A \,\} \cup$
$\{\, (p_{(A,B)}, b) \mid (A,B) \in Y_L \ \wedge \ b \in B \,\} \cup$
$\{\, (i_L, t) \mid t \in T_I \,\} \cup$
$\{\, (t, o_L) \mid t \in T_O \,\}$ arcs



From the initial place to each initial transition

From any final transition to the final place

8. $\alpha(L) = (P_L, T_L, F_L, i_L)$ net

(remind that T_L is the set of all events)

The α-Algorithm

$$L_1 = [\langle a,b,c,d\rangle^3, \langle a,c,b,d\rangle^2, \langle a,e,d\rangle]$$

1. $T_L = \{\ t \in T \ \mid\ \exists_{\sigma \in L}\ t \in \sigma\ \}$ transitions

2. $T_I = \{\ t \in T \ \mid\ \exists_{\sigma \in L}\ t = \mathit{first}(\sigma)\ \}$ start events

3. $T_O = \{\ t \in T \ \mid\ \exists_{\sigma \in L}\ t = \mathit{last}(\sigma)\ \}$ end events

4. $X_L = \left\{ (A,B) \ \middle|\ \begin{array}{lll} A,B \subseteq T_L & \wedge\ \ A,B \neq \emptyset & \wedge \\ \forall_{a \in A} \forall_{b \in B} & a \rightarrow_L b & \wedge \\ \forall_{a_1,a_2 \in A} & a_1 \#_L a_2 & \wedge \\ \forall_{b_1,b_2 \in B} & b_1 \#_L b_2 & \end{array} \right\}$ decision points

5. $Y_L = \left\{ (A,B) \in X_L \ \middle|\ \forall_{(A',B') \in X_L} \begin{array}{c} A \subseteq A' \ \wedge\ B \subseteq B' \\ \Rightarrow \\ (A',B') = (A,B) \end{array} \right\}$ max. dec. points

6. $P_L = \{\ p_{(A,B)} \ \mid\ (A,B) \in Y_L\ \} \cup \{\ i_L, o_L\ \}$ places

7. $F_L = \{\ (a, p_{(A,B)}) \ \mid\ (A,B) \in Y_L \ \wedge\ a \in A\ \} \cup$
   $\{\ (p_{(A,B)}, b) \ \mid\ (A,B) \in Y_L \ \wedge\ b \in B\ \} \cup$
   $\{\ (i_L, t) \ \mid\ t \in T_I\ \} \cup$
   $\{\ (t, o_L) \ \mid\ t \in T_O\ \}$ arcs

8. $\alpha(L) = (P_L, T_L, F_L, i_L)$ net

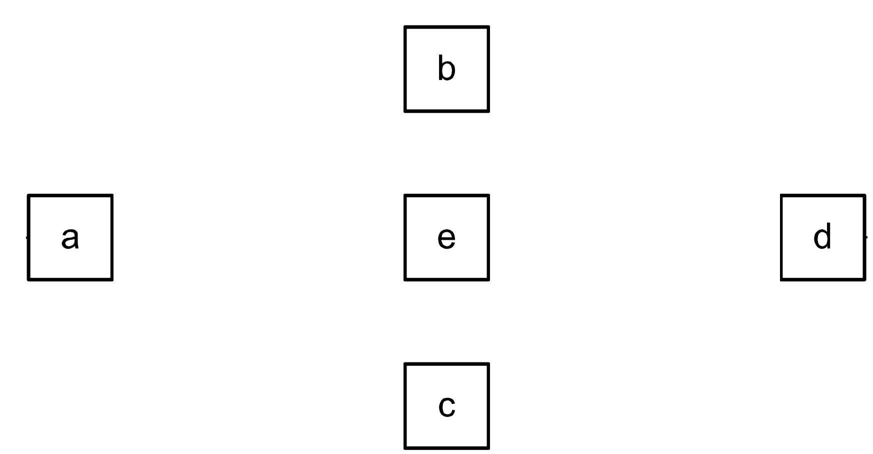
$T_L = \{a,b,c,d,e\}$

$T_I = \{a\}$

$T_O = \{d\}$

$X_{L_1} = \{(\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{e\}), (\{a\},\{b,e\}), (\{a\},\{c,e\}),$
$(\{b\},\{d\}), (\{c\},\{d\}), (\{e\},\{d\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$

$Y_{L_1} = \{(\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$



The α-Algorithm

$$L_1 = [\langle a,b,c,d\rangle^3, \langle a,c,b,d\rangle^2, \langle a,e,d\rangle]$$

1. $T_L = \{\ t \in T \mid \exists_{\sigma \in L}\ t \in \sigma\ \}$ transitions
2. $T_I = \{\ t \in T \mid \exists_{\sigma \in L}\ t = \mathit{first}(\sigma)\ \}$ start events
3. $T_O = \{\ t \in T \mid \exists_{\sigma \in L}\ t = \mathit{last}(\sigma)\ \}$ end events
4. $X_L = \left\{ (A,B) \;\middle|\; \begin{array}{lll} A,B \subseteq T_L & \wedge\ \ A,B \neq \emptyset & \wedge \\ \forall_{a\in A}\forall_{b\in B} & a \rightarrow_L b & \wedge \\ \forall_{a_1,a_2\in A} & a_1 \#_L a_2 & \wedge \\ \forall_{b_1,b_2\in B} & b_1 \#_L b_2 & \end{array} \right\}$ decision points
5. $Y_L = \left\{ (A,B) \in X_L \;\middle|\; \forall_{(A',B')\in X_L} \begin{array}{c} A \subseteq A' \ \wedge\ B \subseteq B' \\ \Rightarrow \\ (A',B') = (A,B) \end{array} \right\}$ max. dec. points
6. $P_L = \{\ p_{(A,B)} \mid (A,B) \in Y_L\ \} \cup \{\ i_L, o_L\ \}$ places
7. $F_L = \{\ (a, p_{(A,B)}) \mid (A,B) \in Y_L \ \wedge\ a \in A\ \} \cup$
   $\{\ (p_{(A,B)}, b) \mid (A,B) \in Y_L \ \wedge\ b \in B\ \} \cup$
   $\{\ (i_L, t) \mid t \in T_I\ \} \cup$
   $\{\ (t, o_L) \mid t \in T_O\ \}$ arcs
8. $\alpha(L) = (P_L, T_L, F_L, i_L)$ net

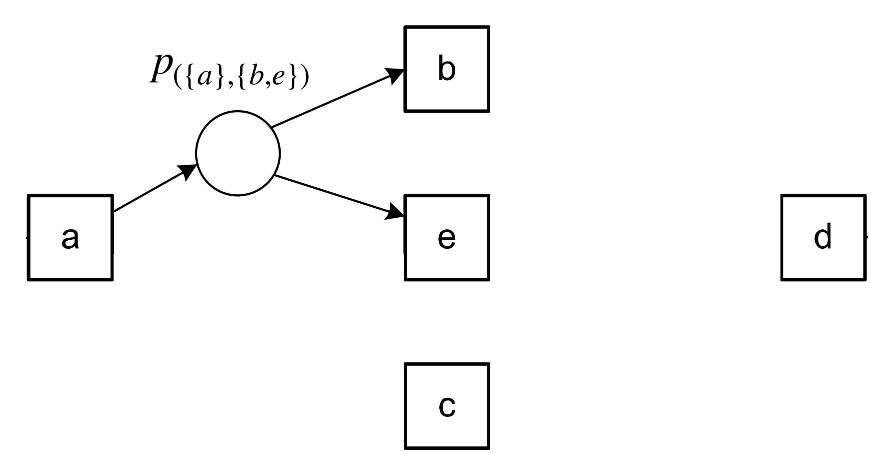
$T_L = \{a,b,c,d,e\}$

$T_I = \{a\}$

$T_O = \{d\}$

$X_{L_1} = \{(\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{e\}), (\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b\},\{d\}), (\{c\},\{d\}), (\{e\},\{d\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$

$Y_{L_1} = \{(\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$



The α-Algorithm

$L_1 = [\langle a, b, c, d\rangle^3, \langle a, c, b, d\rangle^2, \langle a, e, d\rangle]$

1. $T_L = \{\ t \in T \ \mid\ \exists_{\sigma \in L}\ t \in \sigma\ \}$ transitions
2. $T_I = \{\ t \in T \ \mid\ \exists_{\sigma \in L}\ t = \mathit{first}(\sigma)\ \}$ start events
3. $T_O = \{\ t \in T \ \mid\ \exists_{\sigma \in L}\ t = \mathit{last}(\sigma)\ \}$ end events
4. $X_L = \left\{ (A,B) \ \middle|\ \begin{array}{lll} A,B \subseteq T_L & \wedge \quad A,B \neq \emptyset & \wedge \\ \forall_{a \in A} \forall_{b \in B} & a \rightarrow_L b & \wedge \\ \forall_{a_1,a_2 \in A} & a_1 \#_L a_2 & \wedge \\ \forall_{b_1,b_2 \in B} & b_1 \#_L b_2 & \end{array} \right\}$ decision points
5. $Y_L = \left\{ (A,B) \in X_L \ \middle|\ \forall_{(A',B') \in X_L} \begin{array}{c} A \subseteq A' \ \wedge\ B \subseteq B' \\ \Rightarrow \\ (A',B') = (A,B) \end{array} \right\}$ max. dec. points
6. $P_L = \{\ p_{(A,B)} \ \mid\ (A,B) \in Y_L\ \} \cup \{\ i_L, o_L\ \}$ places
7. $F_L = \{\ (a, p_{(A,B)}) \ \mid\ (A,B) \in Y_L \ \wedge\ a \in A\ \} \cup$
   $\{\ (p_{(A,B)}, b) \ \mid\ (A,B) \in Y_L \ \wedge\ b \in B\ \} \cup$
   $\{\ (i_L, t) \ \mid\ t \in T_I\ \} \cup$
   $\{\ (t, o_L) \ \mid\ t \in T_O\ \}$ arcs
8. $\alpha(L) = (P_L, T_L, F_L, i_L)$ net

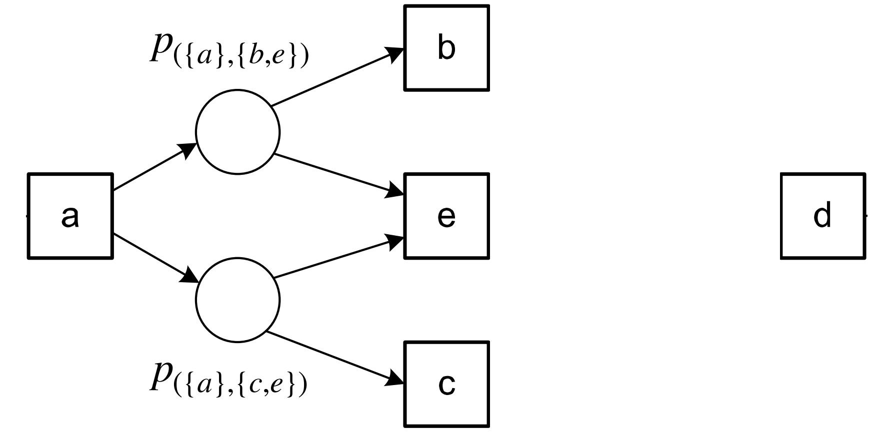
$T_L = \{a, b, c, d, e\}$

$T_I = \{a\}$

$T_O = \{d\}$

$X_{L_1} = \{(\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{e\}), (\{a\},\{b,e\}), (\{a\},\{c,e\}),$
$(\{b\},\{d\}), (\{c\},\{d\}), (\{e\},\{d\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$

$Y_{L_1} = \{(\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$



The α-Algorithm

$L_1 = [\langle a,b,c,d\rangle^3, \langle a,c,b,d\rangle^2, \langle a,e,d\rangle]$

1. $T_L = \{\, t \in T \mid \exists_{\sigma \in L}\; t \in \sigma \,\}$ transitions
2. $T_I = \{\, t \in T \mid \exists_{\sigma \in L}\; t = \mathit{first}(\sigma) \,\}$ start events
3. $T_O = \{\, t \in T \mid \exists_{\sigma \in L}\; t = \mathit{last}(\sigma) \,\}$ end events
4. $X_L = \left\{ (A,B) \;\middle|\; \begin{array}{lll} A,B \subseteq T_L & \wedge\;\; A,B \neq \emptyset & \wedge \\ \forall_{a\in A}\forall_{b \in B} & a \rightarrow_L b & \wedge \\ \forall_{a_1,a_2 \in A} & a_1 \#_L a_2 & \wedge \\ \forall_{b_1,b_2 \in B} & b_1 \#_L b_2 & \end{array} \right\}$ decision points
5. $Y_L = \left\{ (A,B) \in X_L \;\middle|\; \forall_{(A',B') \in X_L} \begin{array}{c} A \subseteq A' \;\wedge\; B \subseteq B' \\ \Rightarrow \\ (A',B') = (A,B) \end{array} \right\}$ max. dec. points
6. $P_L = \{\, p_{(A,B)} \mid (A,B) \in Y_L \,\} \cup \{\, i_L, o_L \,\}$ places
7. $F_L = \{\, (a, p_{(A,B)}) \mid (A,B) \in Y_L \;\wedge\; a \in A \,\} \cup$
   $\{\, (p_{(A,B)}, b) \mid (A,B) \in Y_L \;\wedge\; b \in B \,\} \cup$
   $\{\, (i_L, t) \mid t \in T_I \,\} \cup$
   $\{\, (t, o_L) \mid t \in T_O \,\}$ arcs
8. $\alpha(L) = (P_L, T_L, F_L, i_L)$ net

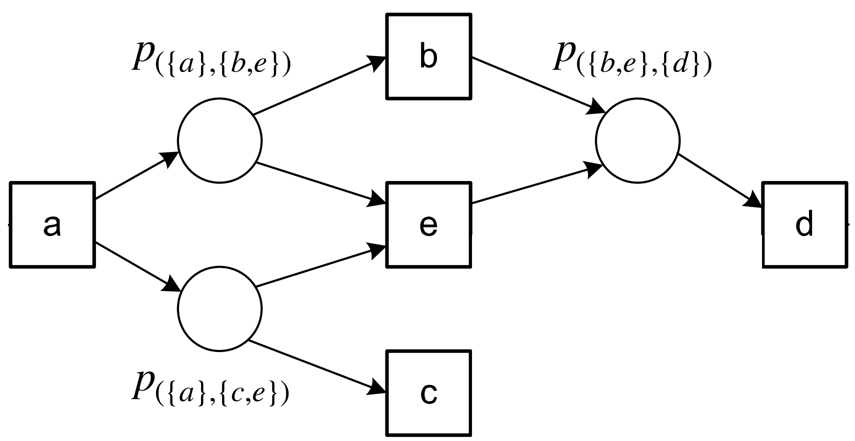
$T_L = \{a,b,c,d,e\}$

$T_I = \{a\}$

$T_O = \{d\}$

$X_{L_1} = \{(\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{e\}), (\{a\},\{b,e\}), (\{a\},\{c,e\}),$
$(\{b\},\{d\}), (\{c\},\{d\}), (\{e\},\{d\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$

$Y_{L_1} = \{(\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$



The α-Algorithm

$L_1 = [\langle a,b,c,d\rangle^3, \langle a,c,b,d\rangle^2, \langle a,e,d\rangle]$

1. $T_L = \{\ t \in T \ |\ \exists_{\sigma \in L}\ t \in \sigma\ \}$ transitions
2. $T_I = \{\ t \in T \ |\ \exists_{\sigma \in L}\ t = \mathit{first}(\sigma)\ \}$ start events
3. $T_O = \{\ t \in T \ |\ \exists_{\sigma \in L}\ t = \mathit{last}(\sigma)\ \}$ end events
4. $X_L = \left\{ (A,B)\ \middle|\ \begin{array}{lll} A,B \subseteq T_L & \wedge\quad A,B \neq \emptyset & \wedge \\ \forall_{a \in A}\forall_{b \in B} & a \rightarrow_L b & \wedge \\ \forall_{a_1,a_2 \in A} & a_1 \#_L a_2 & \wedge \\ \forall_{b_1,b_2 \in B} & b_1 \#_L b_2 & \end{array} \right\}$ decision points
5. $Y_L = \left\{ (A,B) \in X_L\ \middle|\ \forall_{(A',B') \in X_L} \begin{array}{c} A \subseteq A' \ \wedge\ B \subseteq B' \\ \Rightarrow \\ (A',B') = (A,B) \end{array} \right\}$ max. dec. points
6. $P_L = \{\ p_{(A,B)}\ |\ (A,B) \in Y_L\ \} \cup \{\ i_L, o_L\ \}$ places
7. $F_L = \{\ (a, p_{(A,B)})\ |\ (A,B) \in Y_L\ \wedge\ a \in A\ \} \cup$
   $\{\ (p_{(A,B)}, b)\ |\ (A,B) \in Y_L\ \wedge\ b \in B\ \} \cup$
   $\{\ (i_L, t)\ |\ t \in T_I\ \} \cup$
   $\{\ (t, o_L)\ |\ t \in T_O\ \}$ arcs
8. $\alpha(L) = (P_L, T_L, F_L, i_L)$ net

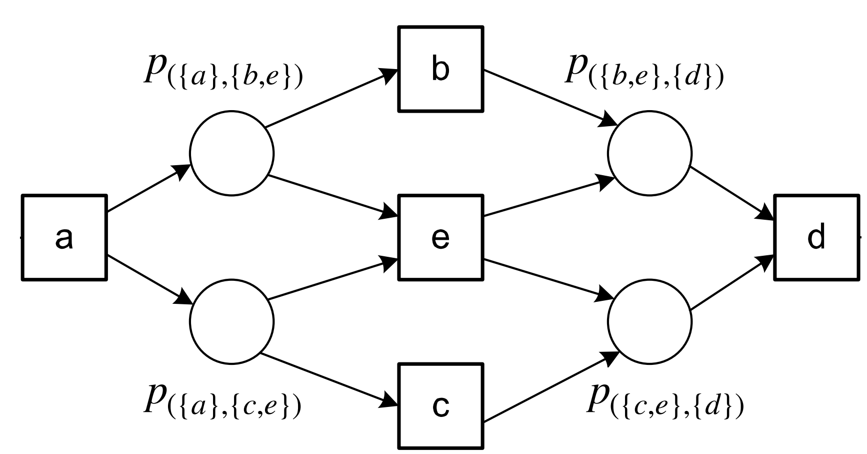
$T_L = \{a,b,c,d,e\}$

$T_I = \{a\}$

$T_O = \{d\}$

$X_{L_1} = \{(\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{e\}), (\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b\},\{d\}), (\{c\},\{d\}), (\{e\},\{d\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$

$Y_{L_1} = \{(\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$



The α-Algorithm

$L_1 = [\langle a,b,c,d\rangle^3, \langle a,c,b,d\rangle^2, \langle a,e,d\rangle]$

1. $T_L = \{\ t \in T \ \mid\ \exists_{\sigma \in L}\ t \in \sigma\ \}$ transitions
2. $T_I = \{\ t \in T \ \mid\ \exists_{\sigma \in L}\ t = \mathit{first}(\sigma)\ \}$ start events
3. $T_O = \{\ t \in T \ \mid\ \exists_{\sigma \in L}\ t = \mathit{last}(\sigma)\ \}$ end events

$T_L = \{a,b,c,d,e\}$

$T_I = \{a\}$

$T_O = \{d\}$

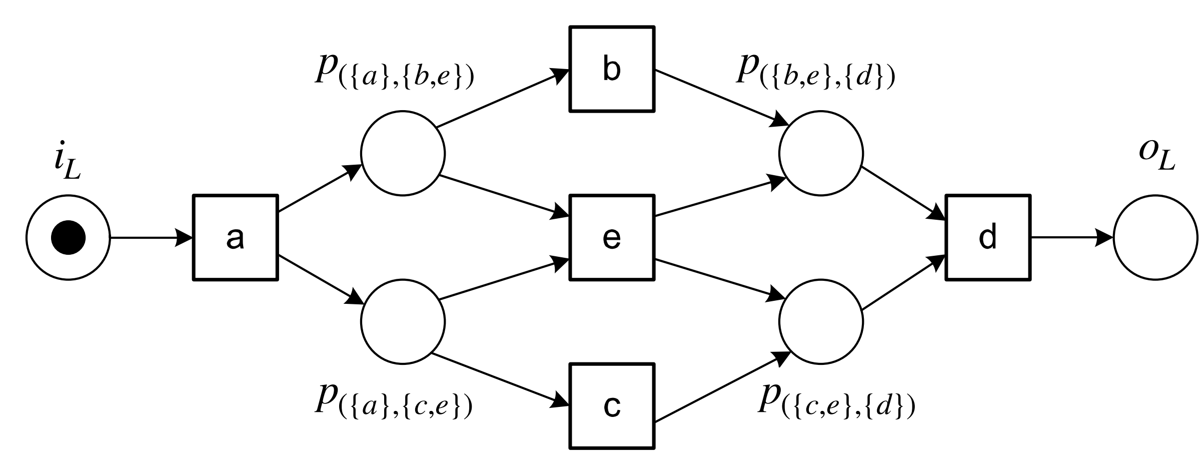
4. $X_L = \left\{ (A,B) \ \middle|\ \begin{array}{lll} A,B \subseteq T_L & \wedge\quad A,B \neq \emptyset & \wedge \\ \forall_{a \in A} \forall_{b \in B} & a \rightarrow_L b & \wedge \\ \forall_{a_1,a_2 \in A} & a_1 \#_L a_2 & \wedge \\ \forall_{b_1,b_2 \in B} & b_1 \#_L b_2 & \end{array} \right\}$ decision points

$X_{L_1} = \{(\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{e\}), (\{a\},\{b,e\}), (\{a\},\{c,e\}),$
$(\{b\},\{d\}), (\{c\},\{d\}), (\{e\},\{d\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$

5. $Y_L = \left\{ (A,B) \in X_L \ \middle|\ \forall_{(A',B') \in X_L} \begin{array}{c} A \subseteq A' \ \wedge\ B \subseteq B' \\ \Rightarrow \\ (A',B') = (A,B) \end{array} \right\}$ max. dec. points

$Y_{L_1} = \{(\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$

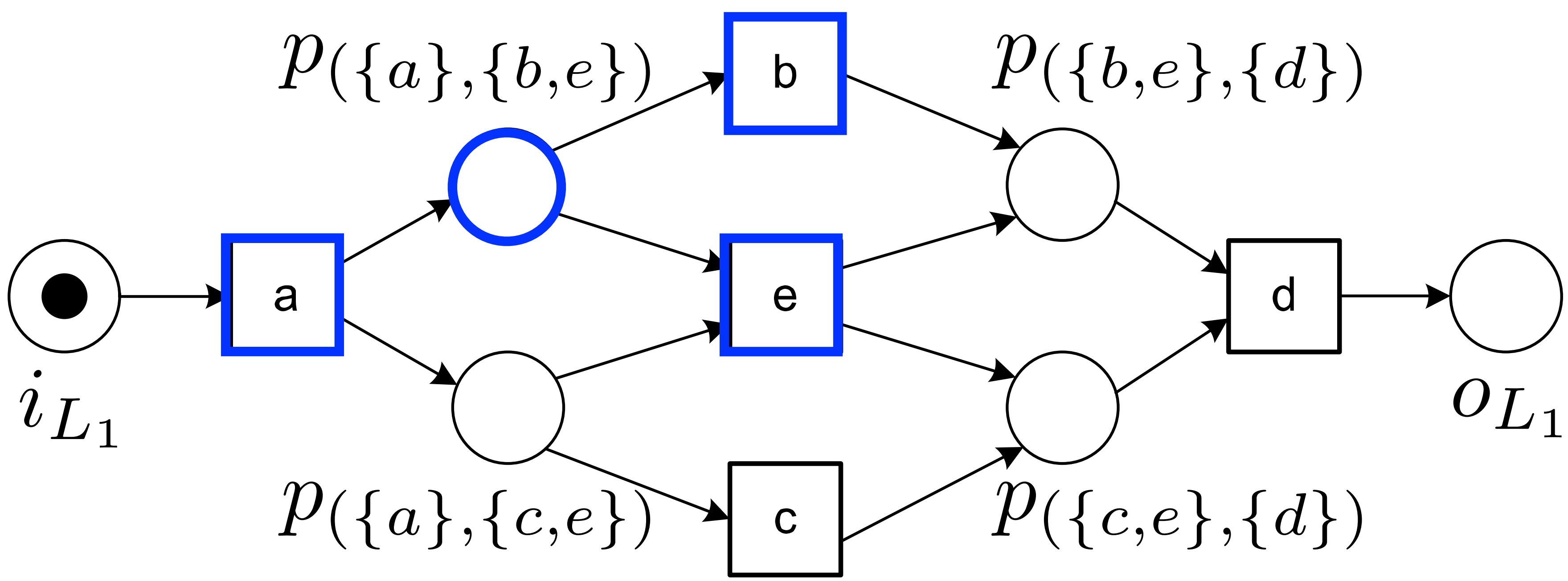
6. $P_L = \{\ p_{(A,B)} \ \mid\ (A,B) \in Y_L\ \} \cup \{\ i_L, o_L\ \}$ places
7. $F_L = \{\ (a, p_{(A,B)}) \ \mid\ (A,B) \in Y_L \ \wedge\ a \in A\ \} \cup$
   $\{\ (p_{(A,B)}, b) \ \mid\ (A,B) \in Y_L \ \wedge\ b \in B\ \} \cup$
   $\{\ (i_L, t) \ \mid\ t \in T_I\ \} \cup$
   $\{\ (t, o_L) \ \mid\ t \in T_O\ \}$ arcs
8. $\alpha(L) = (P_L, T_L, F_L, i_L)$ net



Steps 6-7: Example

$$L_1 = [\langle a,b,c,d\rangle^3, \langle a,c,b,d\rangle^2, \langle a,e,d\rangle]$$

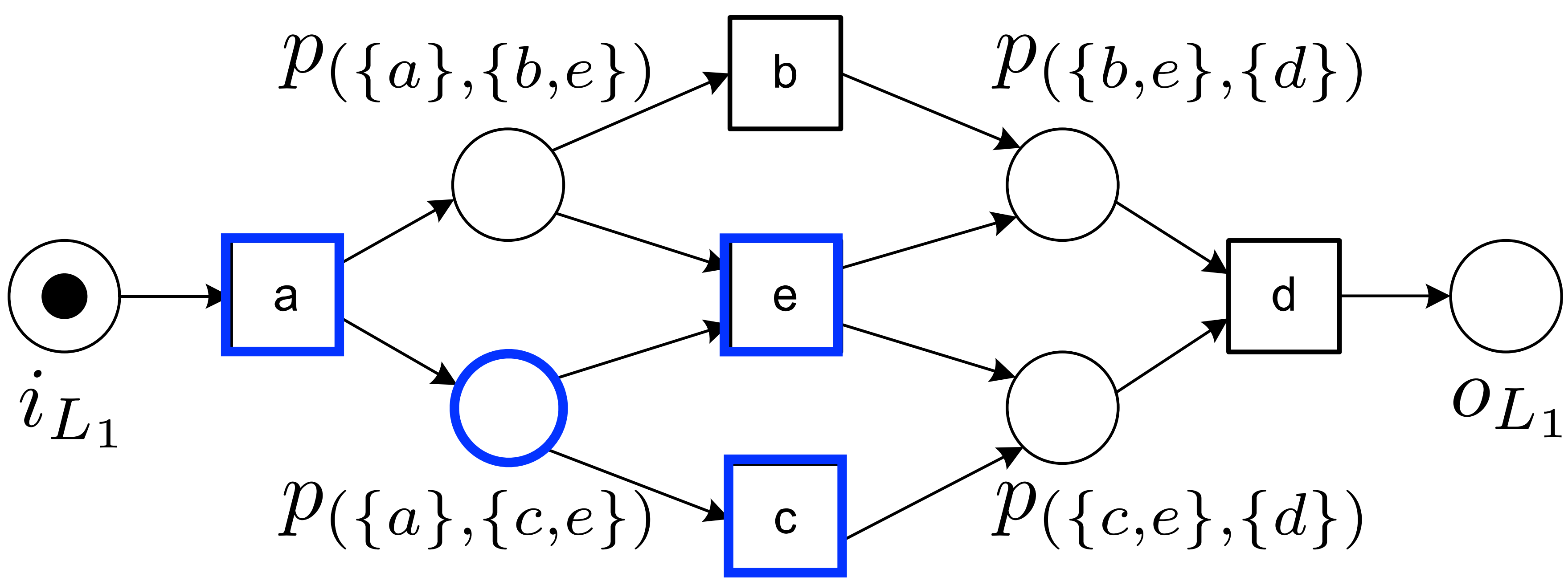
$$Y_{L_1} = \{\boxed{(\{a\},\{b,e\})}, (\{a\},\{c,e\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$$



Steps 6-7: Example

$$L_1 = [\langle a, b, c, d \rangle^3, \langle a, c, b, d \rangle^2, \langle a, e, d \rangle]$$

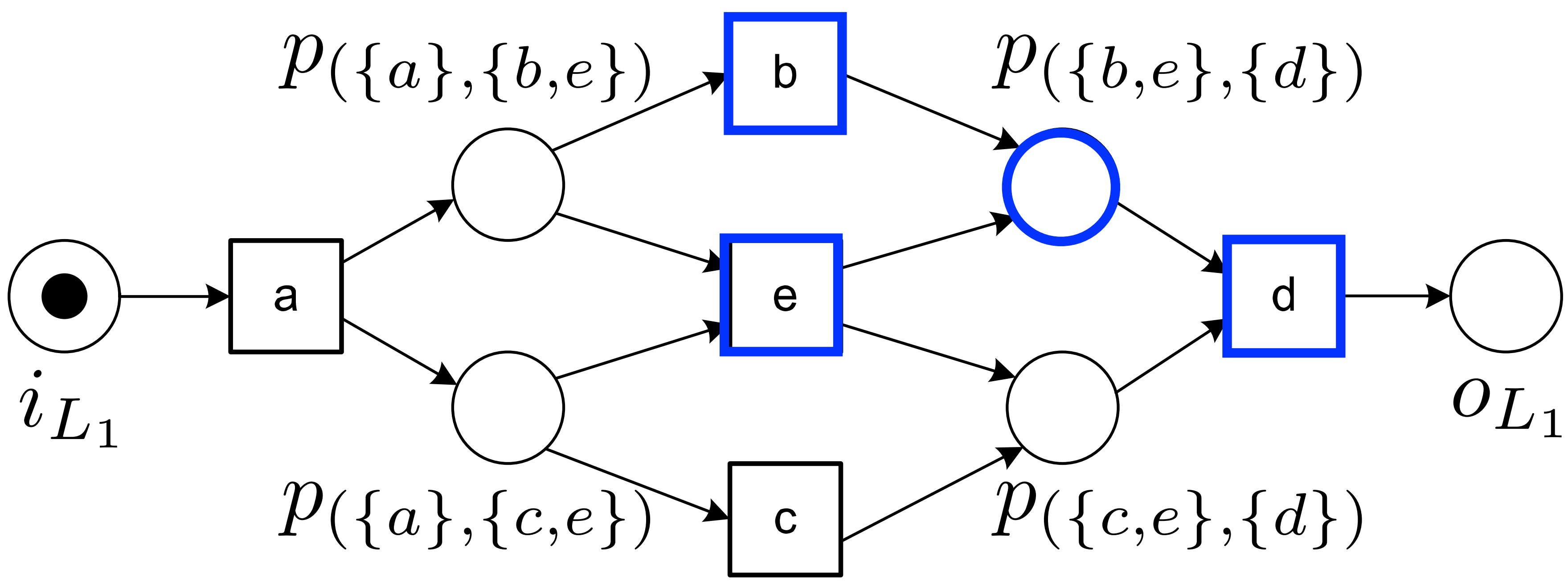
$$Y_{L_1} = \{(\{a\}, \{b, e\}), \boxed{(\{a\}, \{c, e\})}, (\{b, e\}, \{d\}), (\{c, e\}, \{d\})\}$$



Steps 6-7: Example

$$L_1 = [\langle a, b, c, d \rangle^3, \langle a, c, b, d \rangle^2, \langle a, e, d \rangle]$$

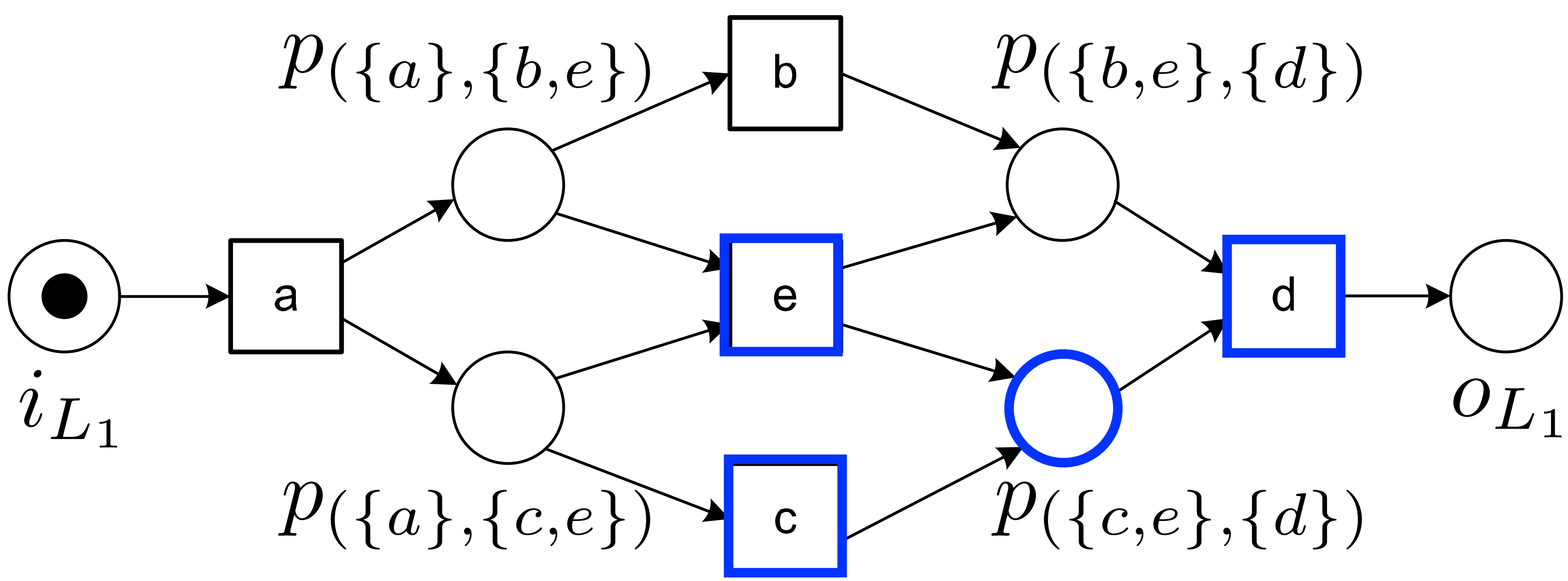
$$Y_{L_1} = \Big\{ \big(\{a\}, \{b, e\}\big), \big(\{a\}, \{c, e\}\big), \boxed{\big(\{b, e\}, \{d\}\big)}, \big(\{c, e\}, \{d\}\big) \Big\}$$



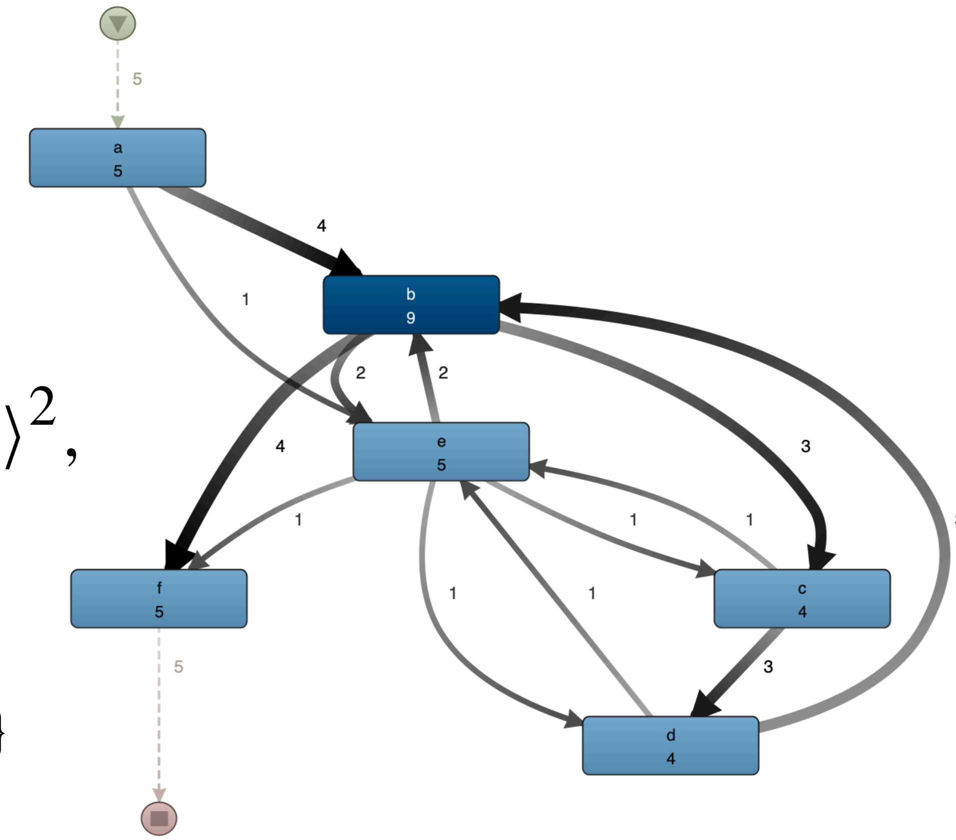
Steps 6-7: Example

$$L_1 = [\langle a, b, c, d\rangle^3, \langle a, c, b, d\rangle^2, \langle a, e, d\rangle]$$

$$Y_{L_1} = \{(\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$$



Another Example



$$L_5 = [\langle a,b,e,f\rangle^2, \langle a,b,e,c,d,b,f\rangle^3, \langle a,b,c,e,d,b,f\rangle^2, \langle a,b,c,d,e,b,f\rangle^4, \langle a,e,b,c,d,b,f\rangle^3]$$

	a	b	c	d	e	f
a	$\#$	$\rightarrow$	$\#$	$\#$	$\rightarrow$	$\#$
b	$\leftarrow$	$\#$	$\rightarrow$	$\leftarrow$	$\parallel$	$\rightarrow$
c	$\#$	$\leftarrow$	$\#$	$\rightarrow$	$\parallel$	$\#$
d	$\#$	$\rightarrow$	$\leftarrow$	$\#$	$\parallel$	$\#$
e	$\leftarrow$	$\parallel$	$\parallel$	$\parallel$	$\#$	$\rightarrow$
f	$\#$	$\leftarrow$	$\#$	$\#$	$\leftarrow$	$\#$

$T_L = \{a,b,c,d,e,f\}$

$T_I = \{a\}$

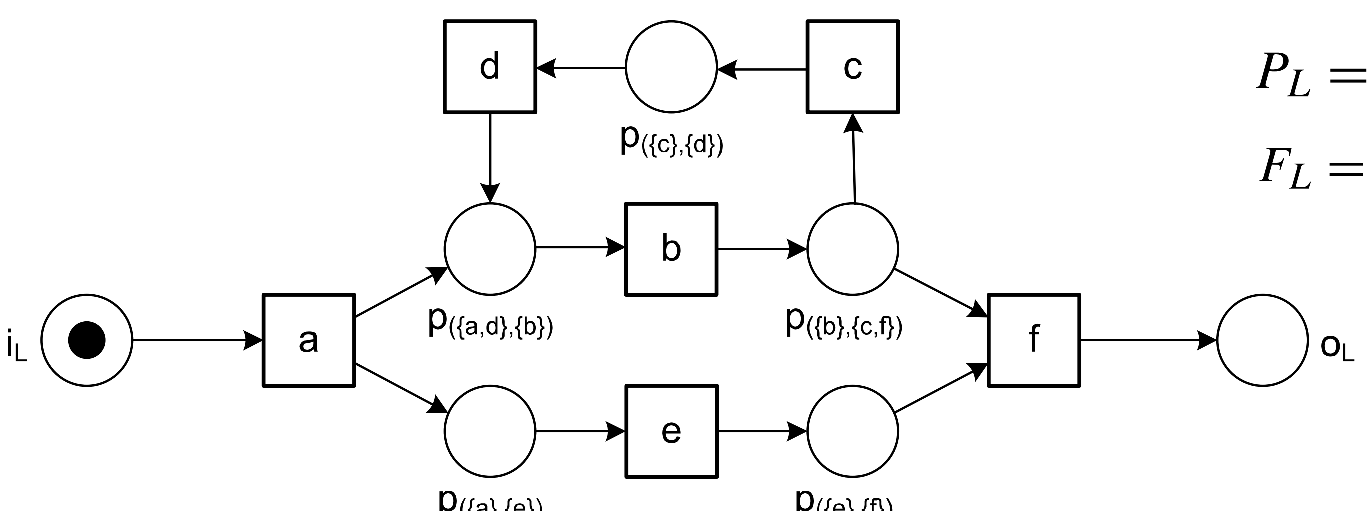
$T_O = \{f\}$

$X_L = \{(\{a\},\{b\}), (\{a\},\{e\}), (\{b\},\{c\}), (\{b\},\{f\}), (\{c\},\{d\}), (\{d\},\{b\}), (\{e\},\{f\}), (\{a,d\},\{b\}), (\{b\},\{c,f\})\}$

$Y_L = \{(\{a\},\{e\}), (\{c\},\{d\}), (\{e\},\{f\}), (\{a,d\},\{b\}), (\{b\},\{c,f\})\}$

$P_L = \{p_{(\{a\},\{e\})}, p_{(\{c\},\{d\})}, p_{(\{e\},\{f\})}, p_{(\{a,d\},\{b\})}, p_{(\{b\},\{c,f\})}, i_L, o_L\}$

$F_L = \{(a, p_{(\{a\},\{e\})}), (p_{(\{a\},\{e\})}, e), (c, p_{(\{c\},\{d\})}), (p_{(\{c\},\{d\})}, d), (e, p_{(\{e\},\{f\})}), (p_{(\{e\},\{f\})}, f), (a, p_{(\{a,d\},\{b\})}), (d, p_{(\{a,d\},\{b\})}), (p_{(\{a,d\},\{b\})}, b), (b, p_{(\{b\},\{c,f\})}), (p_{(\{b\},\{c,f\})}, c), (p_{(\{b\},\{c,f\})}, f), (i_L, a), (f, o_L)\}$



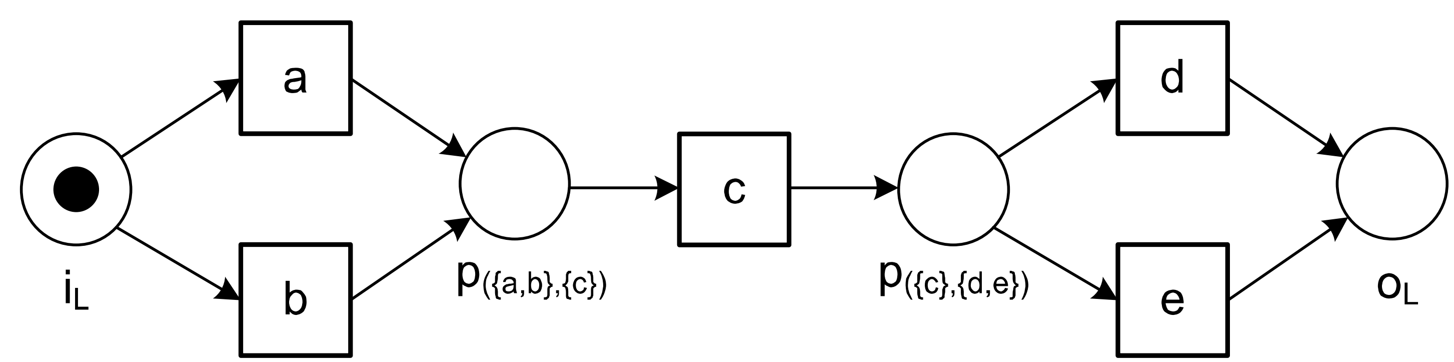
$\alpha(L) = (P_L, T_L, F_L)$

Exercises

$$L_4 = [\langle a,c,d\rangle^{45}, \langle b,c,d\rangle^{42}, \langle a,c,e\rangle^{38}, \langle b,c,e\rangle^{22}]$$

	a	b	c	d	e
a	$\#$	$\#$	$\rightarrow$	$\#$	$\#$
b	$\#$	$\#$	$\rightarrow$	$\#$	$\#$
c	$\leftarrow$	$\leftarrow$	$\#$	$\rightarrow$	$\rightarrow$
d	$\#$	$\#$	$\leftarrow$	$\#$	$\#$
e	$\#$	$\#$	$\leftarrow$	$\#$	$\#$

Check in full autonomy that the footprint matrix corresponds to the log and that the net below is the one discovered by the alpha-algorithm



Exercises

$$L_4 = [\langle a, c, d\rangle^{45}, \langle b, c, d\rangle^{42}, \langle a, c, e\rangle^{38}, \langle b, c, e\rangle^{22}]$$

$a >_{L_4} c$ $\quad$ $b >_{L_4} c$ $\quad$ $a >_{L_4} c$ $\quad$ $b >_{L_4} c$

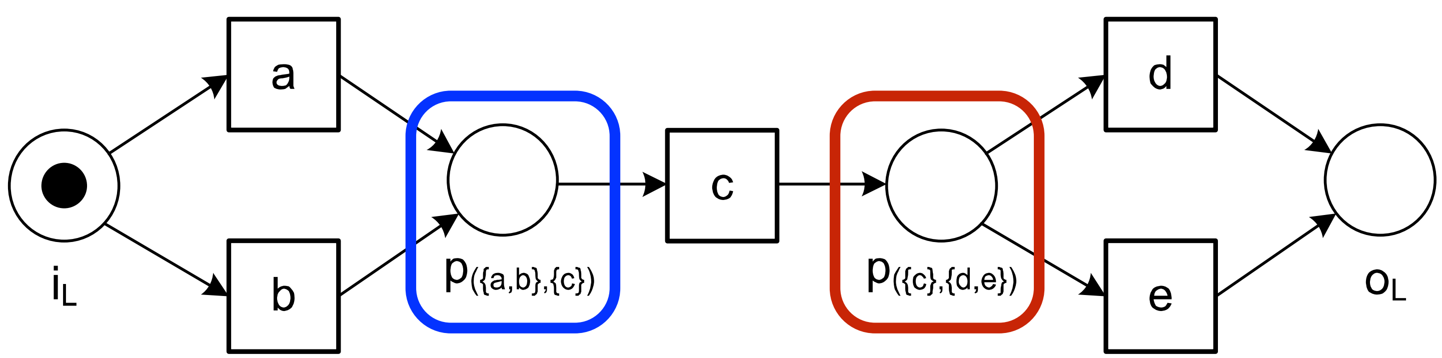
$c >_{L_4} d$ $\quad$ $c >_{L_4} d$ $\quad$ $c >_{L_4} e$ $\quad$ $c >_{L_4} e$

	a	b	c	d	e
a	$\#$	$\#$	$\boxed{\rightarrow}$	$\#$	$\#$
b	$\#$	$\#$	$\boxed{\rightarrow}$	$\#$	$\#$
c	$\boxed{\leftarrow}$	$\boxed{\leftarrow}$	$\#$	$\boxed{\rightarrow}$	$\boxed{\rightarrow}$
d	$\#$	$\#$	$\boxed{\leftarrow}$	$\#$	$\#$
e	$\#$	$\#$	$\boxed{\leftarrow}$	$\#$	$\#$

Exercises

$$L_4 = [\langle a,c,d\rangle^{45}, \langle b,c,d\rangle^{42}, \langle a,c,e\rangle^{38}, \langle b,c,e\rangle^{22}]$$

	a	b	c	d	e
a	$\#$	$\#$	$\rightarrow$	$\#$	$\#$
b	$\#$	$\#$	$\rightarrow$	$\#$	$\#$
c	$\leftarrow$	$\leftarrow$	$\#$	$\rightarrow$	$\rightarrow$
d	$\#$	$\#$	$\leftarrow$	$\#$	$\#$
e	$\#$	$\#$	$\leftarrow$	$\#$	$\#$

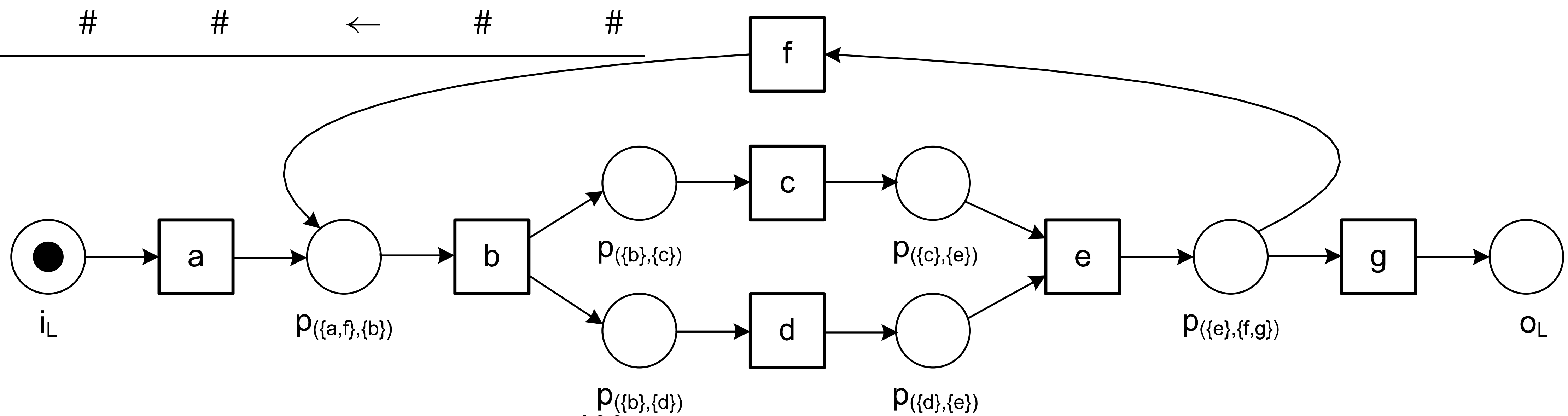


Exercises

$$L_3 = [\langle a,b,c,d,e,f,b,d,c,e,g\rangle, \langle a,b,d,c,e,g\rangle^2, \langle a,b,c,d,e,f,b,c,d,e,f,b,d,c,e,g\rangle]$$

	a	b	c	d	e	f	g
a	$\#$	$\rightarrow$	$\#$	$\#$	$\#$	$\#$	$\#$
b	$\leftarrow$	$\#$	$\rightarrow$	$\rightarrow$	$\#$	$\leftarrow$	$\#$
c	$\#$	$\leftarrow$	$\#$	$\Vert$	$\rightarrow$	$\#$	$\#$
d	$\#$	$\leftarrow$	$\Vert$	$\#$	$\rightarrow$	$\#$	$\#$
e	$\#$	$\#$	$\leftarrow$	$\leftarrow$	$\#$	$\rightarrow$	$\rightarrow$
f	$\#$	$\rightarrow$	$\#$	$\#$	$\leftarrow$	$\#$	$\#$
g	$\#$	$\#$	$\#$	$\#$	$\leftarrow$	$\#$	$\#$

Check in full autonomy that the footprint matrix corresponds to the log and that the net below is the one discovered by the alpha-algorithm



Exercises

$c >_{L_3} d \quad f >_{L_3} b \quad c >_{L_3} e \quad d >_{L_3} c$

$b >_{L_3} c \quad e >_{L_3} f \quad d >_{L_3} c \quad b >_{L_3} d \quad e >_{L_3} g$

$a >_{L_3} b \quad d >_{L_3} e \quad b >_{L_3} d \quad e >_{L_3} g \quad a >_{L_3} b \quad c >_{L_3} e$

$$L_3 = [\langle a,b,c,d,e,f,b,d,c,e,g\rangle, \langle a,b,d,c,e,g\rangle^2, \langle a,b,c,d,e,f,b,c,d,e,f,b,d,c,e,g\rangle]$$

$a >_{L_3} b \quad d >_{L_3} e \quad b >_{L_3} c \quad e >_{L_3} f \quad d >_{L_3} c$

$b >_{L_3} c \quad e >_{L_3} f \quad c >_{L_3} d \quad f >_{L_3} b \quad c >_{L_3} e$

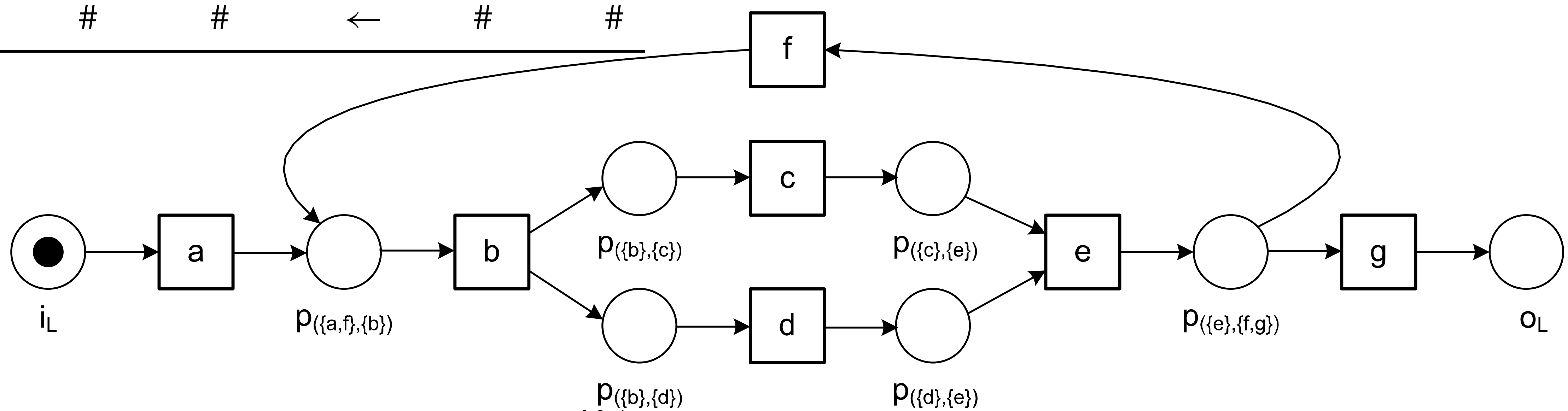
$c >_{L_3} d \quad f >_{L_3} b \quad d >_{L_3} e \quad b >_{L_3} d \quad e >_{L_3} g$

	a	b	c	d	e	f	g
a	#	$\rightarrow$	#	#	#	#	#
b	$\leftarrow$	#	$\rightarrow$	$\rightarrow$	#	$\leftarrow$	#
c	#	$\leftarrow$	#	$\parallel$	$\rightarrow$	#	#
d	#	$\leftarrow$	$\parallel$	#	$\rightarrow$	#	#
e	#	#	$\leftarrow$	$\leftarrow$	#	$\rightarrow$	$\rightarrow$
f	#	$\rightarrow$	#	#	$\leftarrow$	#	#
g	#	#	#	#	$\leftarrow$	#	#

Exercises

$$L_3 = [\langle a,b,c,d,e,f,b,d,c,e,g\rangle, \langle a,b,d,c,e,g\rangle^2,$$
$$\langle a,b,c,d,e,f,b,c,d,e,f,b,d,c,e,g\rangle]$$

	a	b	c	d	e	f	g
a	$\#$	$\rightarrow$	$\#$	$\#$	$\#$	$\#$	$\#$
b	$\leftarrow$	$\#$	$\rightarrow$	$\rightarrow$	$\#$	$\leftarrow$	$\#$
c	$\#$	$\leftarrow$	$\#$	$\parallel$	$\rightarrow$	$\#$	$\#$
d	$\#$	$\leftarrow$	$\parallel$	$\#$	$\rightarrow$	$\#$	$\#$
e	$\#$	$\#$	$\leftarrow$	$\leftarrow$	$\#$	$\rightarrow$	$\rightarrow$
f	$\#$	$\rightarrow$	$\#$	$\#$	$\leftarrow$	$\#$	$\#$
g	$\#$	$\#$	$\#$	$\#$	$\leftarrow$	$\#$	$\#$

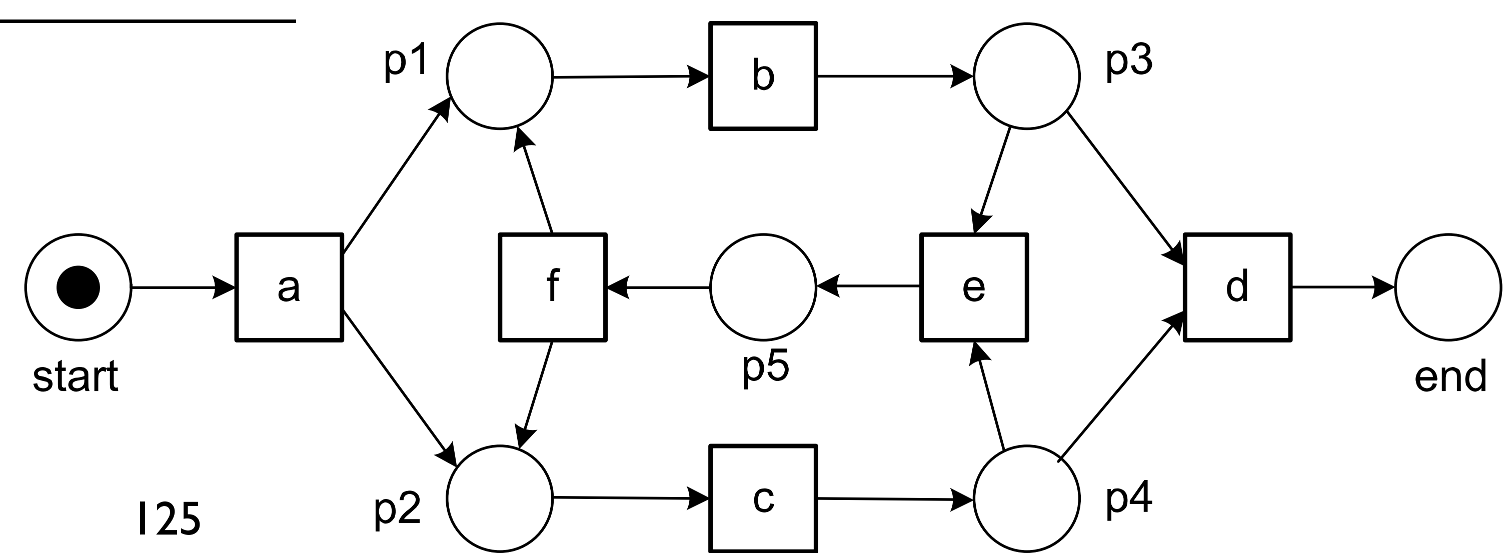


Exercises

$$L_2 = [\langle a,b,c,d\rangle^3, \langle a,c,b,d\rangle^4, \langle a,b,c,e,f,b,c,d\rangle^2, \langle a,b,c,e,f,c,b,d\rangle, \langle a,c,b,e,f,b,c,d\rangle^2, \langle a,c,b,e,f,b,c,e,f,c,b,d\rangle]$$

	a	b	c	d	e	f
a	#	$\rightarrow$	$\rightarrow$	#	#	#
b	$\leftarrow$	#	$\parallel$	$\rightarrow$	$\rightarrow$	$\leftarrow$
c	$\leftarrow$	$\parallel$	#	$\rightarrow$	$\rightarrow$	$\leftarrow$
d	#	$\leftarrow$	$\leftarrow$	#	#	#
e	#	$\leftarrow$	$\leftarrow$	#	#	$\rightarrow$
f	#	$\rightarrow$	$\rightarrow$	#	$\leftarrow$	#

Check in full autonomy that the footprint matrix corresponds to the log and that the net below is the one discovered by the alpha-algorithm



Exercises

$c >_{L_2} d$ $\quad b >_{L_2} d$ $\quad c >_{L_2} e$ $\quad b >_{L_2} c$ $\quad c >_{L_2} e$ $\quad c >_{L_2} b$

$b >_{L_2} c$ $\quad c >_{L_2} b$ $\quad b >_{L_2} c$ $\quad f >_{L_2} b$ $\quad b >_{L_2} c$ $\quad f >_{L_2} c$

$a >_{L_2} b$ $\quad a >_{L_2} c$ $\quad a >_{L_2} b$ $\quad e >_{L_2} f$ $\quad c >_{L_2} d$ $\quad a >_{L_2} b$ $\quad e >_{L_2} f$ $\quad b >_{L_2} d$

$$L_2 = [\langle a,b,c,d\rangle^3, \langle a,c,b,d\rangle^4, \langle a,b,c,e,f,b,c,d\rangle^2, \langle a,b,c,e,f,c,b,d\rangle,$$

$$\langle a,c,b,e,f,b,c,d\rangle^2, \langle a,c,b,e,f,b,c,e,f,c,b,d\rangle]$$

$a >_{L_2} c$ $\quad e >_{L_2} f$ $\quad c >_{L_2} d$ $\quad a >_{L_2} c$ $\quad e >_{L_2} f$ $\quad c >_{L_2} e$ $\quad c >_{L_2} b$

$c >_{L_2} b$ $\quad f >_{L_2} b$ $\quad c >_{L_2} b$ $\quad f >_{L_2} b$ $\quad e >_{L_2} f$ $\quad b >_{L_2} d$

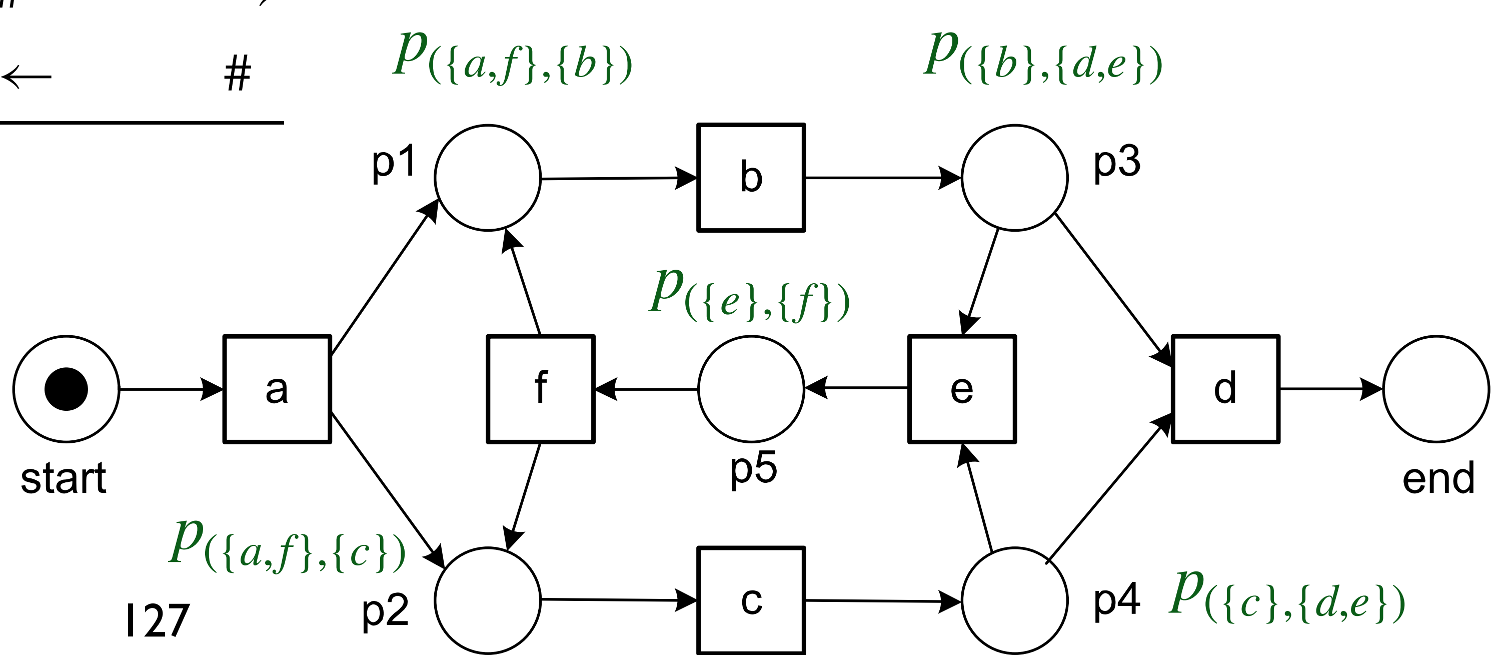
$b >_{L_2} e$ $\quad b >_{L_2} c$ $\quad b >_{L_2} e$ $\quad b >_{L_2} c$ $\quad f >_{L_2} c$

	a	b	c	d	e	f
a	#	$\rightarrow$	$\rightarrow$	#	#	#
b	$\leftarrow$	#	$\parallel$	$\rightarrow$	$\rightarrow$	$\leftarrow$
c	$\leftarrow$	$\parallel$	#	$\rightarrow$	$\rightarrow$	$\leftarrow$
d	#	$\leftarrow$	$\leftarrow$	#	#	#
e	#	$\leftarrow$	$\leftarrow$	#	#	$\rightarrow$
f	#	$\rightarrow$	$\rightarrow$	#	$\leftarrow$	#

Exercises

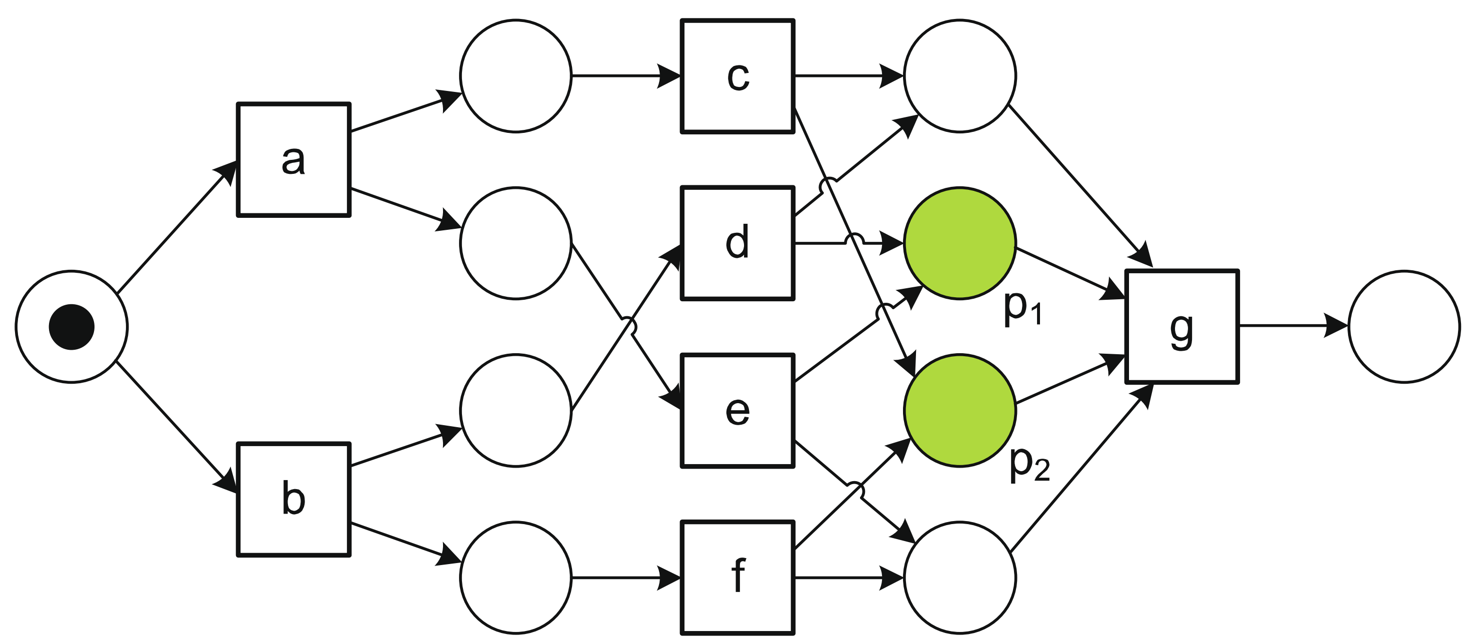
$$L_2 = [\langle a,b,c,d\rangle^3, \langle a,c,b,d\rangle^4, \langle a,b,c,e,f,b,c,d\rangle^2, \langle a,b,c,e,f,c,b,d\rangle, \langle a,c,b,e,f,b,c,d\rangle^2, \langle a,c,b,e,f,b,c,e,f,c,b,d\rangle]$$

	a	b	c	d	e	f
a	#	$\rightarrow$	$\rightarrow$	#	#	#
b	$\leftarrow$	#	$\|$	$\rightarrow$	$\rightarrow$	$\leftarrow$
c	$\leftarrow$	$\|$	#	$\rightarrow$	$\rightarrow$	$\leftarrow$
d	#	$\leftarrow$	$\leftarrow$	#	#	#
e	#	$\leftarrow$	$\leftarrow$	#	#	$\rightarrow$
f	#	$\rightarrow$	$\rightarrow$	#	$\leftarrow$	#



Limitation: Implicit Dependencies

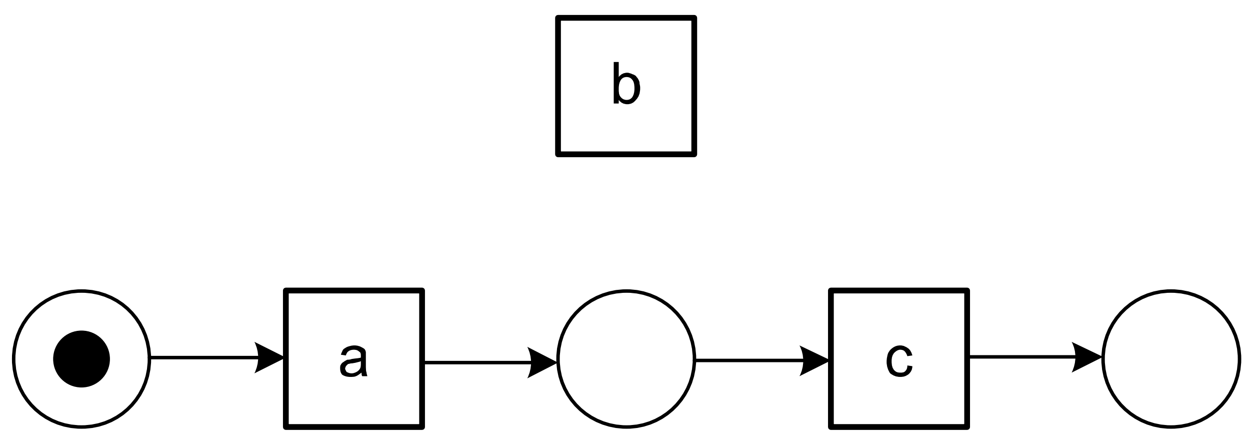
$$L_6 = [\langle a, c, e, g\rangle^2, \langle a, e, c, g\rangle^3, \langle b, d, f, g\rangle^2, \langle b, f, d, g\rangle^4]$$



p_1 and p_2 are redundant

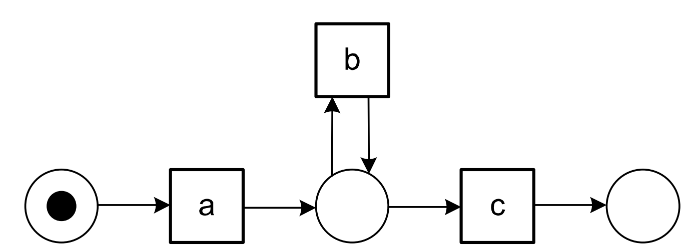
Limitation: Short Loop

$$L_7 = \left[\langle a, c\rangle^2, \langle a, b, c\rangle^3, \langle a, b, b, c\rangle^2, \langle a, b, b, b, b, c\rangle^1\right]$$



b is disconnected from the model

Expected net:



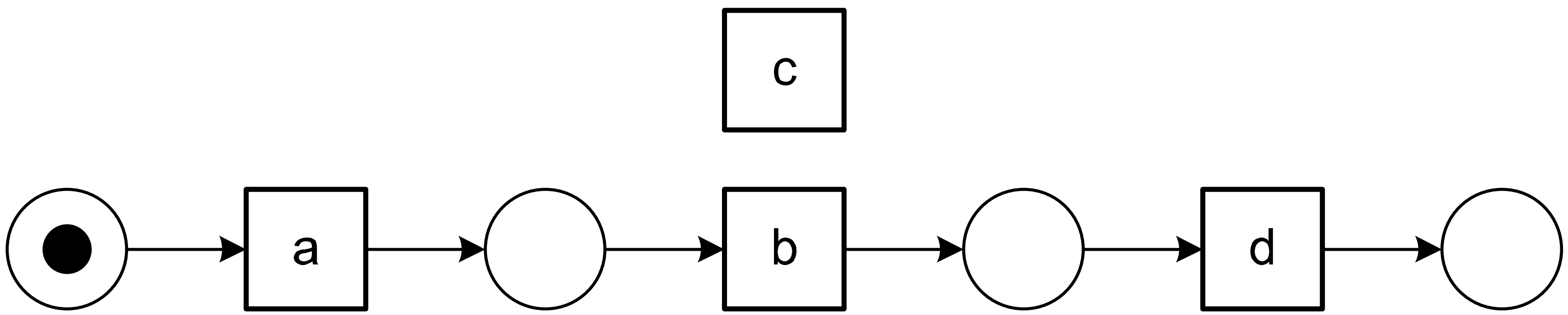
Limitation: Short Loop

$$L_8 = [\langle a,b,d\rangle^3, \langle a,b,c,b,d\rangle^2, \langle a,b,c,b,c,b,d\rangle]$$

$a \rightarrow_{L_8} b,$

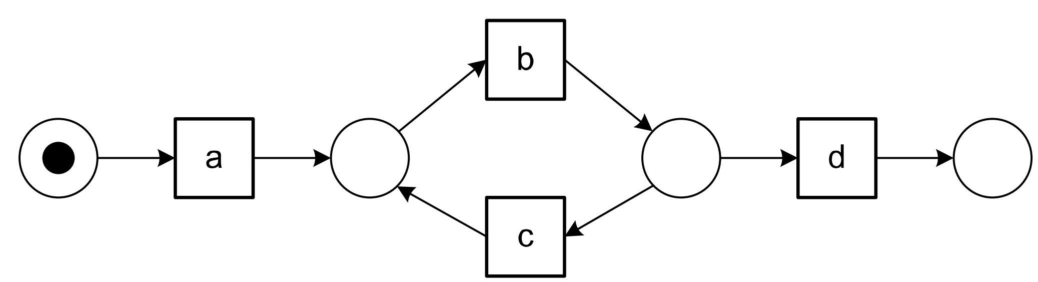
$b \rightarrow_{L_8} d,$

$b \parallel_{L_8} c.$



c is disconnected from the model

Expected net:

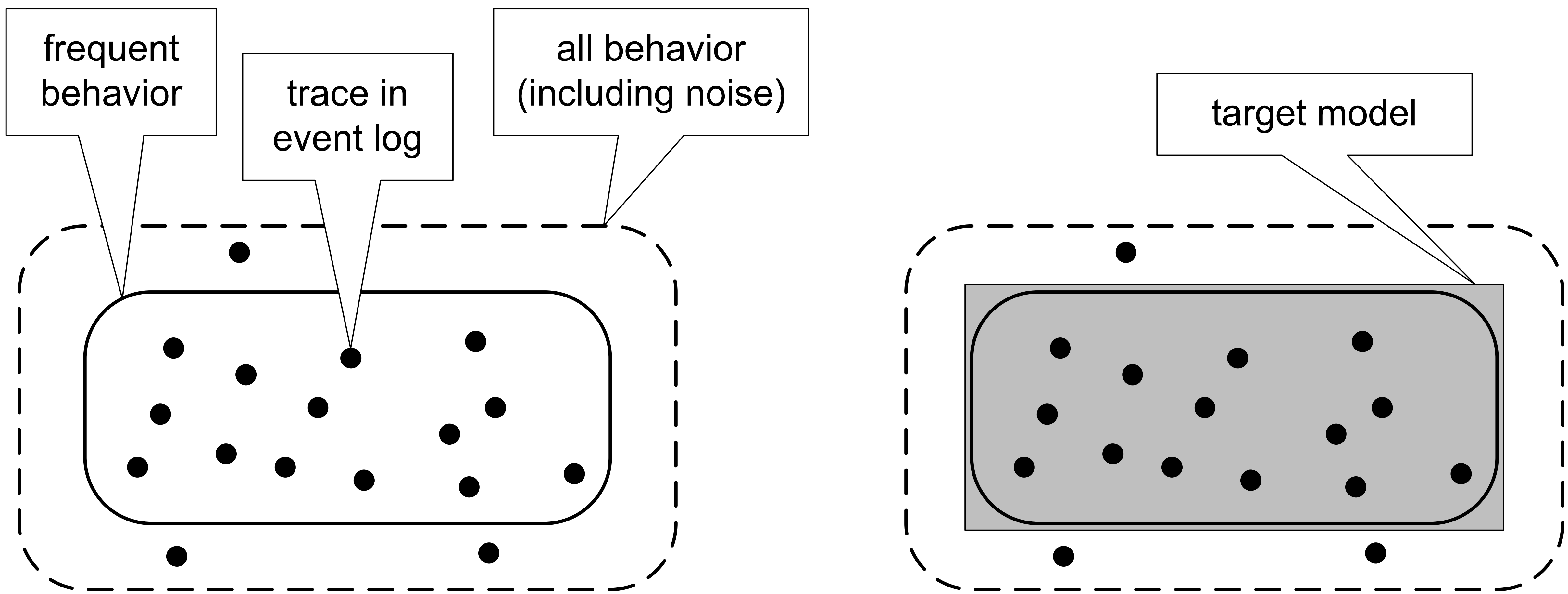


Limitation: Noise

We use the term “noise” to refer to rare and infrequent behaviour rather than errors related to event logging.

For example, frequencies are not taken into account by the α-algorithm (should we disregard less frequent traces?).

Limitation: Noise



Limitation: Noise