



**PSC 2024/25 (375AA, 9CFU)**

Principles for Software Composition

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## 14 - HOFL Denotational Semantics

# Interpretation Domains

# Interpretation Domains

$$D_{int} \triangleq \mathbb{Z}_\perp$$

$$D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_\perp$$

to distinguish:  
pair of divergent terms  
from divergent pair

$$D_{\tau_1 \rightarrow \tau_2} \triangleq [D_{\tau_1} \rightarrow D_{\tau_2}]_\perp$$

to distinguish:  
takes arg and diverge  
from divergence without taking arg

## Example

$$D_{int * int} \triangleq (\mathbb{Z}_\perp \times \mathbb{Z}_\perp)_\perp$$

$$\mathbf{rec} \ p. \ p \quad (\mathbf{rec} \ x. \ x, \mathbf{rec} \ y. \ y)$$

$$\perp_{D_{int * int}} \quad (\perp_{D_{int}}, \perp_{D_{int}})$$

## Example

$$D_{int \rightarrow int} \triangleq [\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp$$

$$\mathbf{rec} \ f. \ f \quad \lambda x. \mathbf{rec} \ y. \ y$$

$$\perp_{D_{int \rightarrow int}} \quad \lambda d. \perp_{D_{int}}$$

# Interpretation Domains

$$D_{int} \triangleq \mathbb{Z}_\perp \quad D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_\perp \quad D_{\tau_1 \rightarrow \tau_2} \triangleq [D_{\tau_1} \rightarrow D_{\tau_2}]_\perp$$

Equivalently:  $D_\tau \triangleq (V_\tau)_\perp$

$$V_{int} \triangleq \mathbb{Z}$$

$$V_{\tau_1 * \tau_2} \triangleq D_{\tau_1} \times D_{\tau_2} = (V_{\tau_1})_\perp \times (V_{\tau_2})_\perp$$

$$V_{\tau_1 \rightarrow \tau_2} \triangleq [D_{\tau_1} \rightarrow D_{\tau_2}] = [(V_{\tau_1})_\perp \rightarrow (V_{\tau_2})_\perp]$$



# Interpretation Function

$$t : \tau \quad \begin{array}{c} \llbracket t \rrbracket \rho \in D_\tau \\ / \\ \text{environment} \end{array} \quad \rho : \text{Var} \rightarrow \bigcup_{\tau \in \mathcal{T}} D_\tau$$

type consistent  
assignment of  
values to variables

$$x : \tau \Rightarrow \rho(x) \in D_\tau$$

we define the interpretation function by structural recursion

# Denotational Semantics

# Constants

$$\underbrace{\underbrace{[n]}_{int}}_{D_{int} = \mathbb{Z}_\perp} \rho \triangleq \underbrace{\underbrace{[n]}_{\mathbb{Z}}}_{\mathbb{Z}_\perp}$$

# Variables

$$\underbrace{\underbrace{[[x]]}_{\tau}}_{D_\tau} \rho \triangleq \underbrace{\rho(x)}_{D_\tau}$$

$$x : \tau \Rightarrow \rho(x) \in D_\tau$$

# Arithmetic ops

to prove:  $\underline{\text{op}}_{\perp}$  is monotone and continuous

$$\text{op} \in \{+, -, \times\}$$

$$\underbrace{\underbrace{\underbrace{[t_1]_{\text{int}}}_{\text{int}} \text{ op } \underbrace{[t_2]_{\text{int}}}_{\text{int}}}_{\text{int}}}_{D_{\text{int}} = \mathbb{Z}_{\perp}} \rho \triangleq \underbrace{\underbrace{[t_1]_{\text{int}}}_{D_{\text{int}} = \mathbb{Z}_{\perp}} \text{ op }_{\perp} \underbrace{[t_2]_{\text{int}}}_{D_{\text{int}} = \mathbb{Z}_{\perp}}}_{D_{\text{int}} = \mathbb{Z}_{\perp}} \rho$$

$$\underline{\text{op}}_{\perp} : \mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}$$

$$v_1 \underline{\text{op}}_{\perp} v_2 \triangleq \begin{cases} [n_1 \text{ op } n_2] & \text{if } v_1 = [n_1] \text{ and } v_2 = [n_2] \\ \perp_{\mathbb{Z}_{\perp}} & \text{otherwise (} v_1 = \perp_{\mathbb{Z}_{\perp}} \text{ or } v_2 = \perp_{\mathbb{Z}_{\perp}} \text{)} \end{cases}$$

9, called *strict extension*

# Conditionals

to prove:  $\text{Cond}_\tau$  is monotone and continuous

$$\underbrace{\underbrace{\underbrace{[\![t]\!]_\rho}_{\text{int}}}_{\tau} \text{ then } \underbrace{[\![t_1]\!]_\rho}_{\tau} \text{ else } \underbrace{[\![t_2]\!]_\rho}_{\tau}}_{D_\tau} \triangleq \text{Cond}_\tau \left( \underbrace{\underbrace{[\![t]\!]_\rho}_{\text{int}}}_{D_{\text{int}} = \mathbb{Z}_\perp}, \underbrace{\underbrace{[\![t_1]\!]_\rho}_{\tau}}_{D_\tau}, \underbrace{\underbrace{[\![t_2]\!]_\rho}_{\tau}}_{D_\tau} \right)$$

$$\text{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \rightarrow D_\tau$$

$$\text{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = [0] \\ d_2 & \text{otherwise (} v = [n] \text{ with } n \neq 0) \end{cases}$$

# Pairing

$$D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_{\perp}$$

$$\begin{array}{c} \llbracket (t_1, t_2) \rrbracket \rho \triangleq \llbracket ( \llbracket t_1 \rrbracket \rho, \llbracket t_2 \rrbracket \rho ) \rrbracket \\ \underbrace{\underbrace{\underbrace{\tau_1} \quad \underbrace{\tau_2}}_{\tau_1 * \tau_2}}_{D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_{\perp}} \qquad \underbrace{\underbrace{\underbrace{\tau_1} \quad \underbrace{\tau_2}}_{D_{\tau_1} \times D_{\tau_2}}}_{D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_{\perp}} \end{array}$$

# Projections

Equivalently:  $\llbracket \mathbf{fst}( t ) \rrbracket \rho \triangleq \mathbf{let} \ d \leftarrow \llbracket t \rrbracket \rho. \ \pi_1(d)$

$$\llbracket \mathbf{fst}( t ) \rrbracket \rho \triangleq \pi_1^* \left( \llbracket t \rrbracket \rho \right)$$

$\underbrace{\underbrace{\tau_1 * \tau_2}_{\tau_1}}_{D_{\tau_1}} \quad \underbrace{D_{\tau_1} \times D_{\tau_2} \rightarrow D_{\tau_1} \quad \tau_1 * \tau_2}_{D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_{\perp}}}_{(D_{\tau_1} \times D_{\tau_2})_{\perp} \rightarrow D_{\tau_1}} \quad \underbrace{\quad}_{D_{\tau_1}}$

$$\llbracket \mathbf{snd}( t ) \rrbracket \rho \triangleq \pi_2^* \left( \llbracket t \rrbracket \rho \right)$$



# Abstraction

$$D_{\tau_1 \rightarrow \tau_2} \triangleq [D_{\tau_1} \rightarrow D_{\tau_2}]_{\perp}$$

$$\begin{array}{c}
 \underbrace{\underbrace{[\lambda x. t]}_{\tau_1} \rho}_{\tau_2} \triangleq \left[ \underbrace{\lambda d.}_{D_{\tau_1}} \underbrace{[[t]] \rho [d/x]}_{\tau_2} \right] \\
 \underbrace{\tau_1 \rightarrow \tau_2}_{D_{\tau_1 \rightarrow \tau_2} = [D_{\tau_1} \rightarrow D_{\tau_2}]_{\perp}} \qquad \underbrace{\qquad \qquad \qquad}_{D_{\tau_2}} \\
 \underbrace{\qquad \qquad \qquad}_{[D_{\tau_1} \rightarrow D_{\tau_2}]} \\
 \underbrace{\qquad \qquad \qquad}_{[D_{\tau_1} \rightarrow D_{\tau_2}]_{\perp}}
 \end{array}$$

# Application (lazy)

Equivalently:

$$\llbracket t \ t_0 \rrbracket \rho \triangleq (\lambda \varphi. \varphi(\llbracket t_0 \rrbracket \rho))^* (\llbracket t \rrbracket \rho)$$

$$\llbracket t \ t_0 \rrbracket \rho \triangleq \mathbf{let} \ \varphi \leftarrow \llbracket t \rrbracket \rho. \ \varphi(\llbracket t_0 \rrbracket \rho)$$

$\underbrace{\underbrace{\tau_0 \rightarrow \tau \ \tau_0}_{\tau}}_{D_\tau}$ 
 $\underbrace{V_{\tau_0 \rightarrow \tau} = [D_{\tau_0} \rightarrow D_\tau] \ \tau_0 \rightarrow \tau \ [D_{\tau_0} \rightarrow D_\tau] \ \tau_0}_{D_{\tau_0 \rightarrow \tau} = (V_{\tau_0 \rightarrow \tau})_\perp} \underbrace{\quad}_{D_\tau}$ 
 $\underbrace{\quad}_{D_\tau}$

# Recursion

$$\underbrace{\underbrace{\underbrace{\mathbf{rec} \ x. \ t}_{\tau} \ \rho}_{\tau}}_{D_\tau} \triangleq \underbrace{\underbrace{\underbrace{[t]}_{\tau} \ \rho}_{D_\tau} \left[ \underbrace{\underbrace{\mathbf{rec} \ x. \ t}_{D_\tau} \ \rho}_{D_\tau} / x \right]}_{D_\tau}$$

# Recursion

$$\begin{array}{c}
 \underbrace{\underbrace{\underbrace{\mathbf{rec} \ x. \ t}_{\tau} \ \rho}_{\tau}}_{D_{\tau}} \triangleq \underbrace{\underbrace{\mathit{fix} \ \lambda d.}_{[[D_{\tau} \rightarrow D_{\tau}] \rightarrow D_{\tau}]} \ \underbrace{\underbrace{\underbrace{[t] \ \rho}_{\tau} \ \underbrace{[d/x]}_{D_{\tau}}}_{\tau}}_{D_{\tau}}}_{[D_{\tau} \rightarrow D_{\tau}]}_{D_{\tau}}
 \end{array}$$

# Recap

$$\llbracket n \rrbracket \rho \triangleq \lfloor n \rfloor$$

$$\llbracket x \rrbracket \rho \triangleq \rho(x)$$

$$\llbracket t_1 \text{ op } t_2 \rrbracket \rho \triangleq \llbracket t_1 \rrbracket \rho \text{ op}_{\perp} \llbracket t_2 \rrbracket \rho$$

$$\llbracket \mathbf{if } t \text{ then } t_1 \text{ else } t_2 \rrbracket \rho \triangleq \text{Cond}_{\tau}(\llbracket t \rrbracket \rho, \llbracket t_1 \rrbracket \rho, \llbracket t_2 \rrbracket \rho)$$

$$\llbracket (t_1, t_2) \rrbracket \rho \triangleq \lfloor (\llbracket t_1 \rrbracket \rho, \llbracket t_2 \rrbracket \rho) \rfloor$$

$$\llbracket \mathbf{fst}(t) \rrbracket \rho \triangleq \pi_1^*(\llbracket t \rrbracket \rho)$$

$$\llbracket \mathbf{snd}(t) \rrbracket \rho \triangleq \pi_2^*(\llbracket t \rrbracket \rho)$$

$$\llbracket \lambda x. t \rrbracket \rho \triangleq \lfloor \lambda d. \llbracket t \rrbracket \rho^{[d/x]} \rfloor$$

$$\llbracket t t_0 \rrbracket \rho \triangleq \mathbf{let } \varphi \leftarrow \llbracket t \rrbracket \rho. \varphi(\llbracket t_0 \rrbracket \rho)$$

$$\llbracket \mathbf{rec } x. t \rrbracket \rho \triangleq \mathit{fix } \lambda d. \llbracket t \rrbracket \rho^{[d/x]}$$

# Example

$$f \stackrel{\text{def}}{=} \lambda x : \text{int}. 3$$

$$[[\lambda x. t]]\rho \triangleq [\lambda d. [[t]]\rho^{[d/x]}] \qquad [[n]]\rho \triangleq [n]$$

$$[[f]]\rho = [[\lambda x. 3]]\rho = [\lambda d. [[3]]\rho^{[d/x]}] = [\lambda d. [3]]$$

# Example

$g \stackrel{\text{def}}{=} \lambda x : \text{int}. \text{if } x \text{ then } 3 \text{ else } 3$

$$\llbracket \lambda x. t \rrbracket \rho \triangleq [ \lambda d. \llbracket t \rrbracket \rho [d/x] ]$$

$$\begin{aligned} \llbracket g \rrbracket \rho &= \llbracket \lambda x. \text{if } x \text{ then } 3 \text{ else } 3 \rrbracket \rho \\ &= [ \lambda d. \llbracket \text{if } x \text{ then } 3 \text{ else } 3 \rrbracket \rho [d/x] ] \\ &= [ \lambda d. \text{Cond}(d, [3], [3]) ] \\ &= [ \lambda d. \text{let } x \Leftarrow d. [3] ] \end{aligned}$$

$$\llbracket f \rrbracket \rho \neq \llbracket g \rrbracket \rho$$

$$[ \lambda d. [3] ]$$

# Example

$$h \stackrel{\text{def}}{=} \mathbf{rec} \ y : \mathit{int} \rightarrow \mathit{int}. \ \lambda x : \mathit{int}. \ 3$$

$$\begin{aligned} \llbracket h \rrbracket \rho &= \llbracket \mathbf{rec} \ y. \ \lambda x. \ 3 \rrbracket \rho && \llbracket \mathbf{rec} \ x. \ t \rrbracket \rho \triangleq \mathit{fix} \ \lambda d. \ \llbracket t \rrbracket \rho [d/x] \\ &= \mathit{fix} \ \lambda d_y. \ \llbracket \lambda x. \ 3 \rrbracket \rho [d_y/y] && \llbracket \lambda x. \ t \rrbracket \rho \triangleq [ \ \lambda d. \ \llbracket t \rrbracket \rho [d/x] \ ] \\ &= \mathit{fix} \ \lambda d_y. \ [ \ \lambda d_x. \ \llbracket 3 \rrbracket \rho [d_y/y, d_x/x] \ ] \\ &= \mathit{fix} \ \lambda d_y. \ [ \ \lambda d_x. \ [3] \ ] && \Gamma_h = \lambda d_y. \ [ \ \lambda d_x. \ [3] \ ] \end{aligned}$$

$$d_0 = \Gamma_h^0(\perp_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp}) = \perp_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp}$$

$$d_1 = \Gamma_h(d_0) = (\lambda d_y. [ \ \lambda d_x. \ [3] \ ]) \perp = [ \ \lambda d_x. \ [3] \ ]$$

$$d_2 = \Gamma_h(d_1) = (\lambda d_y. [ \ \lambda d_x. \ [3] \ ]) [ \ \lambda d_x. \ [3] \ ] = [ \ \lambda d_x. \ [3] \ ] = d_1$$



# Example

$$h \stackrel{\text{def}}{=} \mathbf{rec} \ y : \mathit{int} \rightarrow \mathit{int}. \ \lambda x : \mathit{int}. \ 3$$

$$\llbracket h \rrbracket \rho = \llbracket \mathbf{rec} \ y. \ \lambda x. \ 3 \rrbracket \rho$$

$$= \mathbf{fix} \ \lambda d_y. \ \llbracket \lambda x. \ 3 \rrbracket \rho [d_y / y]$$

$$= \mathbf{fix} \ \lambda d_y. \ \llbracket \lambda d_x. \ \llbracket 3 \rrbracket \rho [d_y / y, d_x / x] \rrbracket$$

$$= \mathbf{fix} \ \lambda d_y. \ \llbracket \lambda d_x. \ \llbracket 3 \rrbracket \rrbracket$$

$$\Gamma_h = \lambda d_y. \ \llbracket \lambda d_x. \ \llbracket 3 \rrbracket \rrbracket$$

$$d_0 = \Gamma_h^0(\perp_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp}) = \perp_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp}$$

$$d_1 = \Gamma_h(d_0) = (\lambda d_y. \ \llbracket \lambda d_x. \ \llbracket 3 \rrbracket \rrbracket)_\perp = \llbracket \lambda d_x. \ \llbracket 3 \rrbracket \rrbracket$$

Maximal element in  $[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp$   
we could already stop here

# Example

$$h \stackrel{\text{def}}{=} \mathbf{rec} \ y : \mathit{int} \rightarrow \mathit{int}. \ \lambda x : \mathit{int}. \ 3$$

$$\begin{aligned} \llbracket h \rrbracket \rho &= \llbracket \mathbf{rec} \ y. \ \lambda x. \ 3 \rrbracket \rho \\ &= \mathbf{fix} \ \lambda d_y. \ \llbracket \lambda x. \ 3 \rrbracket \rho [d_y / y] \\ &= \mathbf{fix} \ \lambda d_y. \ \llbracket \lambda d_x. \ \llbracket 3 \rrbracket \rho [d_y / y, d_x / x] \rrbracket \\ &= \mathbf{fix} \ \lambda d_y. \ \llbracket \lambda d_x. \ \llbracket 3 \rrbracket \rrbracket \end{aligned}$$

$$\llbracket h \rrbracket \rho = \llbracket \lambda d_x. \ \llbracket 3 \rrbracket \rrbracket = \llbracket f \rrbracket \rho$$

# Example

$x : \tau$

$$\begin{aligned} \llbracket \mathbf{rec} \ x. \ x \rrbracket \rho &= \mathit{fix} \ \lambda d_x. \ \llbracket x \rrbracket \rho [d_x / x] \\ &= \mathit{fix} \ \lambda d_x. \ d_x \end{aligned}$$

$$d_0 = \perp_{D_\tau}$$

$$d_1 = (\lambda d_x. d_x) d_0 = d_0 = \perp_{D_\tau}$$

$$\llbracket \mathbf{rec} \ x. \ x \rrbracket \rho = \perp_{D_\tau}$$

$x : \mathit{int} \rightarrow \mathit{int}$

$$\llbracket \mathbf{rec} \ x. \ x \rrbracket \rho = \perp_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]_\perp}$$

$x : \mathit{int} * \mathit{int}$

$$\llbracket \mathbf{rec} \ x. \ x \rrbracket \rho = \perp_{(\mathbb{Z}_\perp \times \mathbb{Z}_\perp)_\perp}$$

# Example

$y : \tau_1 \quad z : \tau_2$

$$\begin{aligned} \llbracket \lambda y. \mathbf{rec} \ z. \ z \rrbracket \rho &= \llbracket \lambda d_y. \llbracket \mathbf{rec} \ z. \ z \rrbracket \rho [d_y / y] \rrbracket \\ &= \llbracket \lambda d_y. \perp_{D_{\tau_2}} \rrbracket \\ &= \llbracket \perp_{[D_{\tau_1} \rightarrow D_{\tau_2}]} \rrbracket \\ &= \llbracket \perp_{V_{\tau_1 \rightarrow \tau_2}} \rrbracket \\ &\neq \perp_{D_{\tau_1 \rightarrow \tau_2}} = \perp_{(V_{\tau_1 \rightarrow \tau_2}) \perp} \end{aligned}$$

$x : int \rightarrow int$

$$\llbracket \mathbf{rec} \ x. \ x \rrbracket \rho = \perp_{[\mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}] \perp} \quad \text{diverges}$$

$y : int, z : int$

$$\llbracket \lambda y. \mathbf{rec} \ z. \ z \rrbracket \rho = \llbracket \perp_{[\mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}]} \rrbracket \quad \text{waits arg and diverges}$$



# Exercise

$x : int * int , y : int , z : int$

$\llbracket \mathbf{rec} x. x \rrbracket \rho \stackrel{?}{=} \llbracket (\mathbf{rec} y. y , \mathbf{rec} z. z) \rrbracket \rho$



diverges

a pair  
of diverging computations

$\perp_{D_{int * int}}$

$\llbracket (\perp_{D_{int}} , \perp_{D_{int}}) \rrbracket$

# Lazy vs Eager

# Eager Application

returns  $\perp$  when  $\llbracket t \rrbracket \rho = \perp$

lazy  $\llbracket t \ t_0 \rrbracket \rho \triangleq \mathbf{let} \ \varphi \leftarrow \llbracket t \rrbracket \rho. \ \varphi(\llbracket t_0 \rrbracket \rho)$

eager  $\llbracket t \ t_0 \rrbracket \rho \triangleq \mathbf{let} \ \varphi \leftarrow \llbracket t \rrbracket \rho. \ \mathbf{let} \ d \leftarrow \llbracket t_0 \rrbracket \rho. \ \varphi(\lfloor d \rfloor)$

returns  $\perp$  when  $\llbracket t \rrbracket \rho = \perp$  or  $\llbracket t_0 \rrbracket \rho = \perp$

# Well-given definitions



# Well-definedness

We must guarantee that all functions we have used are monotone and continuous, so that Kleene's fix point theory is applicable

$\pi_1$   $\pi_2$   $(\cdot)^*$  *apply* *fix*    already considered  
**let**

op<sub>⊥</sub>     $\text{Cond}_\tau$      $\lambda$     to be checked

**TH.**  $\underline{\text{op}}_{\perp}$  is monotone and continuous

$$\underline{\text{op}}_{\perp} : \mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}$$

$$v_1 \underline{\text{op}}_{\perp} v_2 \triangleq \begin{cases} \lfloor n_1 \underline{\text{op}} n_2 \rfloor & \text{if } v_1 = \lfloor n_1 \rfloor \text{ and } v_2 = \lfloor n_2 \rfloor \\ \perp_{\mathbb{Z}_{\perp}} & \text{otherwise (} v_1 = \perp_{\mathbb{Z}_{\perp}} \text{ or } v_2 = \perp_{\mathbb{Z}_{\perp}} \text{)} \end{cases}$$

We omit monotonicity check

Since the domain has only finite chains, it is also continuous

**TH.**  $\text{Cond}_\tau$  is monotone and continuous

$$\text{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \rightarrow D_\tau$$

$$\text{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise (} v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

We omit monotonicity check

We prove continuity on each parameter separately

The first parameter is in  $\mathbb{Z}_\perp$

only finite chains are possible, hence continuity is guaranteed

We prove continuity over the second parameter (next slides)

For the third parameter the proof is analogous and omitted

(continue)

$$\text{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \rightarrow D_\tau$$

$$\text{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = [0] \\ d_2 & \text{otherwise } (v = [n] \text{ with } n \neq 0) \end{cases}$$

Continuity over the second parameter

take  $v \in \mathbb{Z}_\perp, d \in D_\tau, \{d_i\}_{i \in \mathbb{N}} \subseteq D_\tau$

we want to prove  $\text{Cond}_\tau \left( v, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = \bigsqcup_{i \in \mathbb{N}} \text{Cond}_\tau(v, d_i, d)$

we proceed by case analysis on  $v$

$$\begin{array}{l} \perp_{\mathbb{Z}_\perp} \\ [0] \\ [n], n \neq 0 \end{array}$$

(continue)

$$\text{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \rightarrow D_\tau$$

$$\text{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

$$v = \perp_{\mathbb{Z}_\perp}$$

$$\text{Cond}_\tau \left( \perp_{\mathbb{Z}_\perp}, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = \perp_{D_\tau} = \bigsqcup_{i \in \mathbb{N}} \perp_{D_\tau} = \bigsqcup_{i \in \mathbb{N}} \text{Cond}_\tau(\perp_{\mathbb{Z}_\perp}, d_i, d)$$

(continue)

$$\text{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \rightarrow D_\tau$$

$$\text{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

$$v = \lfloor 0 \rfloor$$

$$\text{Cond}_\tau \left( \lfloor 0 \rfloor, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = \bigsqcup_{i \in \mathbb{N}} d_i = \bigsqcup_{i \in \mathbb{N}} \text{Cond}_\tau(\lfloor 0 \rfloor, d_i, d)$$

(continue)

$$\text{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \rightarrow D_\tau$$

$$\text{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise (} v = \lfloor n \rfloor \text{ with } n \neq 0 \text{)} \end{cases}$$

$$v = \lfloor n \rfloor, n \neq 0$$

$$\text{Cond}_\tau \left( \lfloor n \rfloor, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = d = \bigsqcup_{i \in \mathbb{N}} d = \bigsqcup_{i \in \mathbb{N}} \text{Cond}_\tau(\lfloor n \rfloor, d_i, d)$$

## TH. lambda abstraction is monotone and continuous

$t : \tau$        $\lambda d. \llbracket t \rrbracket \rho[d/x]$  is continuous

we focus on the stronger property

$\lambda \tilde{d}. \llbracket t \rrbracket \rho[\tilde{d}/\tilde{x}]$  is continuous

the proof is by structural induction on  $t$

(try on your own)

**Corollary**     $t : \tau_0 \rightarrow \tau$        $fix \lambda d. \llbracket t \rrbracket \rho[d/x]$  is continuous

(the limit of continuous functions is continuous)



# Main properties

# Substitution lemma

$x, t_0 : \tau_0$   
 $t : \tau$

$$\llbracket t^{t_0 / x} \rrbracket \rho = \llbracket t \rrbracket \rho[\llbracket t_0 \rrbracket \rho / x]$$

environment update

syntactic substitution

the proof is by structural induction on  $t$   
(try on your own)

# Compositionality

The substitution lemma  $\llbracket t^{[t_0/x]} \rrbracket \rho = \llbracket t \rrbracket \rho[\llbracket t_0 \rrbracket \rho / x]$  is important: as it guarantees the compositionality of the denotational semantics

**TH.**  $\llbracket t_1 \rrbracket \rho = \llbracket t_2 \rrbracket \rho \quad \Rightarrow \quad \llbracket t^{[t_1/x]} \rrbracket \rho = \llbracket t^{[t_2/x]} \rrbracket \rho$

*proof.* assume  $\llbracket t_1 \rrbracket \rho = \llbracket t_2 \rrbracket \rho$

$$\begin{array}{ccccc} \llbracket t^{[t_1/x]} \rrbracket \rho = \llbracket t \rrbracket \rho[\llbracket t_1 \rrbracket \rho / x] = \llbracket t \rrbracket \rho[\llbracket t_2 \rrbracket \rho / x] = \llbracket t^{[t_2/x]} \rrbracket \rho & & & & \\ | & & | & & | \\ \text{subs} & & \llbracket t_1 \rrbracket \rho = \llbracket t_2 \rrbracket \rho & & \text{subs} \\ \text{lemma} & & & & \text{lemma} \end{array}$$

# Only free variables matter

**TH.**  $t : \tau$   
 $\forall x \in \text{fv}(t). \rho(x) = \rho'(x) \quad \Rightarrow \quad \llbracket t \rrbracket \rho = \llbracket t \rrbracket \rho'$

the proof is by structural induction on  $t$

(try on your own)

**Corollary**  $t$  closed  $\Rightarrow \quad \forall \rho, \rho'. \llbracket t \rrbracket \rho = \llbracket t \rrbracket \rho'$

## TH. Canonical terms are not bottom

$$c \in C_\tau \Rightarrow \forall \rho. \llbracket c \rrbracket \rho \neq \perp_{D_\tau}$$

*proof.* by rule induction on the rules for canonical terms

$$P(c \in C_\tau) \triangleq \forall \rho. \llbracket c \rrbracket \rho \neq \perp_{D_\tau}$$

$$\frac{}{n \in C_{int}}$$

$$\llbracket n \rrbracket \rho = [n] \neq \perp_{D_{int}}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1 \quad t_0, t_1 \text{ closed}}{(t_0, t_1) \in C_{\tau_0 * \tau_1}}$$

$$\llbracket (t_0, t_1) \rrbracket \rho = [(\llbracket t_0 \rrbracket \rho, \llbracket t_1 \rrbracket \rho)] \neq \perp_{D_{\tau_0 * \tau_1}}$$

$$\frac{\lambda x. t : \tau_0 \rightarrow \tau_1 \quad \lambda x. t \text{ closed}}{\lambda x. t \in C_{\tau_0 \rightarrow \tau_1}}$$

$$\llbracket \lambda x. t \rrbracket \rho = [\lambda d. \llbracket t \rrbracket \rho [d/x]] \neq \perp_{D_{\tau_0 \rightarrow \tau_1}}$$