**PSC 2024/25** (375AA, 9CFU)

Principles for Software Composition

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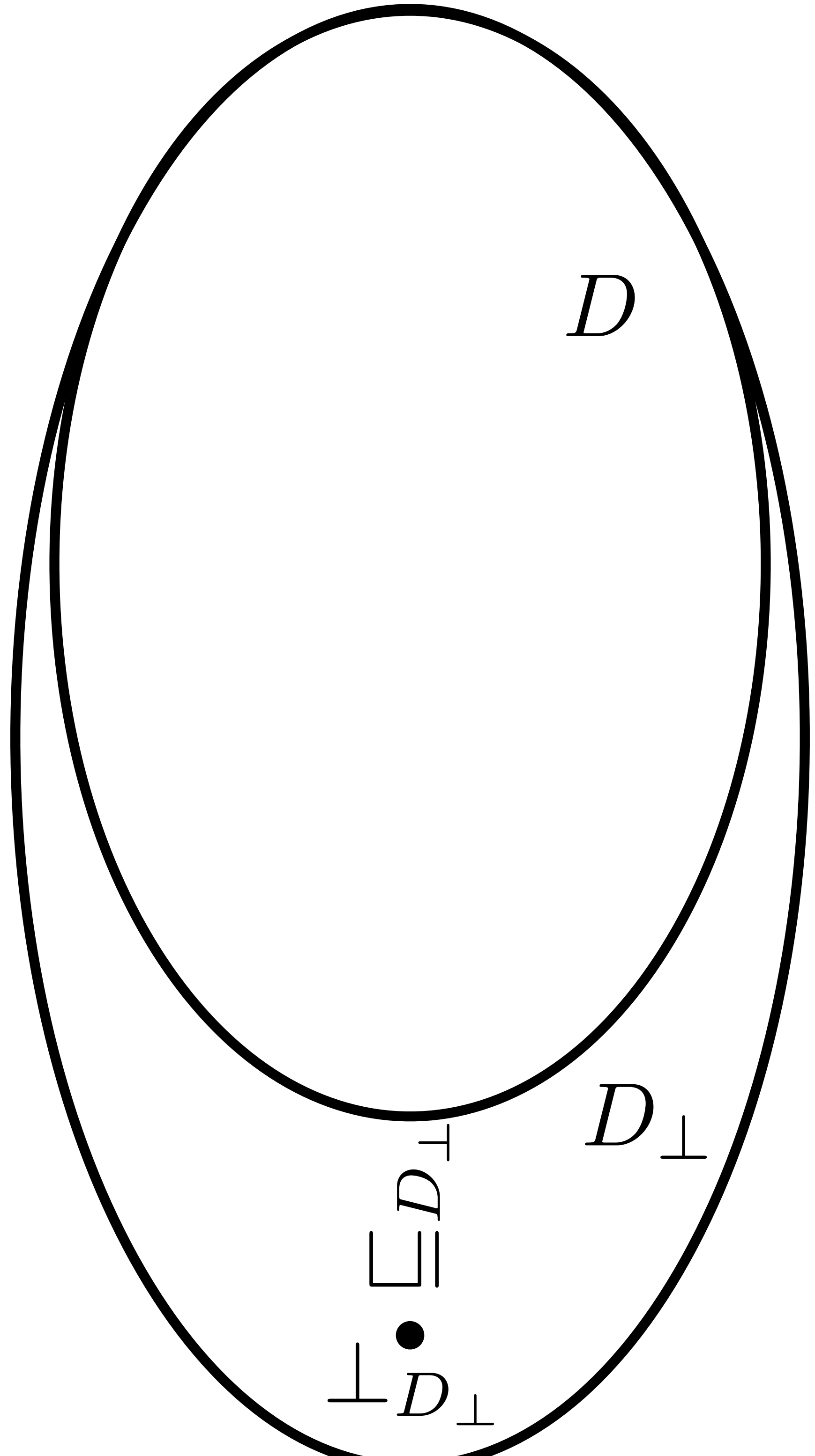
# 13c - Continuity theorems

# Lifted Domains

# Lifted Domains



# Lifted Domains

$\mathcal{D} = (D, \sqsubseteq_D)$ CPO $\quad \Rightarrow \quad \mathcal{D}_\bot = (\ D_\bot\ ,\ \sqsubseteq_{D_\bot}\ )$

$$D_\bot \triangleq \{\bot\} \uplus D$$
$$= \{(0,\bot)\} \cup (\{1\} \times D) = \{(0,\bot)\} \cup \{(1,d) \mid d \in D\}$$

$$\bot_{D_\bot} \triangleq (0, \bot)$$

$$\lfloor \cdot \rfloor : D \to D_\bot$$

lifting function

$$\lfloor d \rfloor \triangleq (1, d)$$

how to order lifted elements?

$$\forall x \in D_\bot.\ \bot_{D_\bot} \sqsubseteq_{D_\bot} x$$

$$\forall d_1, d_2 \in D.\ \lfloor d_1 \rfloor \sqsubseteq_{D_\bot} \lfloor d_2 \rfloor \Leftrightarrow d_1 \sqsubseteq_D d_2$$

# Example

$(\mathbb{Z}, =)$

$$\begin{tikzcd}
\cdots & -1 & 0 & 1 & \cdots \\
 & & \perp_{\mathbb{Z}_\perp} \arrow[ul] \arrow[u] \arrow[ur] & & \mathbb{Z}_\perp
\end{tikzcd}$$

# Lifted Domains

**TH.** $\mathcal{D}_\perp = (\ D_\perp\ ,\ \sqsubseteq_{D_\perp}\ )\ \ \mathrm{CPO}_\perp$

try on your own to prove:
PO,
bottom element,

complete observe that: $\bigsqcup_{i\in\mathbb{N}} \lfloor d_i \rfloor = \left\lfloor \bigsqcup_{i\in\mathbb{N}} d_i \right\rfloor$

it is an upper bound
it is the least upper bound

# Lifting operator

$(D, \sqsubseteq_D)$ CPO

$(E, \sqsubseteq_E)$ $\mathrm{CPO}_\perp$

$$(\cdot)^* : [D \to E] \to [D_\perp \to E]$$

$$\forall f \in [D \to E].\ f^*(x) \triangleq \begin{cases} \perp_E & \text{if } x = \perp_{D_\perp} \\ f(d) & \text{if } x = \lfloor d \rfloor \end{cases}$$

for the definition to be well-given
we need to prove:

$$f \in [D \to E] \quad \Rightarrow \quad f^* \in [D_\perp \to E]$$

$f$ continuous implies $f^*$ continuous

**TH.** the lifting operator is well-defined

*proof.* assume $f$ continuous, take a chain $\{x_n\}_{n\in\mathbb{N}}$ in $D_\perp$

we need to prove $f^*\left(\bigsqcup_{n\in\mathbb{N}} x_n\right) = \bigsqcup_{n\in\mathbb{N}} f^*(x_n)$

if $\forall n \in \mathbb{N}.\ x_n = \perp_{D_\perp}$ then it is obvious

otherwise, let $k = \min\{i \mid x_i \neq \perp_{D_\perp}\}$

then $\forall m \geq k.\ \exists d_m \in D.\ x_m = \lfloor d_m \rfloor$

and by prefix independence of lub

$$\bigsqcup_{n\in\mathbb{N}} x_n = \bigsqcup_{n\in\mathbb{N}} x_{n+k} \qquad \bigsqcup_{n\in\mathbb{N}} f^*(x_n) = \bigsqcup_{n\in\mathbb{N}} f^*(x_{n+k})$$

we can just prove $f^*\left(\bigsqcup_{n\in\mathbb{N}} x_{n+k}\right) = \bigsqcup_{n\in\mathbb{N}} f^*(x_{n+k})$

(see next slide)

## (continue) $f^*\left(\bigsqcup_{n\in\mathbb{N}} x_{n+k}\right) = \bigsqcup_{n\in\mathbb{N}} f^*(x_{n+k})$

$$
\begin{aligned}
f^*\left(\bigsqcup_{n\in\mathbb{N}} x_{n+k}\right) &= f^*\left(\bigsqcup_{n\in\mathbb{N}} \lfloor d_{n+k} \rfloor\right) && \text{by def of } k \\
&= f^*\left(\left\lfloor \bigsqcup_{n\in\mathbb{N}} d_{n+k} \right\rfloor\right) && \text{by lub in a lifted domain} \\
&= f\left(\bigsqcup_{n\in\mathbb{N}} d_{n+k}\right) && \text{by def of lifting} \\
&= \bigsqcup_{n\in\mathbb{N}} f(d_{n+k}) && \text{by continuity of } f \\
&= \bigsqcup_{n\in\mathbb{N}} f^*(\lfloor d_{n+k} \rfloor) && \text{by def of lifting} \\
&= \bigsqcup_{n\in\mathbb{N}} f^*(x_{n+k}) && \text{by def of } k
\end{aligned}
$$

**TH.** $(\cdot)^*$ is monotone

(try to prove on your own)

**TH.** $(\cdot)^*$ is continuous

*proof.* take a chain of continuous functions $\{f_i : D \to E\}_{i\in\mathbb{N}}$

we need to prove $$\left(\bigsqcup_{i\in\mathbb{N}} f_i\right)^* = \bigsqcup_{i\in\mathbb{N}} f_i^*$$

take a generic $x \in D_\perp$

we need to prove $$\left(\bigsqcup_{i\in\mathbb{N}} f_i\right)^* (x) = \left(\bigsqcup_{i\in\mathbb{N}} f_i^*\right)(x)$$

if $x = \perp_{D_\perp}$ it is obvious

if $x = \lfloor d \rfloor$ we have…

(see next slide)

(continue) $\left(\bigsqcup_{i\in\mathbb{N}} f_i\right)^* (\lfloor d \rfloor) = \left(\bigsqcup_{i\in\mathbb{N}} f_i^*\right) (\lfloor d \rfloor)$

$$\left(\bigsqcup_{i\in\mathbb{N}} f_i\right)^* (\lfloor d \rfloor) = \left(\bigsqcup_{i\in\mathbb{N}} f_i\right) (d)$$ by def of lifting

$$= \bigsqcup_{i\in\mathbb{N}} f_i(d)$$ by lub in a functional domain

$$= \bigsqcup_{i\in\mathbb{N}} f_i^*(\lfloor d \rfloor)$$ by def of lifting

$$= \left(\bigsqcup_{i\in\mathbb{N}} f_i^*\right) (\lfloor d \rfloor)$$ by lub in a functional domain

# Let notation (de-lifting)

$(E, \sqsubseteq_E)$ $\quad \mathrm{CPO}_\perp \qquad \lambda x.\ e \in [D \to E] \qquad t \in D_\perp$

$$\mathbf{let}\ x \Leftarrow t.\ e \quad \triangleq \quad \underbrace{\underbrace{(\underbrace{\lambda x.\ e}_{[D \to E]})^*}_{[D_\perp \to E]}\ \underbrace{t}_{D_\perp}}_{E} \quad = \quad \begin{cases} \perp_E & \text{if } t = \perp_{D_\perp} \\ e[^d/_x] & \text{if } t = \lfloor d \rfloor \end{cases}$$

intuitively:

if $t$ is a lifted value $\lfloor d \rfloor$ then we de-lift the value and assign it to $x$ in $e$

otherwise returns $\perp_E$

# Continuity theorems

**TH.** $(D, \sqsubseteq_D)$ $(E_i, \sqsubseteq_{E_i})$ CPO $\quad f : D \to E_1 \times E_2 \qquad g_i \triangleq \pi_i \circ f$

$f$ is continuous iff $g_1, g_2$ are continuous

*proof.* $\Rightarrow$) $f$ is continuous, $\pi_i$ is continuous $\Rightarrow$ $g_i$ is continuous

$\Leftarrow$) note that $\forall d \in D.\ f(d) = (g_1(d), g_2(d))$

assume $g_1, g_2$ are continuous

we want to prove $f$ is continuous

take a chain $\{d_i\}_{i \in \mathbb{N}}$ in $D$

we must prove $f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)$

(see next slide)

## (continue) $f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \bigsqcup_{i\in\mathbb{N}} f(d_i)$

$$f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \left(g_1\left(\bigsqcup_{i\in\mathbb{N}} d_i\right), g_2\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)\right) \quad \text{by def } g_1, g_2$$

$$= \left(\bigsqcup_{i\in\mathbb{N}} g_1(d_i), \bigsqcup_{i\in\mathbb{N}} g_2(d_i)\right) \quad g_1, g_2 \text{ are continuous}$$

$$= \bigsqcup_{i\in\mathbb{N}} (g_1(d_i), g_2(d_i)) \quad \text{by def of lub of pairs}$$

$$= \bigsqcup_{i\in\mathbb{N}} f(d_i) \quad \text{by def } g_1, g_2$$

**TH.** $(D, \sqsubseteq_D)$ $(E, \sqsubseteq_E)$ $(F, \sqsubseteq_F)$ CPO $f : D \times E \to F$

$f_d : E \to F$
$f_d \triangleq \lambda e.\ f(d, e)$

$f_e : D \to F$
$f_e \triangleq \lambda d.\ f(d, e)$

$f$ is continuous iff $\forall d \in D.\ f_d$ are continuous
$\forall e \in E.\ f_e$ are continuous

*proof.* $\Rightarrow)$ assume $f$ is continuous

take a generic $d \in D$

we want to prove $f_d$ is continuous

take a chain $\{e_i\}_{i \in \mathbb{N}}$ in $E$

we prove $f_d\left(\bigsqcup_{i \in \mathbb{N}} e_i\right) = \bigsqcup_{i \in \mathbb{N}} f_d(e_i)$

(see next slide)

$e \in E$

$f_e$

(omitted)

(continue) $f_d\left(\bigsqcup_{i\in\mathbb{N}} e_i\right) = \bigsqcup_{i\in\mathbb{N}} f_d(e_i)$

$$f_d\left(\bigsqcup_{i\in\mathbb{N}} e_i\right) = f\left(d, \bigsqcup_{i\in\mathbb{N}} e_i\right) \qquad \text{by def of } f_d$$

$$= f\left(\bigsqcup_{i\in\mathbb{N}} d, \bigsqcup_{i\in\mathbb{N}} e_i\right) \qquad \text{by lub of constant chain}$$

$$= f\left(\bigsqcup_{i\in\mathbb{N}} (d, e_i)\right) \qquad \text{by lub of pairs}$$

$$= \bigsqcup_{i\in\mathbb{N}} f(d, e_i) \qquad \text{by continuity of } f$$

$$= \bigsqcup_{i\in\mathbb{N}} f_d(e_i) \qquad \text{by def of } f_d$$

**TH.** $(D, \sqsubseteq_D)$ $(E, \sqsubseteq_E)$ $(F, \sqsubseteq_F)$ CPO $\quad f : D \times E \to F$

$$f_d : E \to F \qquad f_d \triangleq \lambda e.\ f(d,e)$$

$$f_e : D \to F \qquad f_e \triangleq \lambda d.\ f(d,e)$$

$f$ is continuous iff $\forall d \in D.\ f_d$ are continuous
$\forall e \in E.\ f_e$ are continuous

$\Leftarrow)$ assume $f_d, f_e$ are continuous for all $d, e$

we want to prove $f$ is continuous

take a chain $\{(d_k, e_k)\}_{k \in \mathbb{N}}$ in $D \times E$

we prove $f\left(\bigsqcup_{k \in \mathbb{N}} (d_k, e_k)\right) = \bigsqcup_{k \in \mathbb{N}} f(d_k, e_k)$

(see next slide)

(continue) $f\left(\bigsqcup_{k\in\mathbb{N}} (d_k, e_k)\right) = \bigsqcup_{k\in\mathbb{N}} f(d_k, e_k)$

| | | |
|---|---|---|
| $f(\sqcup_k (d_k, e_k))$ | $= f(\sqcup_i d_i, \sqcup_j e_j)$ | by def of lub of pairs |
| | $= f_d(\sqcup_j e_j)$ | by def of $f_d$ with $d \triangleq \sqcup_i d_i$ |
| | $= \sqcup_j f_d(e_j)$ | by continuity of $f_d$ |
| | $= \sqcup_j f(d, e_j)$ | by def of $f_d$ |
| | $= \sqcup_j f_{e_j}(d)$ | by def of $f_{e_j}$ |
| | $= \sqcup_j f_{e_j}(\sqcup_i d_i)$ | by def of $d \triangleq \sqcup_i d_i$ |
| | $= \sqcup_j \sqcup_i f_{e_j}(d_i)$ | by continuity of $f_{e_j}$ |
| | $= \sqcup_j \sqcup_i f(d_i, e_j)$ | by def of $f_{e_j}$ |
| | $= \sqcup_k f(d_k, e_k)$ | by switch lemma (applicable?) |

(continue) $f\left(\bigsqcup_{k\in\mathbb{N}}(d_k, e_k)\right) = \bigsqcup_{k\in\mathbb{N}} f(d_k, e_k)$

if $i \leq n \wedge j \leq m$ then $f(d_i, e_j) \sqsubseteq f(d_n, e_m)$ ? ✓

$$\Downarrow$$

$$d_i \sqsubseteq_D d_n \wedge e_j \sqsubseteq_E e_m$$

$$f(d_i, e_j) = f_{d_i}(e_j) \sqsubseteq f_{d_i}(e_m) = f(d_i, e_m) = f_{e_m}(d_i) \sqsubseteq f_{e_m}(d_n) = f(d_n, e_m)$$

$f_{d_i}$ monotone

$f_{e_m}$ monotone

$= \sqcup_j \sqcup_i f(d_i, e_j)$ ✓

$= \sqcup_k f(d_k, e_k)$ by switch lemma (applicable?)

# Some important functions

# Apply

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO

$$apply : [D \to E] \times D \to E$$
$$apply(f, d) \triangleq f(d)$$

**TH.** $apply$ is monotone

(try to prove on your own)

**TH.** $apply$ is continuous

*proof.* from a previous theorem, we prove continuity on each parameter separately $apply_f$ $apply_d$

1. for any $f \in [D \to E]$ $apply_f \triangleq \lambda d.\ f(d)$ is continuous
2. for any $d \in D$ $apply_d \triangleq \lambda f.\ f(d)$ is continuous

(see next slides)

1. for any $f \in [D \to E]$ $\quad apply_f \triangleq \lambda d.\ f(d)$ is continuous

take $f \in [D \to E]$ and a chain $\{d_i\}_{i \in \mathbb{N}}$ in $D$

we want to prove $\quad apply_f \left( \bigsqcup_i d_i \right) = \bigsqcup_i apply_f(d_i)$

$apply_f(\sqcup_i d_i) = apply(f, \sqcup_i d_i)$ by def of $apply_f$

$= f(\sqcup_i d_i)$ by def of $apply$

$= \sqcup_i f(d_i)$ by continuity of $f$

$= \sqcup_i apply(f, d_i)$ by def of $apply$

$= \sqcup_i apply_f(d_i)$ by def of $apply_f$

## 2. for any $d \in D$ $\quad apply_d \triangleq \lambda f.\ f(d)$ is continuous

take $d \in D$ and a chain $\{f_i\}_{i\in\mathbb{N}}$ in $[D \to E]$

we want to prove $apply_d\left(\bigsqcup_i f_i\right) = \bigsqcup_i apply_d(f_i)$

$apply_d(\sqcup_i f_i) = apply(\sqcup_i f_i, d)$ by def of $apply_d$

$= (\sqcup_i f_i)(d)$ by def of $apply$

$= \sqcup_i f_i(d)$ by def of lub of functions

$= \sqcup_i apply(f_i, d)$ by def of $apply$

$= \sqcup_i apply_d(f_i)$ by def of $apply_d$

# Apply: recap

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO

$$apply : [D \to E] \times D \to E$$
$$apply(f, d) \triangleq f(d)$$

$$apply \in [[D \to E] \times D \to E]$$

# Fix

$(D, \sqsubseteq_D)$ CPO$_\perp$

$$\mathit{fix} : [D \to D] \to D$$

$$\mathit{fix} \triangleq \lambda f.\ \bigsqcup_{n \in \mathbb{N}} f^n(\perp_D)$$

**TH.** $\mathit{fix}$ is monotone

(try to prove on your own)

**TH.** $\mathit{fix}$ is continuous

*proof.* $\mathit{fix} \triangleq \lambda f.\ \bigsqcup_{n \in \mathbb{N}} f^n(\perp_D) = \bigsqcup_{n \in \mathbb{N}} \lambda f.\ f^n(\perp_D)$

by def of lub in functional domains

we prove that $\forall n.\ \lambda f.\ f^n(\perp_D)$ is continuous

(by mathematical induction on $n$ )

then $\mathit{fix}$ is continuous because lub of continuous functions

(see next slides)

(continue) $\forall n.\ \lambda f.\ f^n(\bot_D)$

base case: $\lambda f.\ f^0(\bot_D) = \lambda f.\ \bot_D$

is a constant function (continuous)

inductive case: assume $g \triangleq \lambda f.\ f^n(\bot_D)$ is continuous

we want to prove $h \triangleq \lambda f.\ f^{n+1}(\bot_D)$ is continuous

take a chain $\{f_i\}_{i\in\mathbb{N}}$ in $[D \to D]$

we want to prove $h\left(\bigsqcup_{i\in\mathbb{N}} f_i\right) = \bigsqcup_{i\in\mathbb{N}} h(f_i)$

(see next slide)

(continue) $\forall n.\ \lambda f.\ f^n(\bot_D)$ $g \triangleq \lambda f.\ f^n(\bot_D)$ $h \triangleq \lambda f.\ f^{n+1}(\bot_D)$ $h\left(\bigsqcup_{i\in\mathbb{N}} f_i\right) = \bigsqcup_{i\in\mathbb{N}} h(f_i)$

| | |
|---|---|
| $h(\sqcup_i f_i) = (\sqcup_i f_i)^{n+1}(\bot_D)$ | by def of $h$ |
| $= (\sqcup_j f_j)((\sqcup_i f_i)^n(\bot_D))$ | by def of $(\cdot)^{n+1}$ |
| $= (\sqcup_j f_j)(g(\sqcup_i f_i))$ | by def of $g$ |
| $= (\sqcup_j f_j)(\sqcup_i g(f_i))$ | by ind. hyp ($g$ continuous) |
| $= (\sqcup_j f_j)(\sqcup_i f_i^n(\bot_D))$ | by def of $g$ |
| $= \sqcup_j f_j(\sqcup_i f_i^n(\bot_D))$ | by def of lub in functional CPO |
| $= \sqcup_j \sqcup_i\ f_j(f_i^n(\bot_D))$ | by continuity of $f_j$ |
| $= \sqcup_k f_k(f_k^n(\bot_D))$ | by switch lemma |
| $= \sqcup_k f_k^{n+1}(\bot_D)$ | by def of $(\cdot)^{n+1}$ |
| $= \sqcup_k h(f_k)$ | by def of $h$ |

# Fix: recap

$(D, \sqsubseteq_D)$ $\mathrm{CPO}_\perp$

$$\mathit{fix} : [D \to D] \to D$$
$$\mathit{fix} \triangleq \lambda f. \bigsqcup_{n \in \mathbb{N}} f^n(\perp_D)$$

$$\mathit{fix} \in [[D \to D] \to D]$$

# Curry

$(D, \sqsubseteq_D)$

$(E, \sqsubseteq_E)$ CPO

$(F, \sqsubseteq_F)$

$$curry : (D \times E \to F) \to D \to E \to F$$

$$curry\ f\ d\ e \triangleq f(d, e)$$

**TH.** $f$ continuous $\Rightarrow$ $curry(f)$ continuous

(try to prove on your own)

# Curry

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO
$(F, \sqsubseteq_F)$

$curry : (D \times E \to F) \to D \to E \to F$

$curry\ f\ d\ e \triangleq f(d, e)$

**TH.** $f$ continuous $\Rightarrow$ $curry(f)$ continuous

$h \triangleq curry(f) : D \to E \to F$

given $\{d_i\}_{i \in \mathbb{N}}$ $\quad h\ (\sqcup_i d_i) \stackrel{?}{=} \sqcup_i (h\ d_i) : E \to F$

take $e \in E$ $\quad h\ (\sqcup_i d_i)\ e \stackrel{?}{=} (\sqcup_i (h\ d_i))\ e$

$$h\ (\sqcup_i d_i)\ e = curry\ f\ (\sqcup_i d_i)\ e = f((\sqcup_i d_i), e) = f((\sqcup_i d_i), (\sqcup_i e))$$

$$= f(\sqcup_i (d_i, e)) = \sqcup_i f(d_i, e) = \sqcup_i (curry\ f\ d_i\ e)$$

$$= \sqcup_i (h\ d_i\ e) = (\sqcup_i (h\ d_i))\ e$$

# Uncurry

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO
$(F, \sqsubseteq_F)$

$$uncurry : (D \to E \to F) \to (D \times E) \to F$$

$$uncurry\ f\ (d, e) \triangleq f\ d\ e$$

**TH.** $f$ continuous $\Rightarrow$ $uncurry(f)$ continuous

(try to prove on your own)

**TH.** $uncurry$ is the inverse of $curry$

given $g : (D \times E) \to F$ $\quad uncurry(curry(g)) =^? g$

take $(d, e) \in (D \times E)$ $\quad (uncurry(curry(g)))(d, e) =^? g(d, e)$

$$(uncurry(curry(g)))(d, e) = (curry(g))\ d\ e = g(d, e)$$

# Uncurry

$(D, \sqsubseteq_D)$

$(E, \sqsubseteq_E)$ CPO

$(F, \sqsubseteq_F)$

$$uncurry : (D \to E \to F) \to (D \times E) \to F$$

$$uncurry\ f\ (d, e) \triangleq f\ d\ e$$

**TH.** $f$ continuous $\Rightarrow$ $uncurry(f)$ continuous

(try to prove on your own)

**TH.** $uncurry$ is the inverse of $curry$

given $f : D \to E \to F$ $\qquad curry(uncurry(f)) \stackrel{?}{=} f$

take $d \in D, e \in E$ $\qquad (curry(uncurry(f)))\ d\ e \stackrel{?}{=} f\ d\ e$

$$(curry(uncurry(f)))\ d\ e = (uncurry(f))(d, e) = f\ d\ e$$

# Disjoint Union

$$\mathcal{D} = (D, \sqsubseteq_D)$$
$$\mathcal{E} = (E, \sqsubseteq_E)$$

$$\mathrm{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} + \mathcal{E} = (\ D \uplus E\ ,\ \sqsubseteq_{D \uplus E}\ )$$

$$D \uplus E \triangleq \{(0, d) \mid d \in D\} \cup \{(1, e) \mid e \in E\}$$

how to order elements?

is there a bottom element?

is it a complete order?

how to define (continuous) injections?

$$\iota_D : D \to D \uplus E$$
$$\iota_E : E \to D \uplus E$$