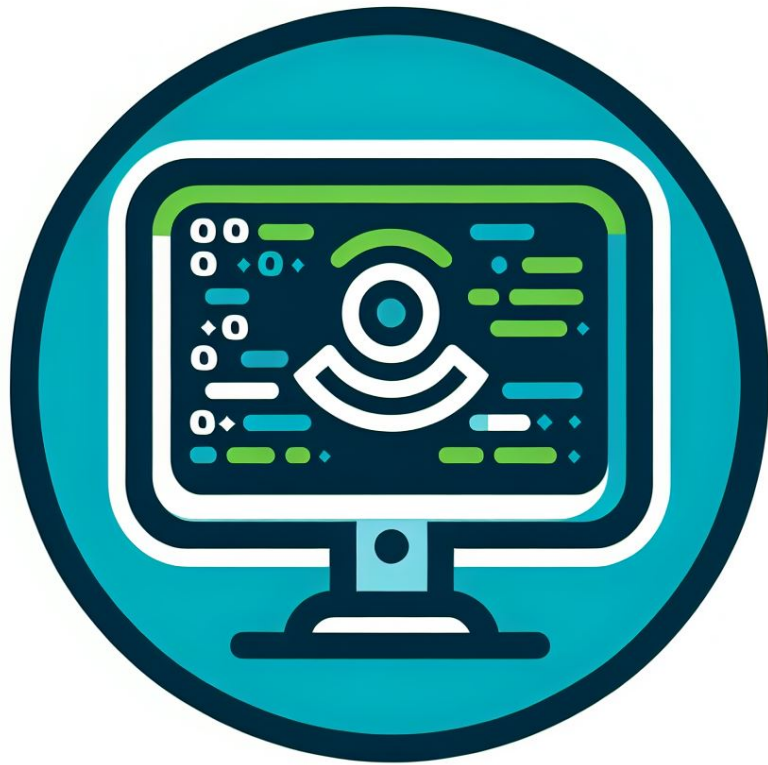




PSC 2024/25 (375AA, 9CFU)

Principles for Software Composition

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<http://www.di.unipi.it/~bruni/>

[http://didawiki.di.unipi.it/doku.php/
magistraleinformatica/psc/start](http://didawiki.di.unipi.it/doku.php/magistraleinformatica/psc/start)

13b - Functional domains

Switch lemma

Switch Lemma

$$\mathcal{E} = (E, \sqsubseteq) \quad \text{CPO}$$

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'} \quad \text{if} \quad n \leq n' \wedge m \leq m'$

$e_{0,0}$

Switch Lemma

$$\mathcal{E} = (E, \sqsubseteq) \quad \text{CPO}$$

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$$e_{0,0} \sqsubseteq e_{0,1}$$

Switch Lemma

$$\mathcal{E} = (E, \sqsubseteq) \quad \text{CPO}$$

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$$e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \cdots$$

Switch Lemma

$$\mathcal{E} = (E, \sqsubseteq) \quad \text{CPO}$$

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

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$$e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \cdots \sqsubseteq e_{0,m} \sqsubseteq \cdots$$

Switch Lemma

$$\mathcal{E} = (E, \sqsubseteq) \quad \text{CPO}$$

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$$\begin{array}{ccccccc} e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{1,m} & \sqsubseteq & \cdots \\ \sqcup & & \sqcup & & \sqcup & & & & \sqcup & & \end{array}$$

$$e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \cdots \sqsubseteq e_{0,m} \sqsubseteq \cdots$$

Switch Lemma

$$\mathcal{E} = (E, \sqsubseteq) \quad \text{CPO}$$

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$$\begin{array}{ccccccc}
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 \sqcup & \sqcup & \sqcup & & \sqcup \\
 e_{2,0} \sqsubseteq e_{2,1} \sqsubseteq e_{2,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots \\
 \sqcup & \sqcup & \sqcup & & \sqcup \\
 e_{1,0} \sqsubseteq e_{1,1} \sqsubseteq e_{1,2} \sqsubseteq \cdots \sqsubseteq e_{1,m} \sqsubseteq \cdots \\
 \sqcup & \sqcup & \sqcup & & \sqcup \\
 e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \cdots \sqsubseteq e_{0,m} \sqsubseteq \cdots
 \end{array}$$

Switch Lemma

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such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$$\begin{array}{ccccccc}
 \vdots & \vdots & \vdots & & \vdots & \ddots & \\
 \sqcup & \sqcup & \sqcup & & \sqcup & & \\
 e_{n,0} \sqsubseteq e_{n,1} \sqsubseteq e_{n,2} \sqsubseteq \cdots \sqsubseteq e_{n,m} \sqsubseteq \cdots & & & & & & \\
 \sqcup & \sqcup & \sqcup & & \sqcup & & \\
 \vdots & \vdots & \vdots & \ddots & \vdots & & \\
 \sqcup & \sqcup & \sqcup & & \sqcup & & \\
 e_{2,0} \sqsubseteq e_{2,1} \sqsubseteq e_{2,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots & & & & & & \\
 \sqcup & \sqcup & \sqcup & & \sqcup & & \\
 e_{1,0} \sqsubseteq e_{1,1} \sqsubseteq e_{1,2} \sqsubseteq \cdots \sqsubseteq e_{1,m} \sqsubseteq \cdots & & & & & & \\
 \sqcup & \sqcup & \sqcup & & \sqcup & & \\
 e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \cdots \sqsubseteq e_{0,m} \sqsubseteq \cdots & & & & & &
 \end{array}$$

Switch Lemma

$$\mathcal{E} = (E, \sqsubseteq) \quad \text{CPO}$$

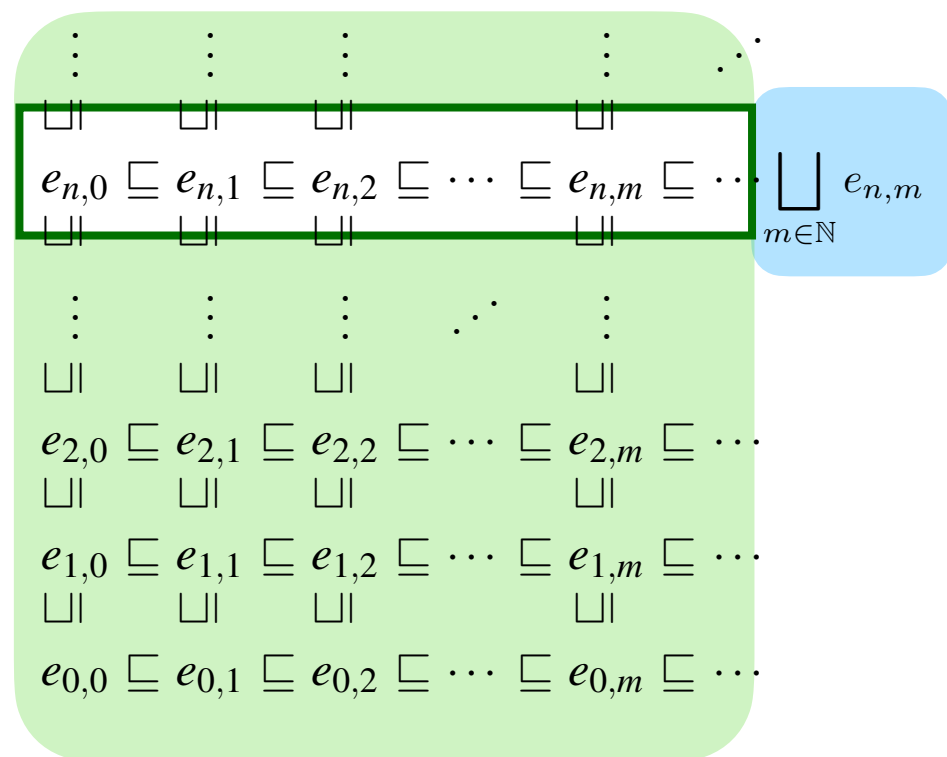
a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

fixed n the set $\{e_{n,m}\}_{m \in \mathbb{N}}$

forms a chain (a row in the picture)

and thus has a lub (E is a CPO)



$$\bigsqcup_{m \in \mathbb{N}} e_{n,m}$$

Switch Lemma

$$\mathcal{E} = (E, \sqsubseteq) \quad \text{CPO}$$

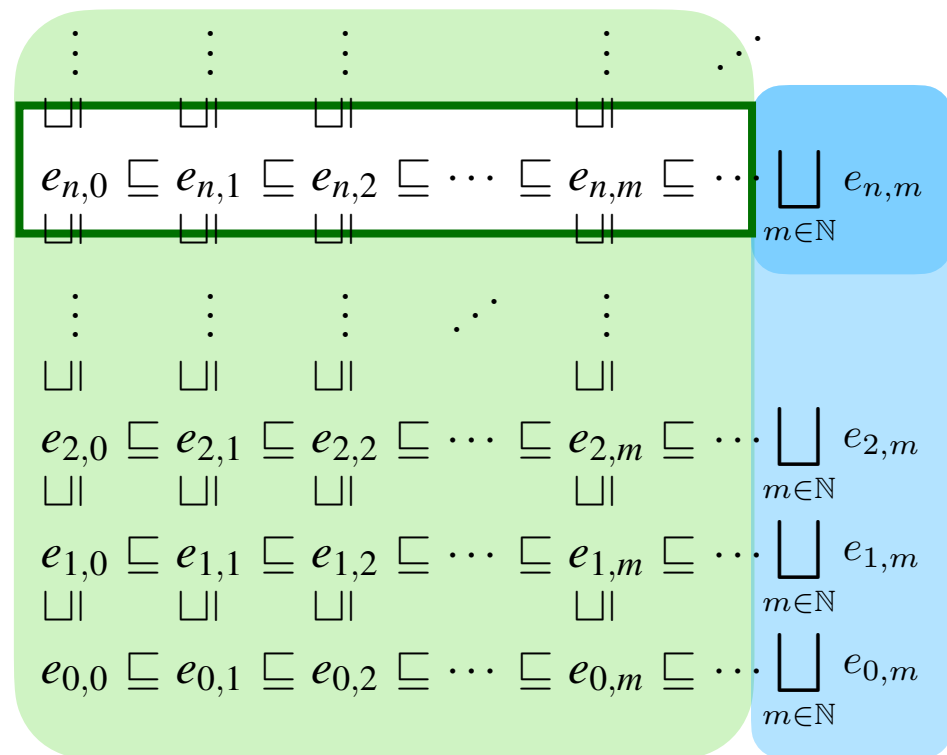
a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

fixed n the set $\{e_{n,m}\}_{m \in \mathbb{N}}$

forms a chain (a row in the picture)

and thus has a lub (E is a CPO)



$$\bigsqcup_{m \in \mathbb{N}} e_{n,m}$$

we form the chain of all row-lubs

$$\left\{ \bigsqcup_{m \in \mathbb{N}} e_{n,m} \right\}_{n \in \mathbb{N}}$$

Switch Lemma

$$\mathcal{E} = (E, \sqsubseteq) \quad \text{CPO}$$

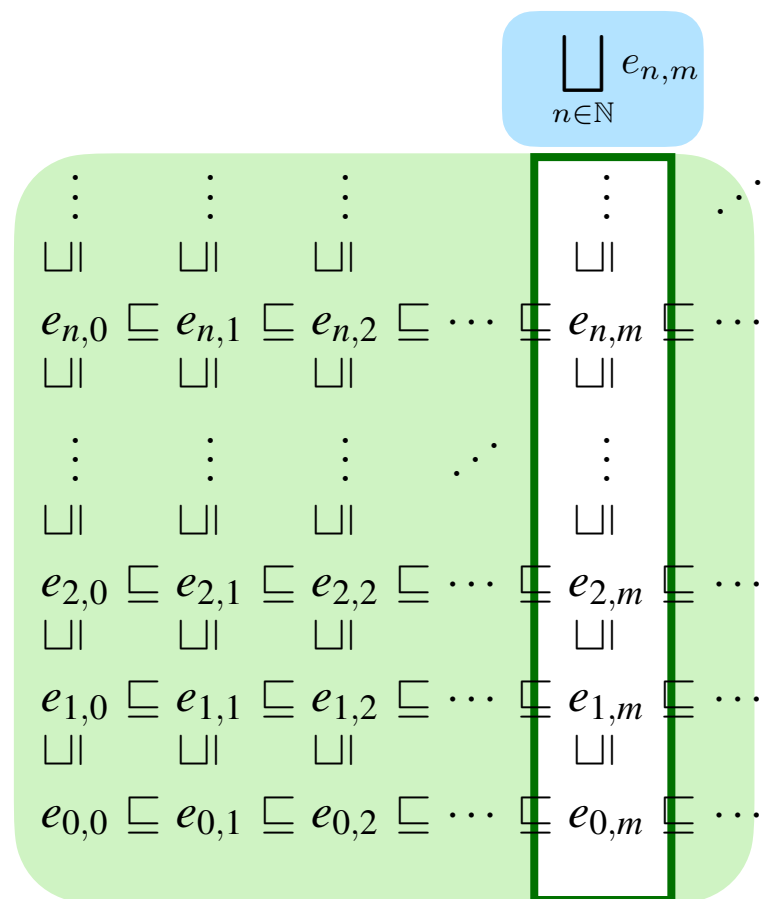
a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

fixed m the set $\{e_{n,m}\}_{n \in \mathbb{N}}$

forms a chain (a column in the picture)

and thus has a lub (E is a CPO)



$$\bigsqcup_{n \in \mathbb{N}} e_{n,m}$$

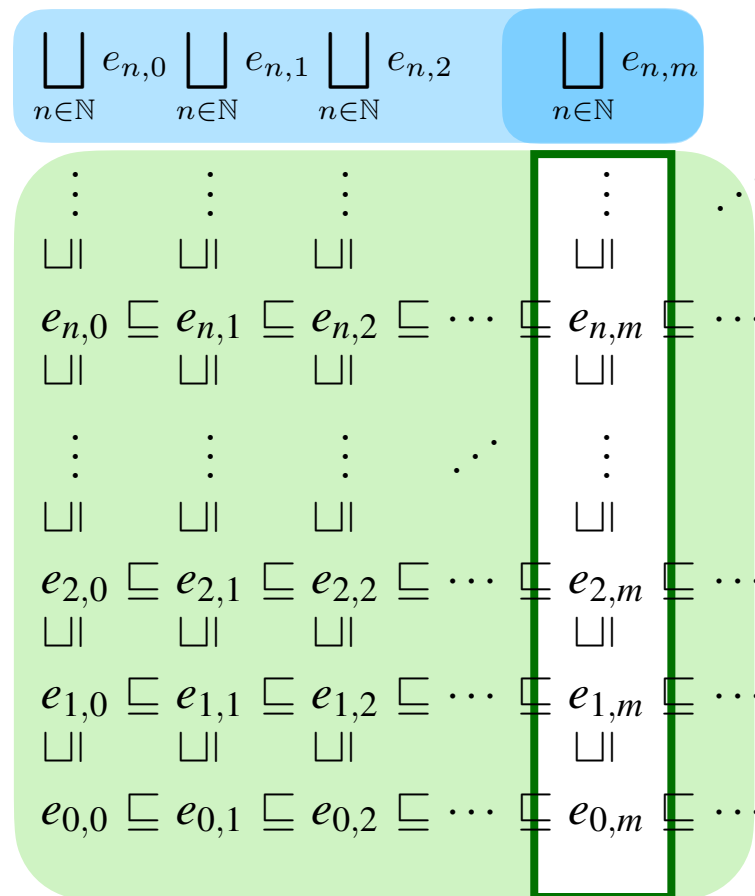
Switch Lemma

$$\mathcal{E} = (E, \sqsubseteq) \quad \text{CPO}$$

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$
 such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

fixed m the set $\{e_{n,m}\}_{n \in \mathbb{N}}$

forms a chain (a column in the picture)
 and thus has a lub (E is a CPO)



$$\bigsqcup_{n \in \mathbb{N}} e_{n,m}$$

we form the chain of all column-lubs

$$\left\{ \bigsqcup_{n \in \mathbb{N}} e_{n,m} \right\}_{m \in \mathbb{N}}$$

Switch Lemma

$$\mathcal{E} = (E, \sqsubseteq) \quad \text{CPO}$$

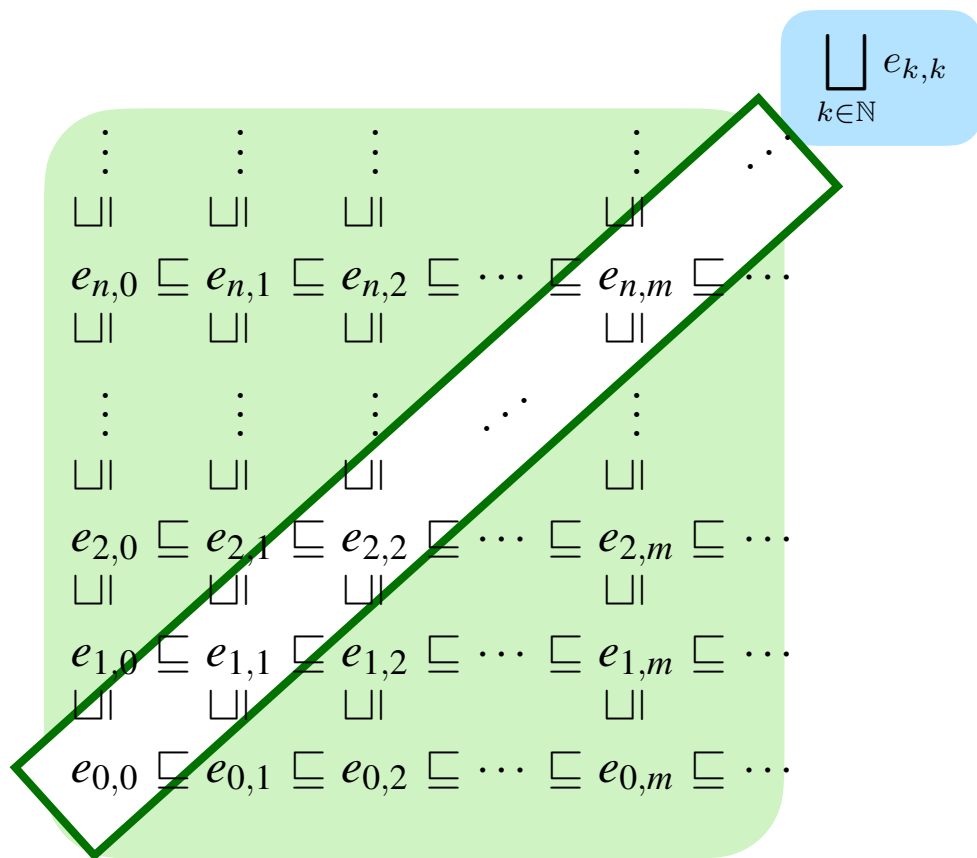
a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

the diagonal elements $\{e_{k,k}\}_{k \in \mathbb{N}}$

also form a chain

and thus has a lub (E is a CPO)



$$\bigsqcup_{k \in \mathbb{N}} e_{k,k}$$

Switch Lemma

$$\mathcal{E} = (E, \sqsubseteq) \quad \text{CPO}$$

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

we prove that
all sets we have seen
have the same
set of upper bounds
and thus the same
least upper bound

$$\begin{array}{ccccccc}
 \vdots & \vdots & \vdots & & \vdots & \ddots & \\
 \sqcup & \sqcup & \sqcup & & \sqcup & & \\
 e_{n,0} \sqsubseteq e_{n,1} \sqsubseteq e_{n,2} \sqsubseteq \cdots \sqsubseteq e_{n,m} \sqsubseteq \cdots & & & & & & \\
 \sqcup & \sqcup & \sqcup & & \sqcup & & \\
 \vdots & \vdots & \vdots & \ddots & \vdots & & \\
 \sqcup & \sqcup & \sqcup & & \sqcup & & \\
 e_{2,0} \sqsubseteq e_{2,1} \sqsubseteq e_{2,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots & & & & & & \\
 \sqcup & \sqcup & \sqcup & & \sqcup & & \\
 e_{1,0} \sqsubseteq e_{1,1} \sqsubseteq e_{1,2} \sqsubseteq \cdots \sqsubseteq e_{1,m} \sqsubseteq \cdots & & & & & & \\
 \sqcup & \sqcup & \sqcup & & \sqcup & & \\
 e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \cdots \sqsubseteq e_{0,m} \sqsubseteq \cdots & & & & & &
 \end{array}$$

$$\bigsqcup_{n \in \mathbb{N}} \bigsqcup_{m \in \mathbb{N}} e_{n,m} = \bigsqcup_{k \in \mathbb{N}} e_{k,k} = \bigsqcup_{m \in \mathbb{N}} \bigsqcup_{n \in \mathbb{N}} e_{n,m}$$

$$\{e_{n,m}\}_{n,m \in \mathbb{N}}$$

$$\left\{ \bigsqcup_{m \in \mathbb{N}} e_{n,m} \right\}_{n \in \mathbb{N}}$$

$$\left\{ \bigsqcup_{n \in \mathbb{N}} e_{n,m} \right\}_{m \in \mathbb{N}}$$

$$\{e_{k,k}\}_{k \in \mathbb{N}}$$

(i)

$$\{e_{n,m}\}_{n,m \in \mathbb{N}}$$

same u.b. as

$$\{e_n\}_{n \in \mathbb{N}}$$

where $e_n = \bigsqcup_{m \in \mathbb{N}} e_{n,m}$

$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\sqcup$	$\sqcup$	$\sqcup$			$\sqcup$
$e_{n,0} \sqsubseteq e_{n,1} \sqsubseteq e_{n,2} \sqsubseteq \cdots \sqsubseteq e_{n,m} \sqsubseteq \cdots$				$\sqsubseteq$	$e_n = \bigsqcup_{m \in \mathbb{N}} e_{n,m}$
$\sqcup$	$\sqcup$	$\sqcup$			$\sqcup$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\sqcup$	$\sqcup$	$\sqcup$			$\sqcup$
$e_{2,0} \sqsubseteq e_{2,1} \sqsubseteq e_{2,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots$				$\sqsubseteq$	$e_2 = \bigsqcup_{m \in \mathbb{N}} e_{2,m}$
$\sqcup$	$\sqcup$	$\sqcup$			$\sqcup$
$e_{1,0} \sqsubseteq e_{1,1} \sqsubseteq e_{1,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots$				$\sqsubseteq$	$e_1 = \bigsqcup_{m \in \mathbb{N}} e_{1,m}$
$\sqcup$	$\sqcup$	$\sqcup$			$\sqcup$
$e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots$				$\sqsubseteq$	$e_0 = \bigsqcup_{m \in \mathbb{N}} e_{0,m}$

(i)

$$\{e_{n,m}\}_{n,m \in \mathbb{N}}$$

same u.b. as

$$\{e_n\}_{n \in \mathbb{N}}$$

where $e_n = \bigsqcup_{m \in \mathbb{N}} e_{n,m}$

1. take an upper bound e of $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

we want to prove it is an upper bound for $\{e_n\}_{n \in \mathbb{N}}$

e

$$\begin{array}{ccccccc} \vdots & \vdots & \vdots & & \vdots & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ e_{n,0} \sqsubseteq e_{n,1} \sqsubseteq e_{n,2} \sqsubseteq \cdots \sqsubseteq e_{n,m} \sqsubseteq \cdots & \sqsubseteq & e_n = \bigsqcup_{m \in \mathbb{N}} e_{n,m} & & & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ e_{2,0} \sqsubseteq e_{2,1} \sqsubseteq e_{2,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots & \sqsubseteq & e_2 = \bigsqcup_{m \in \mathbb{N}} e_{2,m} & & & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ e_{1,0} \sqsubseteq e_{1,1} \sqsubseteq e_{1,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots & \sqsubseteq & e_1 = \bigsqcup_{m \in \mathbb{N}} e_{1,m} & & & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots & \sqsubseteq & e_0 = \bigsqcup_{m \in \mathbb{N}} e_{0,m} & & & & \end{array}$$

take any (row) index n

we prove $e_n \sqsubseteq e$

$$\{e_{n,m}\}_{m \in \mathbb{N}} \subseteq \{e_{n,m}\}_{n,m \in \mathbb{N}}$$

a row

the matrix

e is an u.b. of $\{e_{n,m}\}_{m \in \mathbb{N}}$

$$e_n = \bigsqcup_{m \in \mathbb{N}} e_{n,m} \text{ is the lub}$$

therefore $e_n \sqsubseteq e$

(i)

$$\{e_{n,m}\}_{n,m \in \mathbb{N}}$$

same u.b. as

$$\{e_n\}_{n \in \mathbb{N}}$$

where $e_n = \bigsqcup_{m \in \mathbb{N}} e_{n,m}$

2. take an upper bound e of $\{e_n\}_{n \in \mathbb{N}}$

we want to prove it is an upper bound for $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

e

take any indices n, m

we prove $e_{n,m} \sqsubseteq e$

$$e_{n,m} \sqsubseteq \bigsqcup_{m \in \mathbb{N}} e_{n,m} = e_n \sqsubseteq e$$

one element
of a row

the lub
of that row

$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\sqcup$	$\sqcup$	$\sqcup$		$\sqcup$
$e_{n,0} \sqsubseteq e_{n,1} \sqsubseteq e_{n,2} \sqsubseteq \cdots \sqsubseteq e_{n,m} \sqsubseteq \cdots$				$e_n = \bigsqcup_{m \in \mathbb{N}} e_{n,m}$
$\sqcup$	$\sqcup$	$\sqcup$		$\sqcup$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\sqcup$	$\sqcup$	$\sqcup$		$\sqcup$
$e_{2,0} \sqsubseteq e_{2,1} \sqsubseteq e_{2,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots$				$e_2 = \bigsqcup_{m \in \mathbb{N}} e_{2,m}$
$\sqcup$	$\sqcup$	$\sqcup$		$\sqcup$
$e_{1,0} \sqsubseteq e_{1,1} \sqsubseteq e_{1,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots$				$e_1 = \bigsqcup_{m \in \mathbb{N}} e_{1,m}$
$\sqcup$	$\sqcup$	$\sqcup$		$\sqcup$
$e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots$				$e_0 = \bigsqcup_{m \in \mathbb{N}} e_{0,m}$

(ii)

$\{e_{n,m}\}_{n,m \in \mathbb{N}}$ same u.b. as

$$\left\{ \bigsqcup_{n \in \mathbb{N}} e_{n,m} \right\}_{m \in \mathbb{N}}$$

the proof is analogous to the previous case
(reason by columns, not by rows)

$$\bigsqcup_{n \in \mathbb{N}} e_{n,0} \quad \bigsqcup_{n \in \mathbb{N}} e_{n,1} \quad \bigsqcup_{n \in \mathbb{N}} e_{n,2} \quad \bigsqcup_{n \in \mathbb{N}} e_{n,m}$$

$$\begin{array}{ccccccc} \vdots & \vdots & \vdots & & \vdots & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ e_{n,0} \sqsubseteq e_{n,1} \sqsubseteq e_{n,2} \sqsubseteq \cdots \sqsubseteq e_{n,m} \sqsubseteq \cdots & & & & & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ e_{2,0} \sqsubseteq e_{2,1} \sqsubseteq e_{2,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots & & & & & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ e_{1,0} \sqsubseteq e_{1,1} \sqsubseteq e_{1,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots & & & & & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \cdots \sqsubseteq e_{2,m} \sqsubseteq \cdots & & & & & & \end{array}$$

(iii)

$\{e_{n,m}\}_{n,m \in \mathbb{N}}$

same u.b. as

$\{e_{k,k}\}_{k \in \mathbb{N}}$

1. take an upper bound e of $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

we want to prove it is an upper bound for $\{e_{k,k}\}_{k \in \mathbb{N}}$

e

but this is immediate, because

$$\{e_{k,k}\}_{k \in \mathbb{N}} \subseteq \{e_{n,m}\}_{n,m \in \mathbb{N}}$$

the diagonal

the whole matrix

The diagram shows an infinite matrix with rows and columns indexed by natural numbers. The elements are arranged in a grid where each row n contains elements $e_{n,0}, e_{n,1}, e_{n,2}, \dots, e_{n,m}, \dots$. The diagonal elements $e_{0,0}, e_{1,1}, e_{2,2}, \dots$ are highlighted by a blue diagonal band. The entire matrix is enclosed in a green rounded rectangle, and the diagonal band is highlighted in blue.

$$\begin{array}{ccccccc} \vdots & \vdots & \vdots & & \vdots & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ e_{n,0} \sqsubseteq e_{n,1} \sqsubseteq e_{n,2} \sqsubseteq \dots \sqsubseteq e_{n,m} \sqsubseteq \dots & & & & & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ e_{2,0} \sqsubseteq e_{2,1} \sqsubseteq e_{2,2} \sqsubseteq \dots \sqsubseteq e_{2,m} \sqsubseteq \dots & & & & & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ e_{1,0} \sqsubseteq e_{1,1} \sqsubseteq e_{1,2} \sqsubseteq \dots \sqsubseteq e_{2,m} \sqsubseteq \dots & & & & & & \\ \sqcup & \sqcup & \sqcup & & \sqcup & & \\ e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \dots \sqsubseteq e_{2,m} \sqsubseteq \dots & & & & & & \end{array}$$

(iii)

$\{e_{n,m}\}_{n,m \in \mathbb{N}}$

same u.b. as

$\{e_{k,k}\}_{k \in \mathbb{N}}$

2. take an upper bound e of $\{e_{k,k}\}_{k \in \mathbb{N}}$

we want to prove it is an upper bound for $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

e

take any indices n, m

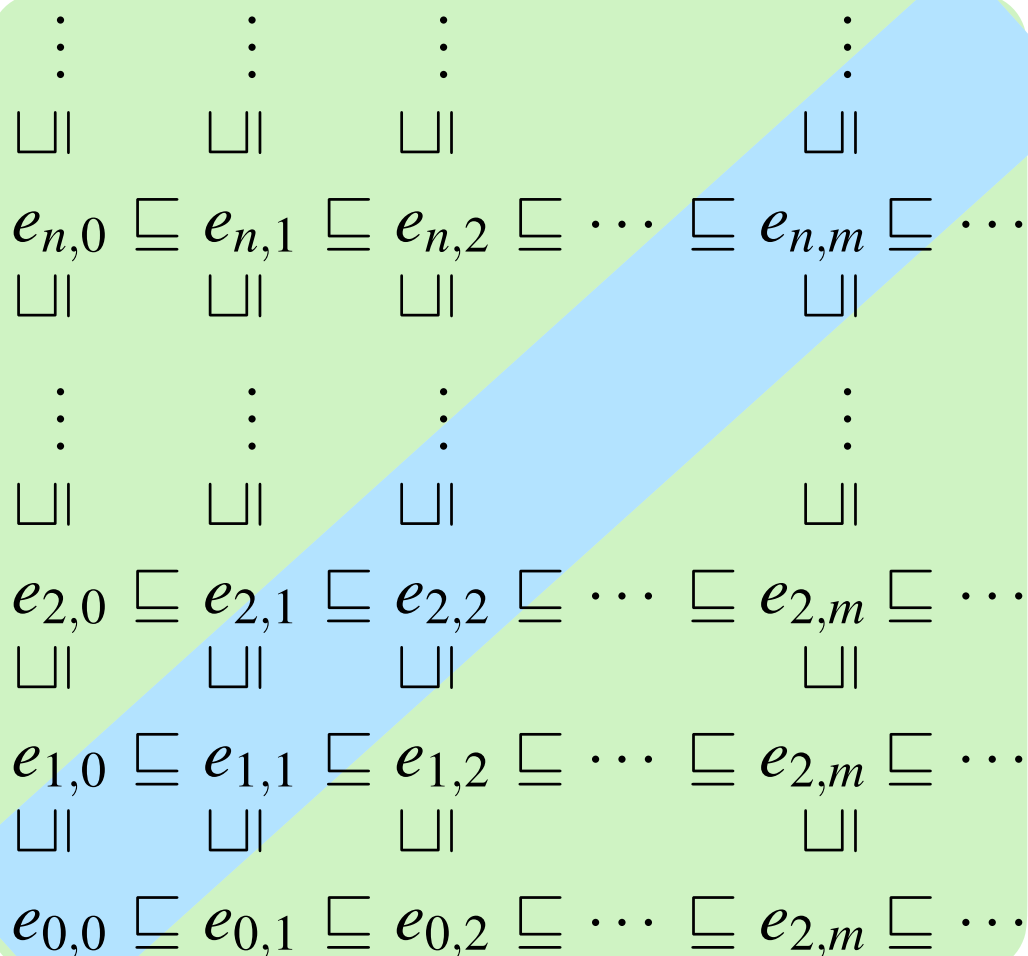
we prove $e_{n,m} \sqsubseteq e$

let $k = \max\{n, m\}$

$e_{n,m} \sqsubseteq e_{k,k} \sqsubseteq e$

$n \leq k \wedge m \leq k$

e is an u.b. of $\{e_{k,k}\}_{k \in \mathbb{N}}$



Switch Lemma: recap

$$\{e_{n,m}\}_{n,m \in \mathbb{N}}$$

$$e_{n,m} \sqsubseteq e_{n',m'} \quad \text{if} \quad n \leq n' \wedge m \leq m'$$

same set of upper bounds as

$$\left\{ \bigsqcup_{m \in \mathbb{N}} e_{n,m} \right\}_{n \in \mathbb{N}} \quad \{e_{k,k}\}_{k \in \mathbb{N}} \quad \left\{ \bigsqcup_{n \in \mathbb{N}} e_{n,m} \right\}_{m \in \mathbb{N}}$$

$$\bigsqcup_{n \in \mathbb{N}} \bigsqcup_{m \in \mathbb{N}} e_{n,m} = \bigsqcup_{k \in \mathbb{N}} e_{k,k} = \bigsqcup_{m \in \mathbb{N}} \bigsqcup_{n \in \mathbb{N}} e_{n,m}$$

Functional domains

Function space

$$\mathcal{D} = (D, \sqsubseteq_D)$$

$$\mathcal{E} = (E, \sqsubseteq_E) \quad \text{CPO}_\perp \Rightarrow [\mathcal{D} \rightarrow \mathcal{E}] = ([D \rightarrow E] , \sqsubseteq_{[D \rightarrow E]})$$

$$D \rightarrow E \triangleq \{ f \mid f : D \rightarrow E \}$$

$$[D \rightarrow E] \triangleq \{ f \mid f : D \rightarrow E , f \text{ continuous} \}$$

how to order functions?

$$f \sqsubseteq_{[D \rightarrow E]} g \quad \text{iff} \quad \forall d \in D. f(d) \sqsubseteq_E g(d)$$

Example

$$f \sqsubseteq_{[D \rightarrow E]} g \quad \text{iff} \quad \forall d \in D. f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}$$

$$f(x) \triangleq 0 \qquad f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}]} g \qquad g(x) \triangleq \begin{cases} 1 & x \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$



not continuous!

$$f(1) = 0 \not\sqsubseteq_{\mathbb{Z}_{\perp}} 1 = g(1)$$

total functions on $\mathbb{Z}_{\perp}$ are not comparable

(unless they are equal)

any total function is maximal in $\mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}$

Example

$$f \sqsubseteq_{[D \rightarrow E]} g \quad \text{iff} \quad \forall d \in D. f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp$$

$$f(x) \triangleq \begin{cases} 1 & x \text{ odd} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases}$$

$$g(x) \triangleq \begin{cases} 1 & x \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]} g$$

not continuous!



Example

$$f \sqsubseteq_{[D \rightarrow E]} g \quad \text{iff} \quad \forall d \in D. f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}$$

$$f(x) \triangleq \begin{cases} 1 & x \text{ odd} \\ \perp_{\mathbb{Z}_{\perp}} & \text{otherwise} \end{cases} \qquad g(x) \triangleq \begin{cases} 0 & x \text{ even} \\ \perp_{\mathbb{Z}_{\perp}} & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}]} g$$



Example

$$f \sqsubseteq_{[D \rightarrow E]} g \quad \text{iff} \quad \forall d \in D. f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}$$

$$f(x) \triangleq \begin{cases} x! & 1 \leq x \leq 10 \\ \perp_{\mathbb{Z}_{\perp}} & \text{otherwise} \end{cases} \quad g(x) \triangleq \begin{cases} x! & 1 \leq x \leq 15 \\ \perp_{\mathbb{Z}_{\perp}} & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}]} g$$



Example

$$f \sqsubseteq_{[D \rightarrow E]} g \quad \text{iff} \quad \forall d \in D. f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp$$

$$f(x, y) \triangleq \begin{cases} x * y & x, y \neq \perp_{\mathbb{Z}_\perp} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases} \quad g(x, y) \triangleq \begin{cases} (x * y)^2 & x, y \neq \perp_{\mathbb{Z}_\perp} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \times \mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]} g$$



Example

$$f \sqsubseteq_{[D \rightarrow E]} g \quad \text{iff} \quad \forall d \in D. f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp$$

$$f(x, y) \triangleq \begin{cases} x * y & x, y \neq \perp_{\mathbb{Z}_\perp} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases} \quad g(x, y) \triangleq \begin{cases} x * y & x, y \neq \perp_{\mathbb{Z}_\perp} \\ 0 & x = \perp_{\mathbb{Z}_\perp} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases}$$

$$f \stackrel{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \times \mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp]} g$$

yes (as functions)

but is g continuous?

$$g(\perp, \perp) = 0 \qquad g(1, 1) = 1$$

not even monotone!

Example

$$f \sqsubseteq_{[D \rightarrow E]} g \quad \text{iff} \quad \forall d \in D. f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp \times \mathbb{Z}_\perp$$

$$f(x) \triangleq (\perp_{\mathbb{Z}_\perp}, x) \qquad f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp \times \mathbb{Z}_\perp]} g \qquad g(x) \triangleq (x, x)$$



Example

$$f \sqsubseteq_{[D \rightarrow E]} g \quad \text{iff} \quad \forall d \in D. f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp \times \mathbb{Z}_\perp$$

$$f(x) \triangleq (\perp_{\mathbb{Z}_\perp}, x) \qquad f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \rightarrow \mathbb{Z}_\perp \times \mathbb{Z}_\perp]} g \qquad g(x) \triangleq (x, \perp_{\mathbb{Z}_\perp})$$



$$f(0) = (\perp_{\mathbb{Z}_\perp}, 0) \not\sqsubseteq_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (0, \perp_{\mathbb{Z}_\perp}) = g(0)$$

Functional CPO

$$[\mathcal{D} \rightarrow \mathcal{E}] = ([D \rightarrow E] , \sqsubseteq_{[D \rightarrow E]})$$

is it a partial order?

reflexivity, antisymmetry, transitivity of $\sqsubseteq_{[D \rightarrow E]}$
follow immediately from those of $\sqsubseteq_E$

is there a bottom element?

let $\perp_{[D \rightarrow E]} = \lambda d. \perp_E$

take any function $f \in [D \rightarrow E]$

for any $d \in D$ we have $\perp_{[D \rightarrow E]} d = \perp_E \sqsubseteq_E f(d)$

Functional CPO (ctd)

$$[\mathcal{D} \rightarrow \mathcal{E}] = ([D \rightarrow E] , \sqsubseteq_{[D \rightarrow E]})$$

is it complete?

first we show that any chain of functions
(not necessarily continuous)
has a limit in $D \rightarrow E$

then we show that the limit in $D \rightarrow E$
of any chain of continuous functions
is also continuous

Functional CPO (ctd)

$\{f_n : D \rightarrow E\}_{n \in \mathbb{N}}$ a chain of functions
(not necessarily continuous)

we prove its lub is $h \triangleq \lambda d. \bigsqcup_{n \in \mathbb{N}} f_n(d)$

$$\text{i.e. } h(d) \triangleq \bigsqcup_{n \in \mathbb{N}} f_n(d)$$

1. it is an upper bound of the chain
2. it is smaller than or equal to any other upper bound

Functional CPO (ctd)

take a chain $\{f_n : D \rightarrow E\}_{n \in \mathbb{N}}$ (not necessarily continuous)

1. $h \triangleq \lambda d. \bigsqcup_{n \in \mathbb{N}} f_n(d)$ is an upper bound of the chain

take any $n \in \mathbb{N}$

for any $d \in D$ $f_n(d) \sqsubseteq_E \bigsqcup_{n \in \mathbb{N}} f_n(d) = h(d)$

therefore $f_n \sqsubseteq_{D \rightarrow E} h$

Functional CPO (ctd)

take a chain $\{f_n : D \rightarrow E\}_{n \in \mathbb{N}}$ (not necessarily continuous)

2. $h \triangleq \lambda d. \bigsqcup_{n \in \mathbb{N}} f_n(d)$ is the least among upper bounds

take any g such that $\forall n. f_n \sqsubseteq_{D \rightarrow E} g$

we want to prove $h \sqsubseteq_{D \rightarrow E} g$

take any $d \in D$ $\forall n. f_n(d) \sqsubseteq_E g(d)$

thus $g(d)$ is an u.b. of $\{f_n(d)\}_{n \in \mathbb{N}}$

and therefore $h(d) = \bigsqcup_{n \in \mathbb{N}} f_n(d) \sqsubseteq_E g(d)$

Functional CPO (ctd)

TH. take a chain $\{f_n : D \rightarrow E\}_{n \in \mathbb{N}}$ of continuous functions
then $h \triangleq \lambda d. \bigsqcup_{n \in \mathbb{N}} f_n(d)$ is continuous

proof. let $\{d_i\}_{i \in \mathbb{N}}$ a chain in D

we prove $h \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) = \bigsqcup_{i \in \mathbb{N}} h(d_i)$

$$\begin{aligned} h \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) &= \bigsqcup_{n \in \mathbb{N}} f_n \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) && \text{by def of } h \\ &= \bigsqcup_{n \in \mathbb{N}} \bigsqcup_{i \in \mathbb{N}} f_n(d_i) && \text{by continuity of } f_n \\ &= \bigsqcup_{i \in \mathbb{N}} \bigsqcup_{n \in \mathbb{N}} f_n(d_i) && \text{by switch lemma (applicable?)} \\ &= \bigsqcup_{i \in \mathbb{N}} h(d_i) && \text{by def of } h \end{aligned}$$

Functional CPO (ctd)

if $n \leq m \wedge i \leq j$ then $f_n(d_i) \sqsubseteq_E f_m(d_j)$? 

$\Downarrow$

$$f_n \sqsubseteq_{[D \rightarrow E]} f_m \wedge d_i \sqsubseteq d_j$$

$$f_n(d_i) \sqsubseteq_E f_n(d_j) \sqsubseteq_E f_m(d_j)$$

$$\begin{array}{c} f_n \\ \text{monotone} \end{array} \quad \begin{array}{c} | \\ f_n \sqsubseteq_{[D \rightarrow E]} f_m \end{array}$$

$$= \bigsqcup_{n \in \mathbb{N}} \bigsqcup_{i \in \mathbb{N}} f_n(d_i)$$



$$= \bigsqcup_{i \in \mathbb{N}} \bigsqcup_{n \in \mathbb{N}} f_n(d_i)$$

by switch lemma (applicable?)

Functional CPO (ctd)

TH. $[\mathcal{D} \rightarrow \mathcal{E}] = ([D \rightarrow E] , \sqsubseteq_{[D \rightarrow E]})$ is complete

proof. take a chain $\{f_n : [D \rightarrow E]\}_{n \in \mathbb{N}}$

$h \triangleq \lambda d. \bigsqcup_{n \in \mathbb{N}} f_n(d)$ is the lub in $D \rightarrow E$
is continuous $h \in [D \rightarrow E]$

since $[D \rightarrow E] \subseteq D \rightarrow E$

h is the lub in $[D \rightarrow E]$

Functional CPO: recap

$$[\mathcal{D} \rightarrow \mathcal{E}] = ([D \rightarrow E] , \sqsubseteq_{[D \rightarrow E]})$$

$$f \sqsubseteq_{[D \rightarrow E]} g \quad \text{iff} \quad \forall d \in D. f(d) \sqsubseteq_E g(d)$$

$$\perp_{[D \rightarrow E]} \triangleq \lambda d. \perp_E$$

$$\bigsqcup_{n \in \mathbb{N}} f_n \triangleq \lambda d. \bigsqcup_{n \in \mathbb{N}} f_n(d)$$

$$f \in [D \rightarrow E], g \in [E \rightarrow F] \quad \Rightarrow \quad g \circ f \in [D \rightarrow F]$$

the composition of continuous functions is continuous