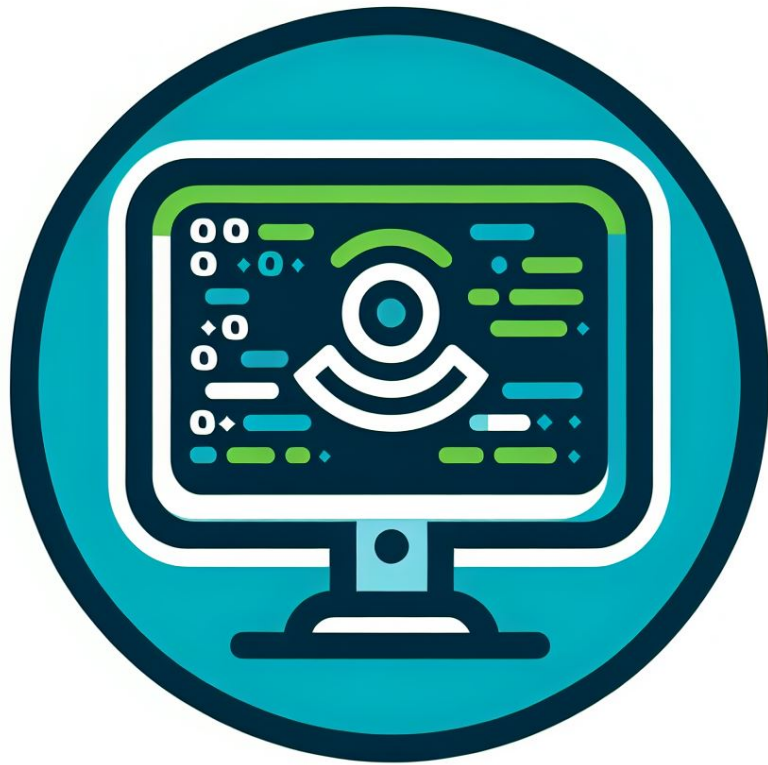




PSC 2024/25 (375AA, 9CFU)

Principles for Software Composition

Roberto Bruni

<http://www.di.unipi.it/~bruni/>

[http://didawiki.di.unipi.it/doku.php/
magistraleinformatica/psc/start](http://didawiki.di.unipi.it/doku.php/magistraleinformatica/psc/start)

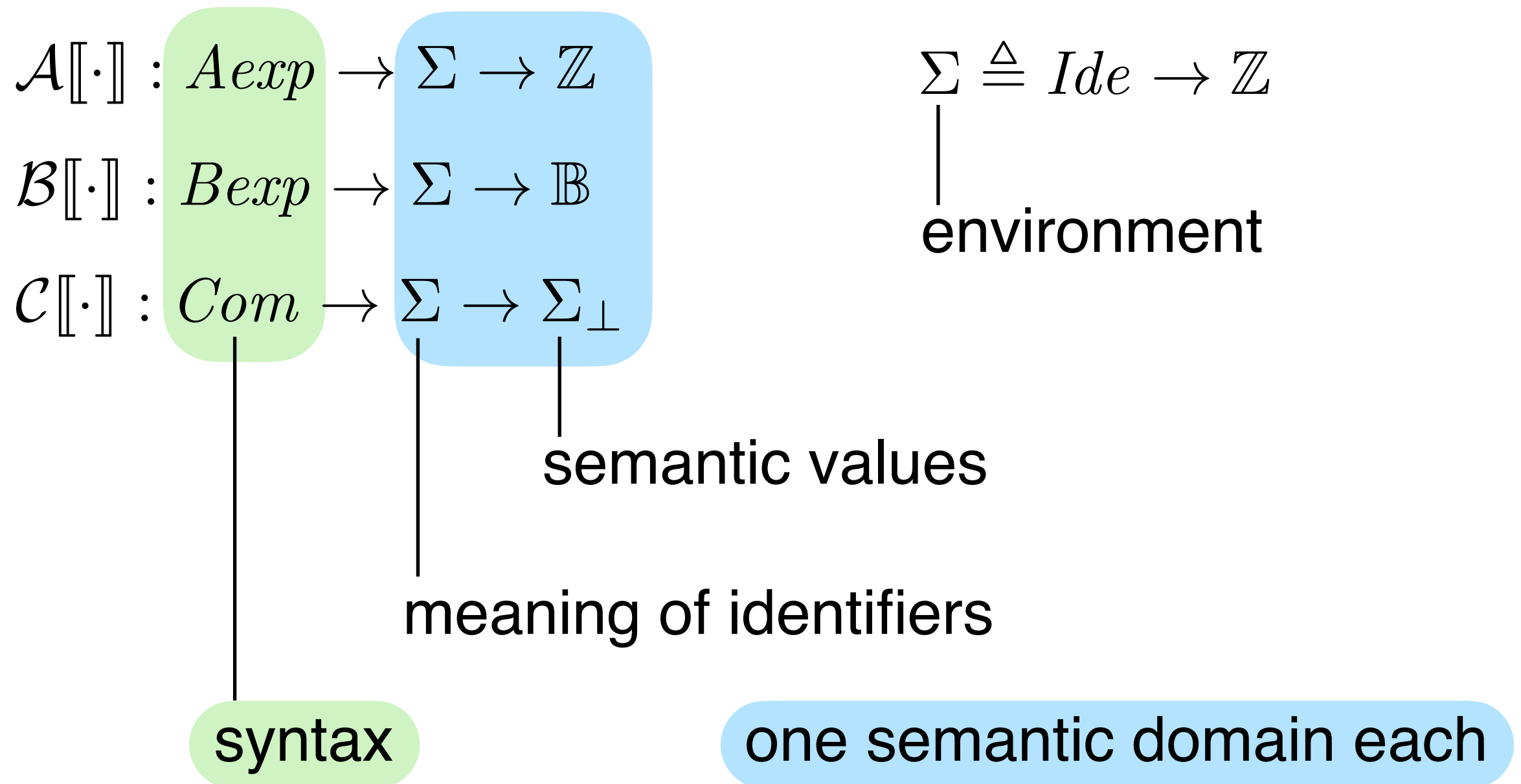
13a - Cartesian domains

HOFL

Towards a denotational semantics

Imp

three syntactic categories (types) $Aexp$, $Bexp$, Com
one interpretation function each



HOF_L

one syntactic category for pre-terms T

infinitely many types $\tau ::= int \mid \tau_0 * \tau_1 \mid \tau_0 \rightarrow \tau_1$

infinitely many categories for typeable terms T_τ

one semantic domain each D_τ

one parametric interpretation function $\llbracket \cdot \rrbracket$

variables also have different types $x : \tau$

the environment must be type-sensitive ρ

Requirements

$t : \tau$ $\llbracket t \rrbracket \rho \in D_\tau$ a domain for each type!

environment $\rho : Var \rightarrow \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $x : \tau \Rightarrow \rho(x) \in D_\tau$

t may diverge (e.g. `rec x. x`) $\Rightarrow D_\tau$ must include a bottom element $\perp_{D_\tau}$

Requirements

$$t : \tau \quad \llbracket t \rrbracket \rho \in D_\tau \quad \text{a domain for each type!}$$

/

environment

$$\rho : Var \rightarrow \bigcup_{\tau \in \mathcal{T}} D_\tau$$

type consistent
assignment of
values to variables

$$x : \tau \Rightarrow \rho(x) \in D_\tau$$

$$\llbracket \mathbf{rec} \ x. \ t \rrbracket \rho = \llbracket t \rrbracket \rho[\llbracket \mathbf{rec} \ x. \ t \rrbracket \rho / x]$$

$$\Gamma_{x,t} \triangleq \lambda d. \llbracket t \rrbracket \rho[d / x]$$

$$\llbracket \mathbf{rec} \ x. \ t \rrbracket \rho = \Gamma_{x,t} (\llbracket \mathbf{rec} \ x. \ t \rrbracket \rho)$$

to solve recursive equations:

$$\llbracket \mathbf{rec} \ x. \ t \rrbracket = \text{fix } \Gamma_{x,t}$$

D_τ must be a $\text{CPO}_\perp$

$\Gamma_{x,t}$ must be continuous

Requirements

$$t : \tau \quad \llbracket t \rrbracket \rho \in D_\tau \quad \text{a domain for each type!}$$

/

environment

$$\rho : Var \rightarrow \bigcup_{\tau \in \mathcal{T}} D_\tau$$

type consistent
assignment of
values to variables

$$x : \tau \Rightarrow \rho(x) \in D_\tau$$

$$\tau ::= int \mid \tau_0 * \tau_1 \mid \tau_0 \rightarrow \tau_1$$

we must be able to combine $\text{CPO}_\perp$
using cartesian product and function spaces

Requirements

$$t : \tau \quad \llbracket t \rrbracket \rho \in D_\tau \quad \text{a domain for each type!}$$

/

environment

$$\rho : Var \rightarrow \bigcup_{\tau \in \mathcal{T}} D_\tau$$

type consistent
assignment of
values to variables

$$x : \tau \Rightarrow \rho(x) \in D_\tau$$

$$\tau ::= int \quad | \quad \tau_0 * \tau_1 \quad | \quad \tau_0 \rightarrow \tau_1$$

choose D_{int}

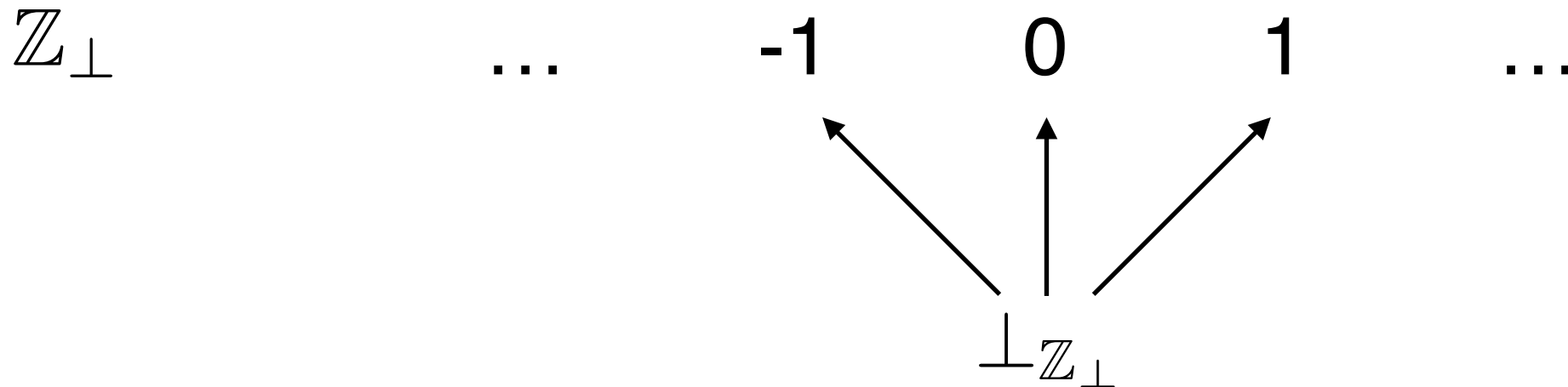
given D_{τ_0}, D_{τ_1} build $D_{\tau_0 * \tau_1} \quad D_{\tau_0 \rightarrow \tau_1}$

Flat domain of Integers

Flat domain of Integers

$\mathbb{Z}$... -1 0 1 ...

Flat domain of Integers



PO: flat order

bottom: any flat order has bottom

completeness: any flat order is complete
(only finite chains are possible)

Strict extensions

$$\text{op} \in \{+, -, \times\}$$

$$\text{op} : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$$

$$\underline{\text{op}}_{\perp} : \mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp} \rightarrow \mathbb{Z}_{\perp}$$

$$1 \underline{+}_{\perp} \perp = \perp$$

$$\perp \underline{\times}_{\perp} 5 = \perp$$

$$\perp \underline{=}_{\perp} \perp = \perp$$

$$v_1 \underline{\text{op}}_{\perp} v_2 \triangleq \begin{cases} v_1 \underline{\text{op}} v_2 & \text{if } v_1, v_2 \in \mathbb{Z} \\ \perp_{\mathbb{Z}_{\perp}} & \text{otherwise } (v_1 = \perp_{\mathbb{Z}_{\perp}} \text{ or } v_2 = \perp_{\mathbb{Z}_{\perp}}) \end{cases}$$

called *strict* extension

to prove: $\underline{\text{op}}_{\perp}$ is monotone and continuous

is $\mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp}$ a $\text{CPO}_{\perp}$?

Cartesian product of domains

Cartesian product

$$\mathcal{D} = (D, \sqsubseteq_D)$$

$$\mathcal{E} = (E, \sqsubseteq_E) \quad \text{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} \times \mathcal{E} = (D \times E, \sqsubseteq_{D \times E})$$

how to order pairs?

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1) \quad \text{iff} \quad d_0 \sqsubseteq_D d_1 \wedge e_0 \sqsubseteq_E e_1$$

example $\mathbb{Z}_\perp \times \mathbb{Z}_\perp$

$$(0, 1) \stackrel{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (1, 2)$$



$$(\perp_{\mathbb{Z}_\perp}, 1) \stackrel{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (1, 1)$$



$$(2, \perp_{\mathbb{Z}_\perp}) \stackrel{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (2, 0)$$



$$(0, \perp_{\mathbb{Z}_\perp}) \stackrel{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (\perp_{\mathbb{Z}_\perp}, 0)$$



Cartesian CPO

$$\mathcal{D} \times \mathcal{E} = (D \times E , \sqsubseteq_{D \times E})$$

is it a partial order?

reflexivity, antisymmetry, transitivity of $\sqsubseteq_{D \times E}$
follow immediately from those of $\sqsubseteq_D \quad \sqsubseteq_E$

is there a bottom element?

let $\perp_{D \times E} = (\perp_D, \perp_E)$

take any pair $(d, e) \in D \times E$

since $\begin{array}{l} \perp_D \sqsubseteq_D d \\ \perp_E \sqsubseteq_E e \end{array}$ then $\perp_{D \times E} = (\perp_D, \perp_E) \sqsubseteq_{D \times E} (d, e)$

Cartesian CPO (ctd)

$$\mathcal{D} \times \mathcal{E} = (D \times E , \sqsubseteq_{D \times E})$$

is it complete?

take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$ we need to find its lub

we prove its lub is $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$

1. it is an upper bound of the chain
2. it is smaller than or equal to any other upper bound

Cartesian CPO (ctd)

$\mathcal{D} \times \mathcal{E} = (D \times E , \sqsubseteq_{D \times E})$ take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$

1. $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$ is an upper bound of the chain

take a generic element of the chain (d_j, e_j)

we have $d_j \sqsubseteq_D \bigsqcup_{i \in \mathbb{N}} d_i$ thus $(d_j, e_j) \sqsubseteq_{D \times E} \left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$
 $e_j \sqsubseteq_E \bigsqcup_{i \in \mathbb{N}} e_i$

Cartesian CPO (ctd)

$\mathcal{D} \times \mathcal{E} = (D \times E , \sqsubseteq_{D \times E})$ take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$

2. $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$ is the least among upper bounds

take a generic upper bound (d, e) : $\forall i \in \mathbb{N}. (d_i, e_i) \sqsubseteq_{D \times E} (d, e)$

by def $\forall i \in \mathbb{N}. d_i \sqsubseteq_D d \wedge \forall i \in \mathbb{N}. e_i \sqsubseteq_E e$

i.e., d is an upper bound of $\{d_i\}_{i \in \mathbb{N}}$ $\Rightarrow$ $\bigsqcup_{i \in \mathbb{N}} d_i \sqsubseteq_D d$
 e is an upper bound of $\{e_i\}_{i \in \mathbb{N}}$ $\Rightarrow$ $\bigsqcup_{i \in \mathbb{N}} e_i \sqsubseteq_E e$

hence $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right) \sqsubseteq_{D \times E} (d, e)$

Cartesian CPO: recap

$$\mathcal{D} \times \mathcal{E} = (D \times E , \sqsubseteq_{D \times E})$$

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1) \quad \text{iff} \quad d_0 \sqsubseteq_D d_1 \wedge e_0 \sqsubseteq_E e_1$$

$$\perp_{D \times E} \triangleq (\perp_D, \perp_E)$$

$$\bigsqcup_{i \in \mathbb{N}} (d_i, e_i) \triangleq \left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$$

is $\mathbb{Z}_\perp \times \mathbb{Z}_\perp$ a $\text{CPO}_\perp$?



Projections

$$\pi_1 : D \times E \rightarrow D$$

$$\pi_1(d, e) = d$$

$$\pi_2 : D \times E \rightarrow E$$

$$\pi_2(d, e) = e$$

TH. projections are monotone

proof. take $(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1)$

we want to prove $\pi_1(d_0, e_0) \sqsubseteq_D \pi_1(d_1, e_1)$

$$\pi_2(d_0, e_0) \sqsubseteq_E \pi_2(d_1, e_1)$$

$$\pi_1(d_0, e_0) = d_0 \sqsubseteq_D d_1 = \pi_1(d_1, e_1)$$

$\Uparrow$

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1)$$

the case of π_2 is analogous

Projections (ctd)

$$\pi_1 : D \times E \rightarrow D$$

$$\pi_1(d, e) = d$$

$$\pi_2 : D \times E \rightarrow E$$

$$\pi_2(d, e) = e$$

TH. projections are continuous

proof. take $\{(d_i, e_i)\}_{i \in \mathbb{N}}$

we want to prove

$$\pi_1 \left(\bigsqcup_{i \in \mathbb{N}} (d_i, e_i) \right) = \bigsqcup_{i \in \mathbb{N}} \pi_1(d_i, e_i)$$

$$\pi_1 \left(\bigsqcup_{i \in \mathbb{N}} (d_i, e_i) \right) = \pi_1 \left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right) = \bigsqcup_{i \in \mathbb{N}} d_i = \bigsqcup_{i \in \mathbb{N}} \pi_1(d_i, e_i)$$

by def
of lub

by def
of π_1

by def
of π_1

the case of π_2 is analogous

Exercise: smashed prod

$$\mathcal{D} = (D, \sqsubseteq_D)$$

$$\mathcal{E} = (E, \sqsubseteq_E) \quad \text{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} \otimes \mathcal{E} = (D \otimes E, \sqsubseteq_{D \otimes E})$$

$$D \otimes E \triangleq \{(d, e) \mid (d, e) \in D \times E, d = \perp_D \Leftrightarrow e = \perp_E\}$$

how to order pairs?

bottom element?

complete order?