PSC 2024/25 (375AA, 9CFU)

Principles for Software Composition

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05b - More Induction

Determinacy by structural induction

Termination

A result can always be returned

Determinacy

Any two results are the same

Termination vs determinacy

	termination?	determinacy?
$x + y$	✓	✓
x^y	✗	✓
$rand()$	✓	✗
$\pm sqrt(x)$	✗	✗

Determinacy of arithmetic expressions

$$a ::= x \mid n \mid a \ \mathsf{op} \ a$$

$$x \in \mathsf{Ide} \qquad \mathsf{op} \in \{+, \times, -\}$$

$$n \in \mathbb{Z} \qquad \mathbb{M} \triangleq \{\sigma \mid \sigma : \mathsf{Ide} \to \mathbb{Z}\}$$

$$\frac{}{\langle x, \sigma\rangle \longrightarrow \sigma(x)} \qquad \frac{}{\langle n, \sigma\rangle \longrightarrow n} \qquad \frac{\langle a_0, \sigma\rangle \longrightarrow n_0 \quad \langle a_1, \sigma\rangle \longrightarrow n_1}{\langle a_0 \ \mathsf{op} \ a_1, \sigma\rangle \longrightarrow n_0 \ op \ n_1}$$

$$P(a) \triangleq \forall \sigma \in \mathbb{M}.\ \forall m, m' \in \mathbb{Z}.\ \langle a, \sigma\rangle \longrightarrow m \wedge \langle a, \sigma\rangle \longrightarrow m' \Rightarrow m = m'$$

$$\forall a.\ P(a)\ ?$$

Structural induction principle

$$\frac{\forall a_0, a_1.\ P(a_0) \wedge P(a_1) \Rightarrow P(a_0\ \mathsf{op}\ a_1)}{\forall a.\ P(a)}$$

$\forall x \in \mathsf{Ide}.\ P(x)$ $\qquad$ $\forall n \in \mathbb{Z}.\ P(n)$

Base case

$\forall x \in \mathsf{Ide}.\ P(x)$

Take a generic $x \in \mathsf{Ide}$

We want to prove

$$P(x) \triangleq \forall \sigma, m, m'.\ \langle x, \sigma \rangle \longrightarrow m \wedge \langle x, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

Take generic σ, m, m' s.t. $\langle x, \sigma \rangle \longrightarrow m$ and $\langle x, \sigma \rangle \longrightarrow m'$

We want to prove $m = m'$

Consider the goal $\langle x, \sigma \rangle \longrightarrow m$

Only the rule $\dfrac{}{\langle x, \sigma \rangle \longrightarrow \sigma(x)}$ is applicable, hence $m = \sigma(x)$

Similarly, since $\langle x, \sigma \rangle \longrightarrow m'$ it must be $m' = \sigma(x)$

and thus we conclude $m = m'$

Base case

$\forall n \in \mathbb{Z}.\ P(n)$

Take a generic $n \in \mathbb{Z}$

We want to prove

$$P(n) \triangleq \forall \sigma, m, m'.\ \langle n, \sigma \rangle \longrightarrow m \wedge \langle n, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

Take generic σ, m, m' s.t. $\langle n, \sigma \rangle \longrightarrow m$ and $\langle n, \sigma \rangle \longrightarrow m'$

We want to prove $m = m'$

Consider the goal $\langle n, \sigma \rangle \longrightarrow m$

Only the rule $\dfrac{}{\langle n, \sigma \rangle \longrightarrow n}$ is applicable, hence $m = n$

Similarly, since $\langle n, \sigma \rangle \longrightarrow m'$ it must be $m' = n$

and thus we conclude $m = m'$

Inductive case

$\forall a_0, a_1.\ P(a_0) \wedge P(a_1) \Rightarrow P(a_0 \text{ op } a_1)$ Take generic a_0, a_1

We assume (inductive hypotheses)

$$P(a_i) \triangleq \forall \sigma, m_i, m_i'.\ \langle a_i, \sigma \rangle \longrightarrow m_i \wedge \langle a_i, \sigma \rangle \longrightarrow m_i' \Rightarrow m_i = m_i'$$

We want to prove

$$P(a_0 \text{ op } a_1) \triangleq \forall \sigma, m, m'.\ \langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m \wedge \langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m' \Rightarrow m = m'$$

Take generic σ, m, m' such that $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m$ and $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m'$

We want to prove $m = m'$

Inductive case (ctd)

Consider the goal $\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow m$

Only the rule $\dfrac{\langle a_0, \sigma\rangle \longrightarrow n_0 \quad \langle a_1, \sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow n_0 \; op \; n_1}$ is applicable

hence $m = n_0 \; op \; n_1$ with $\langle a_0, \sigma\rangle \longrightarrow n_0$ and $\langle a_1, \sigma\rangle \longrightarrow n_1$

Similarly, since $\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow m'$

it must be $m' = n'_0 \; op \; n'_1$ with $\langle a_0, \sigma\rangle \longrightarrow n'_0$ and $\langle a_1, \sigma\rangle \longrightarrow n'_1$

By inductive hypotheses, $n_0 = n'_0$ and $n_1 = n'_1$

and thus we conclude $m = n_0 \; op \; n_1 = n'_0 \; op \; n'_1 = m'$

Many sorted signatures

Different types of expressions

$$e ::= x \mid n \mid e + e \mid e \leq e \mid e \,\&\, e$$

$$((x + 1) \leq (y - 1)) \,\&\, (x \leq 42)$$

$$(((((x + 1) \leq y) - 1) \,\&\, x) \leq 42$$

$$x + (1 \leq (y - (1 \,\&\, (x \leq 42))))$$

Many sorted signatures

$a ::= x \mid n \mid a + a$ arithmetic expressions

$b ::= a \leq a \mid b \,\&\, b$ Boolean expressions

Sorts: identifiers x

numerals n

arithmetic expressions a

Boolean expressions b

Operators:

- $\cdot + \cdot$ takes two arithmetic expressions and returns an arithmetic expression
- $\cdot \leq \cdot$ takes two arithmetic expressions and returns a Boolean expression
- $\cdot \,\&\, \cdot$ takes two Boolean expressions and returns a Boolean expression

Terms over a signature

$S = \{s, ...\}$ a set of sorts

$\Sigma = \{\Sigma_{s_1 ... s_n, s}\}_{s_1, ..., s_n, s \in S}$ a many sorted signature

$$f \in \Sigma_{s_1 ... s_n, s} \qquad f : (s_1 \times \cdots \times s_n) \to s$$

$T_{\Sigma, s}$ denotes the set of all terms of sort s

it is the least set such that:

- if $c \in \Sigma_{\epsilon, s}$, then $c \in T_{\Sigma, s}$
- if $f \in \Sigma_{s_1 ... s_n, s}$ and $\forall i.\ t_i \in T_{\Sigma, s_i}$, then $f(t_1, ..., t_n) \in T_{\Sigma, s}$

$T_\Sigma = \{T_{\Sigma, s}\}_{s \in S}$ denotes the set of all well-sorted terms

Boolean expressions

$x \in \mathsf{Ide} \quad n \in \mathbb{Z} \quad \mathsf{op} \in \{+, \times, -\}$

$v \in \mathbb{B} \quad \mathsf{bop} \in \{\wedge, \vee\} \quad \mathsf{cmp} \in \{<, \leq, >, \geq, =, \neq\}$

$a ::= x \mid n \mid a \ \mathsf{op} \ a$

$b ::= v \mid a \ \mathsf{cmp} \ a \mid \neg b \mid b \ \mathsf{bop} \ b$

$S \triangleq \{\mathsf{Aexp}, \mathsf{Bexp}\}$

$\Sigma_{\epsilon,\mathsf{Aexp}} \triangleq \mathsf{Ide} \cup \mathbb{Z}$

$\Sigma_{\epsilon,\mathsf{Bexp}} \triangleq \mathbb{B}$

$\Sigma_{\mathsf{Bexp},\mathsf{Bexp}} \triangleq \{\neg\}$

$\Sigma_{\mathsf{AexpAexp},\mathsf{Aexp}} \triangleq \{+, \times, -\}$

$\Sigma_{\mathsf{AexpAexp},\mathsf{Bexp}} \triangleq \{<, \leq, >, \geq, =, \neq\}$

$\Sigma_{\mathsf{BexpBexp},\mathsf{Bexp}} \triangleq \{\wedge, \vee\}$

Semantics of expressions

$$a ::= x \mid n \mid a \text{ op } a$$
$$b ::= v \mid a \text{ cmp } a \mid \neg b \mid b \text{ bop } b$$

$$\frac{}{\langle x, \sigma\rangle \longrightarrow \sigma(x)} \quad \frac{}{\langle n, \sigma\rangle \longrightarrow n} \quad \frac{\langle a_0, \sigma\rangle \longrightarrow n_0 \quad \langle a_1, \sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow n_0 \ op \ n_1}$$

$$\frac{}{\langle v, \sigma\rangle \longrightarrow v} \quad \frac{\langle b, \sigma\rangle \longrightarrow v}{\langle \neg b, \sigma\rangle \longrightarrow \neg v} \quad \frac{\langle a_0, \sigma\rangle \longrightarrow n_0 \quad \langle a_1, \sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ cmp } a_1, \sigma\rangle \longrightarrow n_0 \ cmp \ n_1}$$

$$\frac{\langle b_0, \sigma\rangle \longrightarrow v_0 \quad \langle b_1, \sigma\rangle \longrightarrow v_1}{\langle b_0 \text{ bop } b_1, \sigma\rangle \longrightarrow v_0 \ bop \ v_1}$$

Subterms

$$t_i \prec f(t_1, ..., t_n)$$

sometimes, an additional requirement:
sort-preserving relation

$$f \in \Sigma_{s_1 \ldots s_n, s} \ \wedge \ s_i = s$$

One sorted = Unsorted

a special case:

$S = \{*\}$ a singleton set of sorts

$$\Sigma_{\underbrace{*\ldots*}_{n},*} \simeq \Sigma_n$$

Termination of Boolean expressions

Termination of expressions

$$a ::= x \mid n \mid a \text{ op } a$$
$$b ::= v \mid a \text{ cmp } a \mid \neg b \mid b \text{ bop } b$$

$$\frac{}{\langle x, \sigma\rangle \longrightarrow \sigma(x)} \qquad \frac{}{\langle n, \sigma\rangle \longrightarrow n} \qquad \frac{\langle a_0, \sigma\rangle \longrightarrow n_0 \quad \langle a_1, \sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma\rangle \longrightarrow n_0 \; op \; n_1}$$

$$\frac{}{\langle v, \sigma\rangle \longrightarrow v} \qquad \frac{\langle b, \sigma\rangle \longrightarrow v}{\langle \neg b, \sigma\rangle \longrightarrow \neg v} \qquad \frac{\langle a_0, \sigma\rangle \longrightarrow n_0 \quad \langle a_1, \sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ cmp } a_1, \sigma\rangle \longrightarrow n_0 \; cmp \; n_1}$$

$$\frac{\langle b_0, \sigma\rangle \longrightarrow v_0 \quad \langle b_1, \sigma\rangle \longrightarrow v_1}{\langle b_0 \text{ bop } b_1, \sigma\rangle \longrightarrow v_0 \; bop \; v_1}$$

$P(b) \triangleq \forall \sigma \in \mathbb{M}.\ \exists v \in \mathbb{B}.\ \langle b, \sigma\rangle \longrightarrow v$ $\qquad$ $\forall b.\ P(b)$?

Base case

$\forall v \in \mathbb{B}.\ P(v)$ Take a generic $v \in \mathbb{B}$

We want to prove $P(v) \triangleq \forall \sigma.\ \exists u.\ \langle v, \sigma \rangle \longrightarrow u$

the only variable

Take a generic $\sigma \in \mathbb{M}$ and consider the goal $\langle v, \sigma \rangle \longrightarrow u$

By rule $\dfrac{}{\langle v, \sigma \rangle \longrightarrow v}$ we have $\langle v, \sigma \rangle \longrightarrow u \nwarrow_{[u=v]} \square$

And we are done (taking $u = v$)

A surprising base case

$\forall a_0, a_1.\ P(a_0 \text{ cmp } a_1)$ Take generic a_0, a_1

We want to prove $P(a_0 \text{ cmp } a_1) \triangleq \forall \sigma.\ \exists v.\ \langle a_0 \text{ cmp } a_1, \sigma \rangle \longrightarrow v$

Consider the goal $\langle a_0 \text{ cmp } a_1, \sigma \rangle \longrightarrow v$

By rule $\dfrac{\langle a_0, \sigma \rangle \longrightarrow n_0 \quad \langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \text{ cmp } a_1, \sigma \rangle \longrightarrow n_0\ cmp\ n_1}$ we have

$$\langle a_0 \text{ cmp } a_1, \sigma \rangle \longrightarrow v \nwarrow_{[v = n_0\ cmp\ n_1]} \langle a_0, \sigma \rangle \longrightarrow n_0, \langle a_1, \sigma \rangle \longrightarrow n_1$$

By termination of arithmetic expressions, such n_0, n_1 exist

And we are done (taking $v = n_0\ cmp\ n_1$)

Proof obligations

try to complete on your own
the missing inductive cases of the proof

$$\forall b.\ P(b) \Rightarrow P(\neg b)$$

$$\forall b_0, b_1.\ (P(b_0) \wedge P(b_1)) \Rightarrow P(b_0\ \mathsf{bop}\ b_1)$$

Commands

Commands

$$a ::= x \mid n \mid a + a \mid \cdots \qquad b ::= v \mid a \leq a \mid \cdots$$

$$c ::= \mathbf{skip} \mid x := a \mid c\,;c \mid \mathbf{if}\ b\ \mathbf{then}\ c\ \mathbf{else}\ c \mid \mathbf{while}\ b\ \mathbf{do}\ c$$

$S \triangleq \{\mathsf{Aexp}, \mathsf{Bexp}, \mathsf{Com}\}$

$\Sigma_{\epsilon,\mathsf{Aexp}} \triangleq \mathsf{Ide} \cup \mathbb{Z}$

$\Sigma_{\mathsf{AexpAexp},\mathsf{Aexp}} \triangleq \{+, \times, -\}$

$\Sigma_{\epsilon,\mathsf{Bexp}} \triangleq \mathbb{B}$

$\Sigma_{\mathsf{AexpAexp},\mathsf{Bexp}} \triangleq \{<, \leq, >, \geq, =, \neq\}$

$\Sigma_{\mathsf{Bexp},\mathsf{Bexp}} \triangleq \{\neg\}$

$\Sigma_{\mathsf{BexpBexp},\mathsf{Bexp}} \triangleq \{\wedge, \vee\}$

$\Sigma_{\epsilon,\mathsf{Com}} \triangleq \{\mathbf{skip}\}$

$\Sigma_{\mathsf{Aexp},\mathsf{Com}} \triangleq \{x := \mid x \in \mathsf{Ide}\}$

$\Sigma_{\mathsf{ComCom},\mathsf{Com}} \triangleq \{\ ;\ \}$

$\Sigma_{\mathsf{BexpComCom},\mathsf{Com}} \triangleq \{\ \mathbf{if}\ \}$

$\Sigma_{\mathsf{BexpCom},\mathsf{Com}} \triangleq \{\ \mathbf{while}\ \}$

Big-step operational semantics of commands

$\langle a, \sigma \rangle \longrightarrow n$ (the evaluation of a in σ returns the integer n)

$\langle b, \sigma \rangle \longrightarrow v$ (the evaluation of b in σ returns the Boolean v)

$\langle c, \sigma \rangle \longrightarrow \sigma'$ (the evaluation of c in σ returns the memory σ')

Memories

$\mathbb{M} \triangleq \{\, \sigma : \mathsf{Ide} \to \mathbb{Z} \mid \sigma \text{ has finite support} \,\}$

$\{x \in \mathsf{Ide} \mid \sigma(x) \neq 0\}$ is finite

$(n_1/x_1, ..., n_k/x_k) : \mathsf{Ide} \to \mathbb{Z}$

all different

$$(n_1/x_1, ..., n_k/x_k)(x) \triangleq \begin{cases} n_i & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases}$$

$\sigma_0 \triangleq ()$ is the typical initial memory

Memory updates

$$\sigma[n/y](x) \triangleq \begin{cases} n & \text{if } x = y \\ \sigma(x) & \text{otherwise} \end{cases}$$

$\forall \sigma, m, n, y.\ \sigma[m/y][n/y] = \sigma[n/y]$

$$\sigma[m/y][n/y](x) \triangleq \begin{cases} n & \text{if } x = y \\ \sigma[m/y](x) = \sigma(x) & \text{otherwise} \end{cases}$$

$\forall \sigma, m, n, y, z.\ \ y \neq z \ \Rightarrow\ \sigma[n/y][m/z] = \sigma[m/z][n/y]$

we write $\sigma[n/y, m/z]$ in such cases

$(n_1/x_1, ..., n_k/x_k) = \sigma_0[n_1/x_1, ..., n_k/x_k]$

Semantics of commands

$$c ::= \mathbf{skip} \mid x := a \mid c\,;c \mid \mathbf{if}\ b\ \mathbf{then}\ c\ \mathbf{else}\ c \mid \mathbf{while}\ b\ \mathbf{do}\ c$$

$$\frac{}{\langle \mathbf{skip}, \sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle a, \sigma\rangle \longrightarrow n}{\langle x := a, \sigma\rangle \longrightarrow \sigma[n/x]} \qquad \frac{\langle c_0, \sigma\rangle \longrightarrow \sigma'' \quad \langle c_1, \sigma''\rangle \longrightarrow \sigma'}{\langle c_0\,; c_1, \sigma\rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma\rangle \longrightarrow \mathbf{ff} \quad \langle c_1, \sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1, \sigma\rangle \longrightarrow \sigma'} \qquad \frac{\langle b, \sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c_0, \sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1, \sigma\rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma\rangle \longrightarrow \mathbf{ff}}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle b, \sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c, \sigma\rangle \longrightarrow \sigma'' \quad \langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma''\rangle \longrightarrow \sigma'}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma'}$$

Multiple rules for the same construct!

$$\frac{\langle b,\sigma\rangle \longrightarrow \mathbf{ff} \quad \langle c_1,\sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1,\sigma\rangle \longrightarrow \sigma'} \qquad \frac{\langle b,\sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c_0,\sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1,\sigma\rangle \longrightarrow \sigma'}$$

$$\frac{\langle b,\sigma\rangle \longrightarrow \mathbf{ff}}{\langle \mathbf{while}\ b\ \mathbf{do}\ c,\sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle b,\sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c,\sigma\rangle \longrightarrow \sigma'' \quad \langle \mathbf{while}\ b\ \mathbf{do}\ c,\sigma''\rangle \longrightarrow \sigma'}{\langle \mathbf{while}\ b\ \mathbf{do}\ c,\sigma\rangle \longrightarrow \sigma'}$$

$$\frac{}{\langle x,\sigma\rangle \longrightarrow \sigma(x)} \qquad \frac{}{\langle n,\sigma\rangle \longrightarrow n} \qquad \frac{\langle a_0,\sigma\rangle \longrightarrow n_0 \quad \langle a_1,\sigma\rangle \longrightarrow n_1}{\langle a_0\ \mathsf{op}\ a_1,\sigma\rangle \longrightarrow n_0\ op\ n_1}$$

$$\frac{}{\langle \mathbf{skip},\sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle a,\sigma\rangle \longrightarrow n}{\langle x := a,\sigma\rangle \longrightarrow \sigma[n/x]} \qquad \frac{\langle c_0,\sigma\rangle \longrightarrow \sigma'' \quad \langle c_1,\sigma''\rangle \longrightarrow \sigma'}{\langle c_0\,;c_1,\sigma\rangle \longrightarrow \sigma'}$$

$$\frac{}{\langle v,\sigma\rangle \longrightarrow v} \qquad \frac{\langle b,\sigma\rangle \longrightarrow v}{\langle \neg b,\sigma\rangle \longrightarrow \neg v} \qquad \frac{\langle a_0,\sigma\rangle \longrightarrow n_0 \quad \langle a_1,\sigma\rangle \longrightarrow n_1}{\langle a_0\ \mathsf{cmp}\ a_1,\sigma\rangle \longrightarrow n_0\ cmp\ n_1}$$

$$\frac{\langle b_0,\sigma\rangle \longrightarrow v_0 \quad \langle b_1,\sigma\rangle \longrightarrow v_1}{\langle b_0\ \mathsf{bop}\ b_1,\sigma\rangle \longrightarrow v_0\ bop\ v_1}$$

Recursive definition!

one premise is as complex as the conclusion

$$\frac{\langle b,\sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c,\sigma\rangle \longrightarrow \sigma'' \quad \langle \mathbf{while}\ b\ \mathbf{do}\ c,\sigma''\rangle \longrightarrow \sigma'}{\langle \mathbf{while}\ b\ \mathbf{do}\ c,\sigma\rangle \longrightarrow \sigma'}$$

$$\frac{}{\langle x,\sigma\rangle \longrightarrow \sigma(x)} \qquad \frac{}{\langle n,\sigma\rangle \longrightarrow n} \qquad \frac{\langle a_0,\sigma\rangle \longrightarrow n_0 \quad \langle a_1,\sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1,\sigma\rangle \longrightarrow n_0\ op\ n_1}$$

$$\frac{}{\langle \mathbf{skip},\sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle a,\sigma\rangle \longrightarrow n}{\langle x := a,\sigma\rangle \longrightarrow \sigma[n/x]} \qquad \frac{\langle c_0,\sigma\rangle \longrightarrow \sigma'' \quad \langle c_1,\sigma''\rangle \longrightarrow \sigma'}{\langle c_0\,;c_1,\sigma\rangle \longrightarrow \sigma'}$$

$$\frac{}{\langle v,\sigma\rangle \longrightarrow v} \qquad \frac{\langle b,\sigma\rangle \longrightarrow v}{\langle \neg b,\sigma\rangle \longrightarrow \neg v} \qquad \frac{\langle a_0,\sigma\rangle \longrightarrow n_0 \quad \langle a_1,\sigma\rangle \longrightarrow n_1}{\langle a_0 \text{ cmp } a_1,\sigma\rangle \longrightarrow n_0\ cmp\ n_1}$$

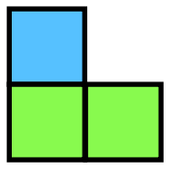
$$\frac{\langle b,\sigma\rangle \longrightarrow \mathbf{ff} \quad \langle c_1,\sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1,\sigma\rangle \longrightarrow \sigma'} \qquad \frac{\langle b,\sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c_0,\sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1,\sigma\rangle \longrightarrow \sigma'}$$

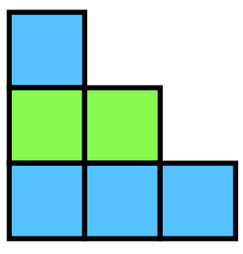
$$\frac{\langle b_0,\sigma\rangle \longrightarrow v_0 \quad \langle b_1,\sigma\rangle \longrightarrow v_1}{\langle b_0 \text{ bop } b_1,\sigma\rangle \longrightarrow v_0\ bop\ v_1}$$

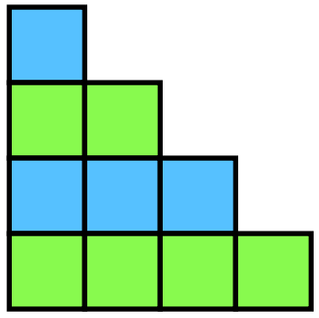
$$\frac{\langle b,\sigma\rangle \longrightarrow \mathbf{ff}}{\langle \mathbf{while}\ b\ \mathbf{do}\ c,\sigma\rangle \longrightarrow \sigma}$$

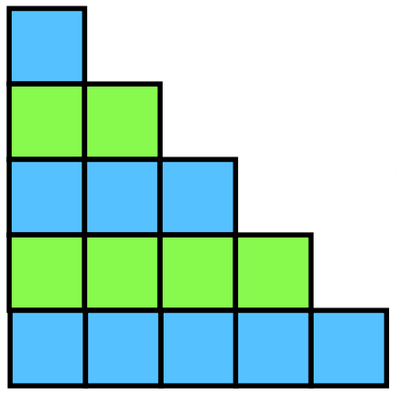
Triangular numbers

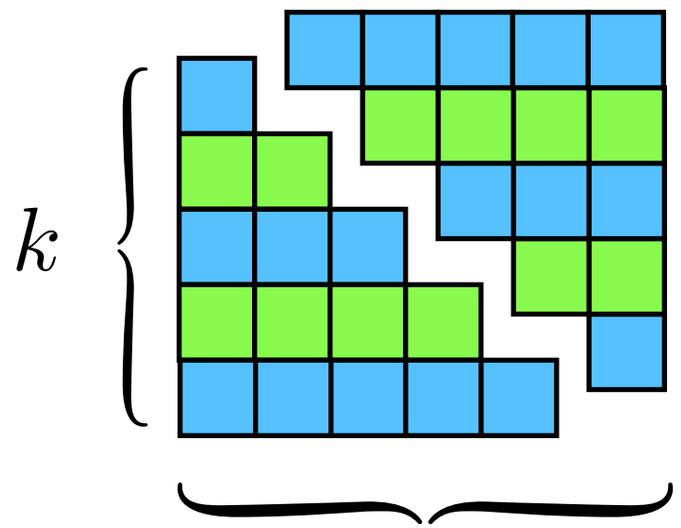
T_1  1

T_2  3

T_3  6

T_4  10

T_5  15



$$T_k \triangleq \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Compute T_k without div

$$
\begin{array}{l}
c_0 \left\{ \right. \\
c_1 \left\{ \right. \\
w \left\{ \right. \\
\\
\end{array}
\begin{array}{l}
t := 0; \\
i := 1; \\
\textbf{while } \overbrace{i \leq k}^{b} \textbf{ do } (\\
\qquad t := t + i; \\
\qquad i := i + 1\ )
\end{array}
\begin{array}{l}
\\
\\
\left. \right\} c_2 \\
\left. \right\} c_3
\end{array}
\left. \right\} c
$$

initial memory

let's find σ such that $\langle (c_0; c_1); w, (2/k) \rangle \longrightarrow \sigma$

let's find σ', σ such that $\begin{cases} \langle c_0; c_1, (2/k) \rangle \longrightarrow \sigma' \\ \langle w, \sigma' \rangle \longrightarrow \sigma \end{cases}$

Let's derive

let's find σ' such that $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$$
\begin{array}{l}
c_0\ \{\ t := 0; \\
c_1\ \{\ i := 1; \quad \overbrace{i \leq k}^{b} \\
w\ \left\{\ \textbf{while}\ i \leq k\ \textbf{do}\ (\right. \\
\qquad\qquad t := t + i; \qquad \left.\right\}\ c_2 \\
\qquad\qquad i := i + 1\ ) \qquad \left.\right\}\ c_3
\end{array}
\qquad \sigma \triangleq (2/n)
$$

$\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$\nwarrow \langle t := 0, (2/k)\rangle \longrightarrow \sigma_1, \langle i := 1, \sigma_1\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma_1=(2/k)[n_1/t]} \langle 0, (2/k)\rangle \longrightarrow n_1, \langle i := 1, (2/k, n_1/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{n_1=0} \langle i := 1, (2/k, 0/t)\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma'=(2/k,0/t)[n_2/i]} \langle 1, (2/k, 0/t)\rangle \longrightarrow n_2$

$\nwarrow_{n_2=1} \square$

$$\sigma' = (2/k, 0/t, 1/i)$$

more concisely: $\langle c_0; c_1, (2/k)\rangle \longrightarrow \sigma'$

$\nwarrow^{*}_{\sigma'=(2/k,0/t,1/i)} \square$

Let's derive

let's find σ such that $\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$$\begin{array}{l} c_0\ \{\ t := 0; \\ c_1\ \{\ i := 1; \\ w\ \left\{\ \begin{array}{l} \textbf{while}\ \overbrace{i \leq k}^{b}\ \textbf{do}\ (\\ \qquad t := t + i; \\ \qquad i := i + 1\ ) \end{array}\right. \begin{array}{l} \} c_2 \\ \} c_3 \end{array} \Big\} c \end{array}$$

$\sigma \triangleq (2/n)$

$\langle w, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 0/t, 1/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 0/t, 1/i)\rangle \longrightarrow \sigma_1, \langle w, \sigma_1\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_1=(2/k,0/t,1/i)[1/t,2/i]} \langle w, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma$

$\nwarrow \langle b, (2/k, 1/t, 2/i)\rangle \longrightarrow \mathbf{tt}, \langle c, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma_2, \langle w, \sigma_2\rangle \longrightarrow \sigma$

$\nwarrow^* \langle c, (2/k, 1/t, 2/i)\rangle \longrightarrow \sigma_2, \langle w, \sigma_2\rangle \longrightarrow \sigma$

$\nwarrow^*_{\sigma_2=(2/k,1/t,2/i)[3/t,3/i]} \langle w, (2/k, 3/t, 3/i)\rangle \longrightarrow \sigma$

$\nwarrow_{\sigma=(2/k,3/t,3/i)} \langle b, (2/k, 3/t, 3/i)\rangle \longrightarrow \mathbf{ff}$

$\nwarrow^* \Box$

$\sigma = (2/k, 3/t, 3/i)$

Divergence

Termination of commands?

$$c ::= \mathbf{skip} \mid x := a \mid c\,;c \mid \mathbf{if}\ b\ \mathbf{then}\ c\ \mathbf{else}\ c \mid \mathbf{while}\ b\ \mathbf{do}\ c$$

$$\frac{}{\langle \mathbf{skip}, \sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle a, \sigma\rangle \longrightarrow n}{\langle x := a, \sigma\rangle \longrightarrow \sigma[n/x]} \qquad \frac{\langle c_0, \sigma\rangle \longrightarrow \sigma'' \quad \langle c_1, \sigma''\rangle \longrightarrow \sigma'}{\langle c_0\,; c_1, \sigma\rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma\rangle \longrightarrow \mathbf{ff} \quad \langle c_1, \sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1, \sigma\rangle \longrightarrow \sigma'} \qquad \frac{\langle b, \sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c_0, \sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1, \sigma\rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma\rangle \longrightarrow \mathbf{ff}}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle b, \sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c, \sigma\rangle \longrightarrow \sigma'' \quad \langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma''\rangle \longrightarrow \sigma'}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma'}$$

$$P(c) \stackrel{\triangle}{=} \forall \sigma \in \mathbb{M}.\ \exists \sigma' \in \mathbb{M}.\ \langle c, \sigma\rangle \longrightarrow \sigma' \qquad \forall c.\ P(c)\ ?$$

Termination?

$P(c) \triangleq \forall \sigma.\ \exists \sigma'.\ \langle c, \sigma \rangle \longrightarrow \sigma'$ $\qquad\qquad$ $\forall c.\ P(c)$?

take $w \triangleq$ **while tt do skip**

$\langle w, \sigma \rangle \longrightarrow \sigma'$

$\nwarrow \langle \mathbf{tt}, \sigma \rangle \longrightarrow \mathbf{tt}, \langle \mathbf{skip}, \sigma \rangle \longrightarrow \sigma'', \langle w, \sigma'' \rangle \longrightarrow \sigma'$

$\nwarrow \langle \mathbf{skip}, \sigma \rangle \longrightarrow \sigma'', \langle w, \sigma'' \rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma''=\sigma} \langle w, \sigma \rangle \longrightarrow \sigma'$

same goal from which we started!
no other options: $\langle \mathbf{tt}, \sigma \rangle \not\rightarrow \mathbf{ff}$
no way to complete the derivation!

$\neg P(w)$ we have found a counterexample to termination

Divergence

$\langle c, \sigma \rangle \not\longrightarrow$ means $\neg \exists \sigma'.\ \langle c, \sigma \rangle \longrightarrow \sigma'$

and we say that c diverges on σ

sometimes divergence is very difficult to detect
(well… it is undecidable)

Divergence

take $w \triangleq \mathbf{while}\ x > 0\ \mathbf{do}\ x := x + 1$

take a generic σ

if $\sigma(x) \leq 0$: $\langle w, \sigma\rangle \longrightarrow \sigma' \quad \nwarrow_{\sigma'=\sigma} \langle x > 0, \sigma\rangle \longrightarrow \mathbf{ff} \qquad \nwarrow^* \square$

hence $\langle w, \sigma\rangle \longrightarrow \sigma$

if $\sigma(x) > 0$: $\langle w, \sigma\rangle \longrightarrow \sigma'$

$\nwarrow \langle x > 0, \sigma\rangle \longrightarrow \mathbf{tt}, \langle x := x + 1, \sigma\rangle \longrightarrow \sigma'', \langle w, \sigma''\rangle \longrightarrow \sigma'$

$\nwarrow^* \langle x := x + 1, \sigma\rangle \longrightarrow \sigma'', \langle w, \sigma''\rangle \longrightarrow \sigma'$

$\nwarrow_{\sigma''=\sigma[n/x]} \langle x + 1, \sigma\rangle \longrightarrow n, \langle w, \sigma[n/x]\rangle \longrightarrow \sigma'$

$\nwarrow^*_{n=\sigma(x)+1} \langle w, \sigma[\sigma(x) + 1/x]\rangle \longrightarrow \sigma'$

not the same goal from which we started!
how can we prove divergence?

$\sigma(x) > 0 \Rightarrow \sigma[\sigma(x) + 1/x](x) = \sigma(x) + 1 > 0$

Proving divergence (if possible)

consider $w \triangleq \mathbf{while}\ b\ \mathbf{do}\ c$

suppose we find a set of memories $S \subseteq \mathbb{M}$ such that

- $\forall \sigma \in S.\ \langle b, \sigma \rangle \longrightarrow \mathbf{tt}$
- $\forall \sigma \in S.\ \forall \sigma' \in \mathbb{M}.\ \langle c, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' \in S$

then we can conclude $\forall \sigma \in S.\ \langle w, \sigma \rangle \not\longrightarrow$

note that if $\langle c, \sigma \rangle \not\longrightarrow$, then $\langle c, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' \in S$ trivially holds

Revisiting the example

take $w \triangleq \mathbf{while}\ x > 0\ \mathbf{do}\ x := x + 1$ take a generic σ

if $\sigma(x) \leq 0$: $\langle w, \sigma \rangle \longrightarrow \sigma$

let $S \triangleq \{\sigma \mid \sigma(x) > 0\}$

- $\forall \sigma \in S.\ \langle x > 0, \sigma \rangle \longrightarrow \mathbf{tt}$ ✅
- $\forall \sigma \in S.\ \forall \sigma'.\ \langle x := x + 1, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' \in S$ ✅

in fact $\langle x := x + 1, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' = \sigma[\sigma(x) + 1/x]$

$$\sigma(x) > 0 \Rightarrow \sigma[\sigma(x) + 1/x](x) = \sigma(x) + 1 > 0$$

therefore, if $\sigma(x) > 0$, then $\langle w, \sigma \rangle \not\longrightarrow$

Exercise

take $w \triangleq \textbf{while}\ x \neq 0\ \textbf{do}\ x := x - 2$

find the set of all and only memories σ such that $\langle w, \sigma \rangle \not\rightarrow$

$S_1 \triangleq \{\sigma \mid \sigma(x) < 0\}$ $\qquad$ $S_2 \triangleq \{\sigma \mid \exists k \in \mathbb{Z}.\ \sigma(x) = 2k + 1\}$

- $\forall \sigma \in S.\ \langle x \neq 0, \sigma \rangle \longrightarrow \textbf{tt}$
- $\forall \sigma \in S.\ \forall \sigma'.\ \langle x := x - 2, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' \in S$

  $\langle x := x - 2, \sigma \rangle \longrightarrow \sigma' \Rightarrow \sigma' = \sigma[\sigma(x) - 2/x]$

Collatz's conjecture: half or triple plus one

$$w \triangleq \textbf{while } x > 1 \textbf{ do } (\ \textbf{if } x\%2 = 0 \textbf{ then } x := x/2 \textbf{ else } x := (3 \times x) + 1\ )$$

$\forall \sigma.\ \sigma(x) \leq 1 \Rightarrow \langle w, \sigma \rangle \longrightarrow \sigma$ ✅

open conjecture: $\forall \sigma.\ \exists \sigma'.\ \langle w, \sigma \rangle \longrightarrow \sigma' \quad \equiv \quad \neg(\exists \sigma.\ \langle w, \sigma \rangle \not\longrightarrow)$

more precisely: $\forall \sigma.\ \sigma(x) > 1 \Rightarrow \langle w, \sigma \rangle \longrightarrow \sigma[1/x]$

Experimental evidence:
as of 2020, the conjecture has been verified for all values up to 2^{68}.
Barina, David. "Convergence verification of the Collatz problem".
The Journal of Supercomputing (2020). doi:10.1007/s11227-020-03368-x

Determinacy?

Determinacy of commands?

$$c ::= \mathbf{skip} \mid x := a \mid c\,;c \mid \mathbf{if}\ b\ \mathbf{then}\ c\ \mathbf{else}\ c \mid \mathbf{while}\ b\ \mathbf{do}\ c$$

$$\frac{}{\langle \mathbf{skip}, \sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle a, \sigma\rangle \longrightarrow n}{\langle x := a, \sigma\rangle \longrightarrow \sigma[n/x]} \qquad \frac{\langle c_0, \sigma\rangle \longrightarrow \sigma'' \quad \langle c_1, \sigma''\rangle \longrightarrow \sigma'}{\langle c_0\,; c_1, \sigma\rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma\rangle \longrightarrow \mathbf{ff} \quad \langle c_1, \sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1, \sigma\rangle \longrightarrow \sigma'} \qquad \frac{\langle b, \sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c_0, \sigma\rangle \longrightarrow \sigma'}{\langle \mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1, \sigma\rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma\rangle \longrightarrow \mathbf{ff}}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma} \qquad \frac{\langle b, \sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c, \sigma\rangle \longrightarrow \sigma'' \quad \langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma''\rangle \longrightarrow \sigma'}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma'}$$

$$P(c) \triangleq \forall \sigma, \sigma_1, \sigma_2.\ \langle c, \sigma\rangle \longrightarrow \sigma_1 \wedge \langle c, \sigma\rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2 \qquad \forall c.\ P(c)\ ?$$

Structural induction principle

$$\frac{\begin{array}{c}\forall x, a.\ P(x := a) \qquad\qquad P(\mathbf{skip}) \\ \forall c_0, c_1.\ P(c_0) \wedge P(c_1) \Rightarrow P(c_0\,;c_1) \\ \forall b, c_0, c_1.\ P(c_0) \wedge P(c_1) \Rightarrow P(\mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1) \\ \forall b, c.\ P(c) \Rightarrow P(\mathbf{while}\ b\ \mathbf{do}\ c)\end{array}}{\forall c \in \mathsf{Com}.\ P(c)}$$

Base case

$\forall x, a.\ P(x := a)$

Take generic $x \in \mathsf{Ide}, a \in \mathsf{Aexp}$

We want to prove

$$P(x := a) \triangleq \forall \sigma, \sigma_1, \sigma_2.\ \langle x := a, \sigma\rangle \longrightarrow \sigma_1 \wedge \langle x := a, \sigma\rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

Take generic $\sigma, \sigma_1, \sigma_2$ s.t. $\langle x := a, \sigma\rangle \longrightarrow \sigma_1$ and $\langle x := a, \sigma\rangle \longrightarrow \sigma_2$

We want to prove $\sigma_1 = \sigma_2$

Consider the goal $\langle x := a, \sigma\rangle \longrightarrow \sigma_1$

Only the rule $\dfrac{\langle a, \sigma\rangle \longrightarrow n}{\langle x := a, \sigma\rangle \longrightarrow \sigma[n/x]}$ is applicable, hence $\sigma_1 = \sigma[n/x]$ with $\langle a, \sigma\rangle \longrightarrow n$

Similarly, since $\langle x := a, \sigma\rangle \longrightarrow \sigma_2$ it must be $\sigma_2 = \sigma[m/x]$ with $\langle a, \sigma\rangle \longrightarrow m$

by determinacy of Aexp we have $n = m$ and thus $\sigma_1 = \sigma_2$

Inductive case

$\forall c_0, c_1.\ P(c_0) \wedge P(c_1) \Rightarrow P(c_0\,;c_1)$

Take generic c_0, c_1

We assume (inductive hypotheses)

$$P(c_i) \triangleq \forall \sigma, \sigma_1, \sigma_2.\ \langle c_i, \sigma\rangle \longrightarrow \sigma_1 \wedge \langle c_i, \sigma\rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

We want to prove

$$P(c_0\,;c_1) \triangleq \forall \sigma, \sigma_1, \sigma_2.\ \langle c_0\,;c_1, \sigma\rangle \longrightarrow \sigma_1 \wedge \langle c_0\,;c_1, \sigma\rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

Take generic $\sigma, \sigma_1, \sigma_2$ such that $\langle c_0\,;c_1, \sigma\rangle \longrightarrow \sigma_1$ and $\langle c_0\,;c_1, \sigma\rangle \longrightarrow \sigma_2$

We want to prove $\sigma_1 = \sigma_2$

Inductive case (ctd)

Consider the goal $\langle c_0 \,;\, c_1, \sigma\rangle \longrightarrow \sigma_1$

Only the rule $\dfrac{\langle c_0, \sigma\rangle \longrightarrow \sigma'' \quad \langle c_1, \sigma''\rangle \longrightarrow \sigma'}{\langle c_0 \,;\, c_1, \sigma\rangle \longrightarrow \sigma'}$ is applicable

hence $\sigma_1 = \sigma_1'$ with $\langle c_0, \sigma\rangle \longrightarrow \sigma_1''$ and $\langle c_1, \sigma_1''\rangle \longrightarrow \sigma_1'$

Similarly, since $\langle c_0 \,;\, c_1, \sigma\rangle \longrightarrow \sigma_2$

it must be $\sigma_2 = \sigma_2'$ with $\langle c_0, \sigma\rangle \longrightarrow \sigma_2''$ and $\langle c_1, \sigma_2''\rangle \longrightarrow \sigma_2'$

By inductive hypothesis $P(c_0)$, we have $\sigma_1'' = \sigma_2''$

and thus $\langle c_1, \sigma_2''\rangle \longrightarrow \sigma_1'$ and $\langle c_1, \sigma_2''\rangle \longrightarrow \sigma_2'$

By inductive hypothesis $P(c_1)$, we then have $\sigma_1' = \sigma_2'$

and thus we conclude $\sigma_1 = \sigma_1' = \sigma_2' = \sigma_2$

Inductive case

$\forall b, c.\ P(c) \Rightarrow P(\mathbf{while}\ b\ \mathbf{do}\ c)$ Take generic b, c

We assume (inductive hypothesis)

$$P(c) \triangleq \forall \sigma, \sigma_1, \sigma_2.\ \langle c, \sigma\rangle \longrightarrow \sigma_1 \wedge \langle c, \sigma\rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

We want to prove

$$P(\mathbf{while}\ b\ \mathbf{do}\ c) \triangleq \forall \sigma, \sigma_1, \sigma_2.\ \langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma_1 \wedge \langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

Take $\sigma, \sigma_1, \sigma_2$ such that $\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma_1$ and $\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma_2$

We want to prove $\sigma_1 = \sigma_2$

By determinacy of boolean expressions, there are two cases

$\langle b, \sigma\rangle \longrightarrow \mathbf{ff}$ $\langle b, \sigma\rangle \longrightarrow \mathbf{tt}$

Inductive case (ctd)

if $\langle b, \sigma \rangle \longrightarrow \mathbf{ff}$

Consider the goal $\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma \rangle \longrightarrow \sigma_1$

Only the rule $\dfrac{\langle b, \sigma \rangle \longrightarrow \mathbf{ff}}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma \rangle \longrightarrow \sigma}$ is applicable hence $\sigma_1 = \sigma$

Similarly, since $\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma \rangle \longrightarrow \sigma_2$ it must be $\sigma_2 = \sigma$

and thus we conclude $\sigma_1 = \sigma = \sigma_2$

Inductive case (ctd)

if $\langle b, \sigma \rangle \longrightarrow \mathbf{tt}$

Consider the goal $\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma \rangle \longrightarrow \sigma_1$

Only the rule $\dfrac{\langle b, \sigma \rangle \longrightarrow \mathbf{tt} \quad \langle c, \sigma \rangle \longrightarrow \sigma'' \quad \langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma'' \rangle \longrightarrow \sigma'}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma \rangle \longrightarrow \sigma'}$ is applicable

hence $\sigma_1 = \sigma_1'$ with $\langle c, \sigma \rangle \longrightarrow \sigma_1''$ and $\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma_1'' \rangle \longrightarrow \sigma_1'$

Similarly, since $\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma \rangle \longrightarrow \sigma_2$

it must be $\sigma_2 = \sigma_2'$ with $\langle c, \sigma \rangle \longrightarrow \sigma_2''$ and $\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma_2'' \rangle \longrightarrow \sigma_2'$

By inductive hypothesis $P(c)$, we have $\sigma_1'' = \sigma_2''$

thus $\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma_2'' \rangle \longrightarrow \sigma_1'$ and $\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma_2'' \rangle \longrightarrow \sigma_2'$

but there is no inductive hypothesis $P(\mathbf{while}\ b\ \mathbf{do}\ c)$!

Recursive definition!

one premise is as much complex as the conclusion

$$\frac{\langle b, \sigma\rangle \longrightarrow \mathbf{tt} \quad \langle c, \sigma\rangle \longrightarrow \sigma'' \quad \langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma''\rangle \longrightarrow \sigma'}{\langle \mathbf{while}\ b\ \mathbf{do}\ c, \sigma\rangle \longrightarrow \sigma'}$$

to close the proof of determinacy
we need a more convenient induction principle:

Rule induction