

MPP 2026/27 (0077A, 9CFU)

Models for Programming Paradigms

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<https://didawiki.di.unipi.it/doku.php/magistraleinformatica/mpp/start>

04 - Logical Systems

Some exercises (unification)

Unification

delete

$\mathcal{G} \cup \{t \stackrel{?}{=} t\}$
becomes
 $\mathcal{G}$

eliminate

$\mathcal{G} \cup \{x \stackrel{?}{=} t\}$
becomes if $x \in \text{vars}(\mathcal{G}) \setminus \text{vars}(t)$
 $\mathcal{G}[x = t] \cup \{x \stackrel{?}{=} t\}$

swap

$\mathcal{G} \cup \{f(t_1, \dots, t_m) \stackrel{?}{=} x\}$
becomes
 $\mathcal{G} \cup \{x \stackrel{?}{=} f(t_1, \dots, t_m)\}$

decompose

$\mathcal{G} \cup \{f(t_1, \dots, t_m) \stackrel{?}{=} f(u_1, \dots, u_m)\}$
becomes
 $\mathcal{G} \cup \{t_1 \stackrel{?}{=} u_1, \dots, t_m \stackrel{?}{=} u_m\}$

occur-check

$\mathcal{G} \cup \{x \stackrel{?}{=} f(t_1, \dots, t_m)\}$
fails if $x \in \text{vars}(f(t_1, \dots, t_m))$

conflict

$\mathcal{G} \cup \{f(t_1, \dots, t_m) \stackrel{?}{=} g(u_1, \dots, u_h)\}$
fails if $f \neq g$ or $m \neq h$

Exercise

$$\{\text{prod}(\textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{prod}(\textcolor{red}{y}, \textcolor{red}{z}, \textcolor{red}{x})\}$$

Exercise

$$\{\text{prod}(\textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{prod}(\textcolor{red}{y}, \textcolor{red}{z}, \textcolor{red}{x})\}$$

Exercise

$$\{\text{prod}(s(x), y, s(z)) \stackrel{?}{=} \text{prod}(y, z, x)\}$$

decompose

Exercise

$$\{\text{prod}(\textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{prod}(\textcolor{red}{y}, \textcolor{red}{z}, \textcolor{red}{x})\}$$

decompose

$$\{\textcolor{blue}{s}(\textcolor{red}{x}) \stackrel{?}{=} \textcolor{red}{y}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{red}{x}\}$$

Exercise

$$\{\text{prod}(s(x), y, s(z)) \stackrel{?}{=} \text{prod}(y, z, x)\}$$

decompose

$$\{s(x) \stackrel{?}{=} y, y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

Exercise

$$\{\text{prod}(s(x), y, s(z)) \stackrel{?}{=} \text{prod}(y, z, x)\}$$

decompose

$$\{s(x) \stackrel{?}{=} y, y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

Exercise

$$\{\text{prod}(s(x), y, s(z)) \stackrel{?}{=} \text{prod}(y, z, x)\}$$

decompose

$$\{s(x) \stackrel{?}{=} y, y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

eliminate

Exercise

$$\{\text{prod}(s(x), y, s(z)) \stackrel{?}{=} \text{prod}(y, z, x)\}$$

decompose

$$\{s(x) \stackrel{?}{=} y, y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

eliminate

$$\{y \stackrel{?}{=} s(x), s(x) \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

Exercise

$$\{\text{prod}(s(x), y, s(z)) \stackrel{?}{=} \text{prod}(y, z, x)\}$$

decompose

$$\{s(x) \stackrel{?}{=} y, y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

eliminate

$$\{y \stackrel{?}{=} s(x), s(x) \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

Exercise

$$\{\text{prod}(s(x), y, s(z)) \stackrel{?}{=} \text{prod}(y, z, x)\}$$

decompose

$$\{s(x) \stackrel{?}{=} y, y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

eliminate

$$\{y \stackrel{?}{=} s(x), s(x) \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), z \stackrel{?}{=} s(x), s(z) \stackrel{?}{=} x\}$$

Exercise

$$\{\text{prod}(s(x), y, s(z)) \stackrel{?}{=} \text{prod}(y, z, x)\}$$

decompose

$$\{s(x) \stackrel{?}{=} y, y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

eliminate

$$\{y \stackrel{?}{=} s(x), s(x) \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), z \stackrel{?}{=} s(x), s(z) \stackrel{?}{=} x\}$$

eliminate

Exercise

$$\{\text{prod}(s(x), y, s(z)) \stackrel{?}{=} \text{prod}(y, z, x)\}$$

decompose

$$\{s(x) \stackrel{?}{=} y, y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

eliminate

$$\{y \stackrel{?}{=} s(x), s(x) \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), z \stackrel{?}{=} s(x), s(z) \stackrel{?}{=} x\}$$

eliminate

$$\{y \stackrel{?}{=} s(x), z \stackrel{?}{=} s(x), s(s(x)) \stackrel{?}{=} x\}$$

Exercise

$$\{\text{prod}(s(x), y, s(z)) \stackrel{?}{=} \text{prod}(y, z, x)\}$$

decompose

$$\{s(x) \stackrel{?}{=} y, y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

eliminate

$$\{y \stackrel{?}{=} s(x), s(x) \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), z \stackrel{?}{=} s(x), s(z) \stackrel{?}{=} x\}$$

eliminate

$$\{y \stackrel{?}{=} s(x), z \stackrel{?}{=} s(x), s(s(x)) \stackrel{?}{=} x\}$$

swap

Exercise

$$\{\text{prod}(s(x), y, s(z)) \stackrel{?}{=} \text{prod}(y, z, x)\}$$

decompose

$$\{s(x) \stackrel{?}{=} y, y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

eliminate

$$\{y \stackrel{?}{=} s(x), s(x) \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), z \stackrel{?}{=} s(x), s(z) \stackrel{?}{=} x\}$$

eliminate

$$\{y \stackrel{?}{=} s(x), z \stackrel{?}{=} s(x), s(s(x)) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), z \stackrel{?}{=} s(x), x \stackrel{?}{=} s(s(x))\}$$

Exercise

$$\{\text{prod}(\textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{prod}(\textcolor{red}{y}, \textcolor{red}{z}, \textcolor{red}{x})\}$$

decompose

$$\{\textcolor{blue}{s}(\textcolor{red}{x}) \stackrel{?}{=} \textcolor{red}{y}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{red}{x}\}$$

swap

$$\{\textcolor{red}{y} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{red}{x}\}$$

eliminate

$$\{\textcolor{red}{y} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{blue}{s}(\textcolor{red}{x}) \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{red}{x}\}$$

swap

$$\{\textcolor{red}{y} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{z} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{red}{x}\}$$

eliminate

$$\{\textcolor{red}{y} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{z} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{blue}{s}(\textcolor{blue}{s}(\textcolor{red}{x})) \stackrel{?}{=} \textcolor{red}{x}\}$$

swap

$$\{\textcolor{red}{y} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{z} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{x} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{blue}{s}(\textcolor{red}{x}))\}$$

occur-check

Exercise

$$\{\text{prod}(s(x), y, s(z)) \stackrel{?}{=} \text{prod}(y, z, x)\}$$

decompose

$$\{s(x) \stackrel{?}{=} y, y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), y \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

eliminate

$$\{y \stackrel{?}{=} s(x), s(x) \stackrel{?}{=} z, s(z) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), z \stackrel{?}{=} s(x), s(z) \stackrel{?}{=} x\}$$

eliminate

$$\{y \stackrel{?}{=} s(x), z \stackrel{?}{=} s(x), s(s(x)) \stackrel{?}{=} x\}$$

swap

$$\{y \stackrel{?}{=} s(x), z \stackrel{?}{=} s(x), x \stackrel{?}{=} s(s(x))\}$$

occur-check



Exercise

$$\{\text{pow}(\textcolor{red}{x}, \textcolor{blue}{s}(\textcolor{red}{y}), \textcolor{red}{x}) \stackrel{?}{=} \text{pow}(\textcolor{blue}{s}(\textcolor{red}{y}), \textcolor{red}{z}, \textcolor{red}{z})\}$$

Exercise

$$\{\text{pow}(\textcolor{red}{x}, \textcolor{blue}{s}(\textcolor{red}{y}), \textcolor{red}{x}) \stackrel{?}{=} \text{pow}(\textcolor{blue}{s}(\textcolor{red}{y}), \textcolor{red}{z}, \textcolor{red}{z})\}$$

Exercise

$$\{\text{pow}(\textcolor{red}{x}, \textcolor{blue}{s}(\textcolor{red}{y}), \textcolor{red}{x}) \stackrel{?}{=} \text{pow}(\textcolor{blue}{s}(\textcolor{red}{y}), \textcolor{red}{z}, \textcolor{red}{z})\}$$

decompose

Exercise

$$\{\text{pow}(x, s(y), x) \stackrel{?}{=} \text{pow}(s(y), z, z)\}$$

decompose

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, x \stackrel{?}{=} z\}$$

Exercise

$$\{\text{pow}(x, s(y), x) \stackrel{?}{=} \text{pow}(s(y), z, z)\}$$

decompose

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, x \stackrel{?}{=} z\}$$

eliminate

Exercise

$$\{\text{pow}(x, s(y), x) \stackrel{?}{=} \text{pow}(s(y), z, z)\}$$

decompose

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, x \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, s(y) \stackrel{?}{=} z\}$$

Exercise

$$\{\text{pow}(x, s(y), x) \stackrel{?}{=} \text{pow}(s(y), z, z)\}$$

decompose

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, x \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, s(y) \stackrel{?}{=} z\}$$

swap

Exercise

$$\{\text{pow}(x, s(y), x) \stackrel{?}{=} \text{pow}(s(y), z, z)\}$$

decompose

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, x \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, s(y) \stackrel{?}{=} z\}$$

swap

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z\}$$

Exercise

$$\{\text{pow}(x, s(y), x) \stackrel{?}{=} \text{pow}(s(y), z, z)\}$$

decompose

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, x \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, s(y) \stackrel{?}{=} z\}$$

swap

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z\}$$

eliminate

Exercise

$$\{\text{pow}(x, s(y), x) \stackrel{?}{=} \text{pow}(s(y), z, z)\}$$

decompose

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, x \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, s(y) \stackrel{?}{=} z\}$$

swap

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} s(y)\}$$

Exercise

$$\{\text{pow}(x, s(y), x) \stackrel{?}{=} \text{pow}(s(y), z, z)\}$$

decompose

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, x \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, s(y) \stackrel{?}{=} z\}$$

swap

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} s(y)\}$$

delete

Exercise

$$\{\text{pow}(x, s(y), x) \stackrel{?}{=} \text{pow}(s(y), z, z)\}$$

decompose

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, x \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, s(y) \stackrel{?}{=} z\}$$

swap

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} s(y)\}$$

delete

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y)\}$$

Exercise

$$\{\text{pow}(x, s(y), x) \stackrel{?}{=} \text{pow}(s(y), z, z)\}$$

decompose

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, x \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, s(y) \stackrel{?}{=} z\}$$

swap

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} s(y)\}$$

delete

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y)\}$$

$$[x = s(y), z = s(y)]$$

Exercise

$$\{\text{pow}(x, s(y), x) \stackrel{?}{=} \text{pow}(s(y), z, z)\}$$

decompose

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, x \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z, s(y) \stackrel{?}{=} z\}$$

swap

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} z\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), s(y) \stackrel{?}{=} s(y)\}$$

delete

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y)\}$$

$$[x = s(y), z = s(y)]$$



Exercise

$$\{\text{div}(\textcolor{red}{x}, \textcolor{blue}{s}(\textcolor{red}{y})) \stackrel{?}{=} \text{div}(\textcolor{red}{z}, \textcolor{red}{x}), \text{div}(\textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{div}(\textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{u}))\}$$

Exercise

$$\{\text{div}(\textcolor{red}{x}, \textcolor{blue}{s}(\textcolor{red}{y})) \stackrel{?}{=} \text{div}(\textcolor{red}{z}, \textcolor{red}{x}), \text{div}(\textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{div}(\textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{u}))\}$$

Exercise

$$\{\text{div}(\textcolor{red}{x}, \textcolor{blue}{s}(\textcolor{red}{y})) \stackrel{?}{=} \text{div}(\textcolor{red}{z}, \textcolor{red}{x}), \text{div}(\textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{div}(\textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{u}))\}$$

decompose decompose

Exercise

$$\{\text{div}(\textcolor{red}{x}, \textcolor{blue}{s}(\textcolor{red}{y})) \stackrel{?}{=} \text{div}(\textcolor{red}{z}, \textcolor{red}{x}), \text{div}(\textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{div}(\textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{u}))\}$$

decompose decompose

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{y}) \stackrel{?}{=} \textcolor{red}{x}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

Exercise

$$\{\text{div}(\textcolor{red}{x}, \textcolor{blue}{s}(\textcolor{red}{y})) \stackrel{?}{=} \text{div}(\textcolor{red}{z}, \textcolor{red}{x}), \text{div}(\textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{div}(\textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{u}))\}$$

decompose

decompose

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{y}) \stackrel{?}{=} \textcolor{red}{x}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

eliminate

Exercise

$$\{\text{div}(\textcolor{red}{x}, \textcolor{blue}{s}(\textcolor{red}{y})) \stackrel{?}{=} \text{div}(\textcolor{red}{z}, \textcolor{red}{x}), \text{div}(\textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{div}(\textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{u}))\}$$

decompose

decompose

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{y}) \stackrel{?}{=} \textcolor{red}{x}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

eliminate

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{y}) \stackrel{?}{=} \textcolor{red}{z}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

Exercise

$$\{\text{div}(\textcolor{red}{x}, \textcolor{blue}{s}(\textcolor{red}{y})) \stackrel{?}{=} \text{div}(\textcolor{red}{z}, \textcolor{red}{x}), \text{div}(\textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{div}(\textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{u}))\}$$

decompose

decompose

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{y}) \stackrel{?}{=} \textcolor{red}{x}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

eliminate

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{y}) \stackrel{?}{=} \textcolor{red}{z}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

swap

Exercise

$$\{\text{div}(\textcolor{red}{x}, \textcolor{blue}{s}(\textcolor{red}{y})) \stackrel{?}{=} \text{div}(\textcolor{red}{z}, \textcolor{red}{x}), \text{div}(\textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{div}(\textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{u}))\}$$

decompose

decompose

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{y}) \stackrel{?}{=} \textcolor{red}{x}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

eliminate

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{y}) \stackrel{?}{=} \textcolor{red}{z}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

swap

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{red}{z} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{y}), \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

Exercise

$$\{\text{div}(\textcolor{red}{x}, \textcolor{blue}{s}(\textcolor{red}{y})) \stackrel{?}{=} \text{div}(\textcolor{red}{z}, \textcolor{red}{x}), \text{div}(\textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{div}(\textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{u}))\}$$

decompose

decompose

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{y}) \stackrel{?}{=} \textcolor{red}{x}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

eliminate

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{y}) \stackrel{?}{=} \textcolor{red}{z}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

swap

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{red}{z} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{y}), \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

eliminate

Exercise

$$\{\text{div}(x, s(y)) \stackrel{?}{=} \text{div}(z, x), \text{div}(y, s(z)) \stackrel{?}{=} \text{div}(u, s(u))\}$$

decompose

decompose

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} x, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} z, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

swap

$$\{x \stackrel{?}{=} z, z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(s(y)) \stackrel{?}{=} s(u)\}$$

Exercise

$$\{\text{div}(x, s(y)) \stackrel{?}{=} \text{div}(z, x), \text{div}(y, s(z)) \stackrel{?}{=} \text{div}(u, s(u))\}$$

decompose

decompose

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} x, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} z, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

swap

$$\{x \stackrel{?}{=} z, z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(s(y)) \stackrel{?}{=} s(u)\}$$

eliminate

Exercise

$$\{\text{div}(x, s(y)) \stackrel{?}{=} \text{div}(z, x), \text{div}(y, s(z)) \stackrel{?}{=} \text{div}(u, s(u))\}$$

decompose

decompose

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} x, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} z, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

swap

$$\{x \stackrel{?}{=} z, z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(s(y)) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(u), z \stackrel{?}{=} s(u), y \stackrel{?}{=} u, s(s(u)) \stackrel{?}{=} s(u)\}$$

Exercise

$$\{\text{div}(x, s(y)) \stackrel{?}{=} \text{div}(z, x), \text{div}(y, s(z)) \stackrel{?}{=} \text{div}(u, s(u))\}$$

decompose

decompose

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} x, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} z, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

swap

$$\{x \stackrel{?}{=} z, z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(s(y)) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(u), z \stackrel{?}{=} s(u), y \stackrel{?}{=} u, s(s(u)) \stackrel{?}{=} s(u)\}$$

decompose

Exercise

$$\{\text{div}(x, s(y)) \stackrel{?}{=} \text{div}(z, x), \text{div}(y, s(z)) \stackrel{?}{=} \text{div}(u, s(u))\}$$

decompose

decompose

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} x, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} z, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

swap

$$\{x \stackrel{?}{=} z, z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(s(y)) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(u), z \stackrel{?}{=} s(u), y \stackrel{?}{=} u, s(s(u)) \stackrel{?}{=} s(u)\}$$

decompose

$$\{x \stackrel{?}{=} s(u), z \stackrel{?}{=} s(u), y \stackrel{?}{=} u, s(u) \stackrel{?}{=} u\}$$

Exercise

$$\{\text{div}(x, s(y)) \stackrel{?}{=} \text{div}(z, x), \text{div}(y, s(z)) \stackrel{?}{=} \text{div}(u, s(u))\}$$

decompose

decompose

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} x, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} z, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

swap

$$\{x \stackrel{?}{=} z, z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(s(y)) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(u), z \stackrel{?}{=} s(u), y \stackrel{?}{=} u, s(s(u)) \stackrel{?}{=} s(u)\}$$

decompose

$$\{x \stackrel{?}{=} s(u), z \stackrel{?}{=} s(u), y \stackrel{?}{=} u, s(u) \stackrel{?}{=} u\}$$

swap

Exercise

$$\{\text{div}(x, s(y)) \stackrel{?}{=} \text{div}(z, x), \text{div}(y, s(z)) \stackrel{?}{=} \text{div}(u, s(u))\}$$

decompose

decompose

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} x, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} z, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

swap

$$\{x \stackrel{?}{=} z, z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(s(y)) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(u), z \stackrel{?}{=} s(u), y \stackrel{?}{=} u, s(s(u)) \stackrel{?}{=} s(u)\}$$

decompose

$$\{x \stackrel{?}{=} s(u), z \stackrel{?}{=} s(u), y \stackrel{?}{=} u, s(u) \stackrel{?}{=} u\}$$

swap

$$\{x \stackrel{?}{=} s(u), z \stackrel{?}{=} s(u), y \stackrel{?}{=} u, u \stackrel{?}{=} s(u)\}$$

Exercise

$$\{\text{div}(\textcolor{red}{x}, \textcolor{blue}{s}(\textcolor{red}{y})) \stackrel{?}{=} \text{div}(\textcolor{red}{z}, \textcolor{red}{x}), \text{div}(\textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \stackrel{?}{=} \text{div}(\textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{u}))\}$$

decompose

decompose

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{y}) \stackrel{?}{=} \textcolor{red}{x}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

eliminate

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{red}{y}) \stackrel{?}{=} \textcolor{red}{z}, \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

swap

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{red}{z}, \textcolor{red}{z} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{y}), \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{z}) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

eliminate

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{y}), \textcolor{red}{z} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{y}), \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{blue}{s}(\textcolor{red}{y})) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

eliminate

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u}), \textcolor{red}{z} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u}), \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{blue}{s}(\textcolor{red}{u})) \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

decompose

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u}), \textcolor{red}{z} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u}), \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{blue}{s}(\textcolor{red}{u}) \stackrel{?}{=} \textcolor{red}{u}\}$$

swap

$$\{\textcolor{red}{x} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u}), \textcolor{red}{z} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u}), \textcolor{red}{y} \stackrel{?}{=} \textcolor{red}{u}, \textcolor{red}{u} \stackrel{?}{=} \textcolor{blue}{s}(\textcolor{red}{u})\}$$

occur-check

Exercise

$$\{\text{div}(x, s(y)) \stackrel{?}{=} \text{div}(z, x), \text{div}(y, s(z)) \stackrel{?}{=} \text{div}(u, s(u))\}$$

decompose

decompose

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} x, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} z, s(y) \stackrel{?}{=} z, y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

swap

$$\{x \stackrel{?}{=} z, z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(z) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(y), z \stackrel{?}{=} s(y), y \stackrel{?}{=} u, s(s(y)) \stackrel{?}{=} s(u)\}$$

eliminate

$$\{x \stackrel{?}{=} s(u), z \stackrel{?}{=} s(u), y \stackrel{?}{=} u, s(s(u)) \stackrel{?}{=} s(u)\}$$

decompose

$$\{x \stackrel{?}{=} s(u), z \stackrel{?}{=} s(u), y \stackrel{?}{=} u, s(u) \stackrel{?}{=} u\}$$

swap

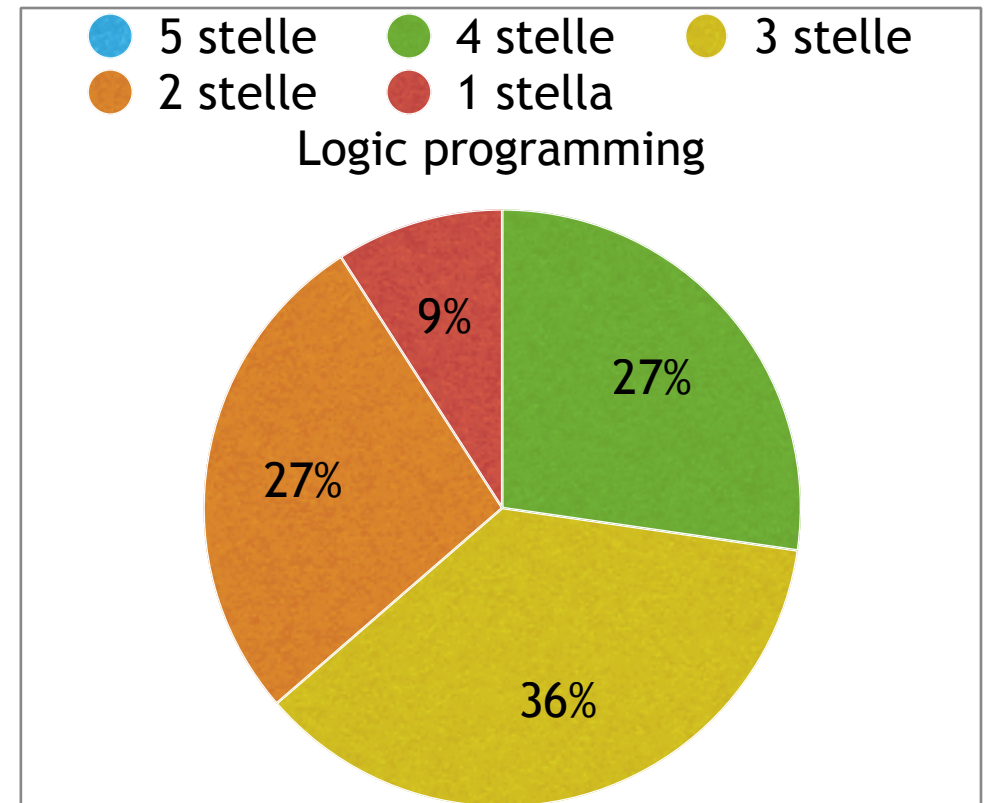
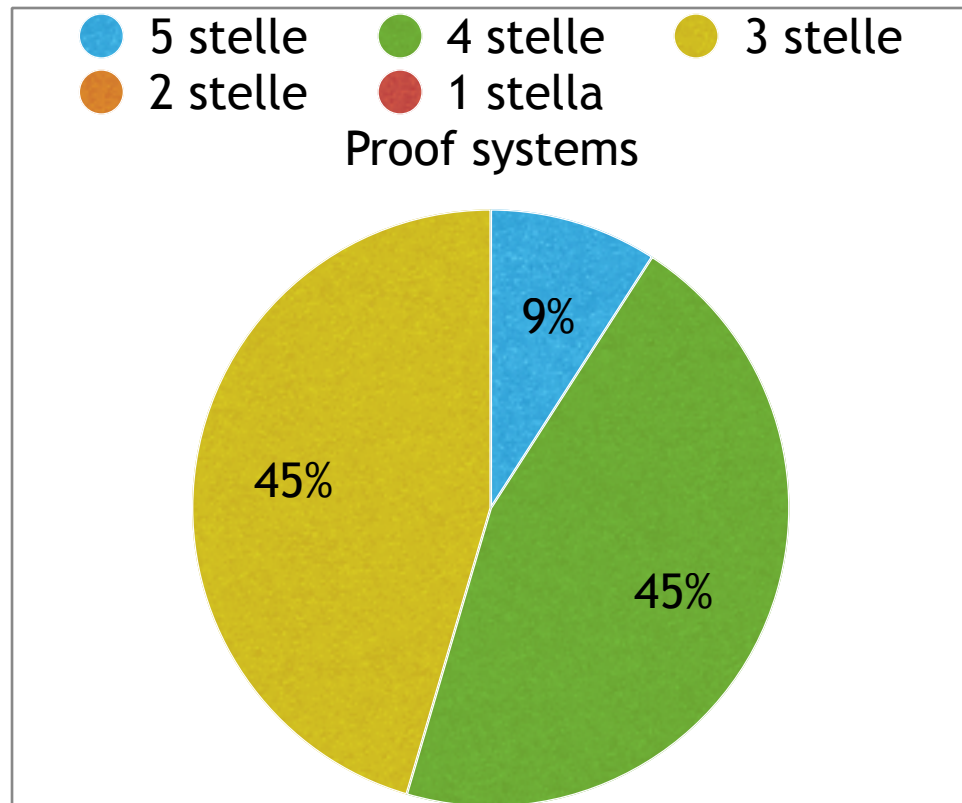
$$\{x \stackrel{?}{=} s(u), z \stackrel{?}{=} s(u), y \stackrel{?}{=} u, u \stackrel{?}{=} s(u)\}$$

occur-check



Inference rules

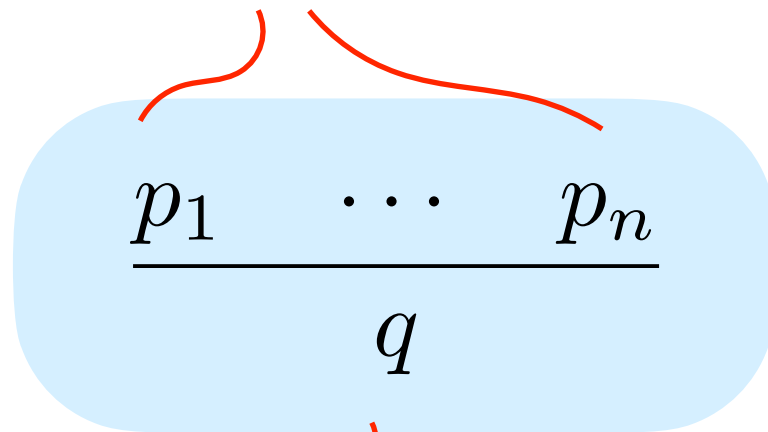
From your forms



(over 11 answers)

Inference rules

premises (one, none, many)



if the premises are valid,
then the conclusion is also valid

conclusion (one)

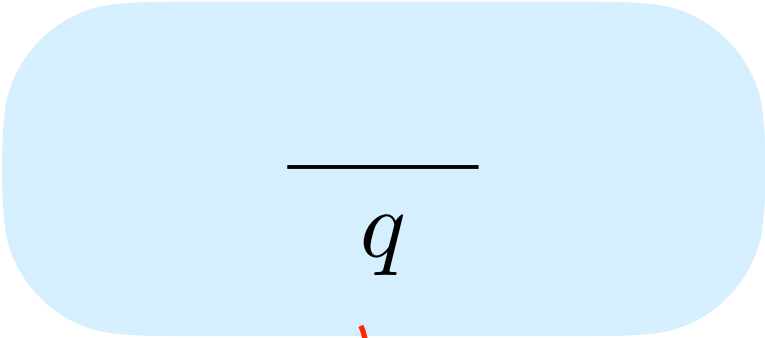
$p_1, \dots, p_n, q$ are formulas

any variable they contain is universally quantified (implicitly)

a rule instance is obtained by applying some ρ to $p_1, \dots, p_n, q$

Axioms

no premises


$$\frac{}{q}$$

the conclusion is valid

conclusion (always valid, it is a fact)

Rule instances

$$(prod) \frac{E_0 \longrightarrow n_0 \quad E_1 \longrightarrow n_1}{E_0 \otimes E_1 \longrightarrow n} \quad n = n_0 \cdot n_1$$

$$\rho \triangleq [E_0 = 1, E_1 = 1 \oplus 2, n_0 = 1, n_1 = 3, n = 3]$$

an instance
of *(prod)*

$$(prod) \frac{1 \longrightarrow 1 \quad 1 \oplus 2 \longrightarrow 3}{1 \otimes (1 \oplus 2) \longrightarrow 3} \quad 3 = 1 \cdot 3$$

Rule instances

$$(prod) \frac{E_0 \longrightarrow n_0 \quad E_1 \longrightarrow n_1}{E_0 \otimes E_1 \longrightarrow n} \quad n = n_0 \cdot n_1$$

$$\rho \triangleq [E_0 = 1, E_1 = 1 \oplus 2, n_0 = 1, n_1 = 3, n = 3]$$

an instance
of *(prod)*

$$(prod) \frac{1 \longrightarrow 1 \quad 1 \oplus 2 \longrightarrow 3}{1 \otimes (1 \oplus 2) \longrightarrow 3} \quad 3 = 1 \cdot 3$$

$$\rho \triangleq [E_0 = 1, E_1 = 1 \oplus 2, n_0 = 3, n_1 = 5, n = 15]$$

this is also
an instance
of *(prod)*!

$$(prod) \frac{1 \longrightarrow 3 \quad 1 \oplus 2 \longrightarrow 5}{1 \otimes (1 \oplus 2) \longrightarrow 15} \quad 15 = 3 \cdot 5$$

Rule instances

$$(prod) \frac{E_0 \longrightarrow n_0 \quad E_1 \longrightarrow n_1}{E_0 \otimes E_1 \longrightarrow n} \quad n = n_0 \cdot n_1$$

$$\rho \triangleq [E_0 = 1, E_1 = 1 \oplus 2, n_0 = 1, n_1 = 3, n = 3]$$

an instance
of *(prod)*

$$(prod) \frac{1 \longrightarrow 1 \quad 1 \oplus 2 \longrightarrow 3}{1 \otimes (1 \oplus 2) \longrightarrow 3} \quad 3 = 1 \cdot 3$$

$$\rho \triangleq [E_0 = 1, E_1 = 1 \oplus 2, n_0 = 3, n_1 = 5, n = 15]$$

this is also
an instance
of *(prod)*!

$$(prod) \frac{1 \longrightarrow 3 \quad 1 \oplus 2 \longrightarrow 5}{1 \otimes (1 \oplus 2) \longrightarrow 15} \quad 15 = 3 \cdot 5$$

but it is unlikely that such premises will be valid

More instances

$$(prod) \frac{E_0 \longrightarrow n_0 \quad E_1 \longrightarrow n_1}{E_0 \otimes E_1 \longrightarrow n} \quad n = n_0 \cdot n_1$$

$\rho \triangleq$

an instance
of *(prod)*

$$(prod) \frac{E \otimes 2 \longrightarrow k \quad E \oplus 1 \longrightarrow 3}{(E \otimes 2) \otimes (E \oplus 1) \longrightarrow 3k}$$

variables can be shared

More instances

$$(prod) \frac{E_0 \longrightarrow n_0 \quad E_1 \longrightarrow n_1}{E_0 \otimes E_1 \longrightarrow n} \quad n = n_0 \cdot n_1$$

$$\rho \triangleq [E_0 = E \otimes 2, E_1 = E \oplus 1, n_0 = k, n_1 = 3, n = 3k]$$

an instance
of *(prod)*

$$(prod) \frac{E \otimes 2 \longrightarrow k \quad E \oplus 1 \longrightarrow 3}{(E \otimes 2) \otimes (E \oplus 1) \longrightarrow 3k}$$

variables can be shared

Logical System

$$R = \left\{ \frac{}{r}, \frac{p_1 \cdots p_n}{q}, \dots \right\}$$

A **logical system** is just a set of axioms and inference rules

if an inference rule contains some variables,
we assume all its instances are in R

Derivation

Given a logical system R , a **derivation in R** , is written

$$d \Vdash_R q \quad d \text{ derives } q \text{ (in } R\text{)}$$

where

- either $d = \left(\frac{}{q} \right) \in R$ is an axiom of R ;
- or $d = \left(\frac{d_1, \dots, d_n}{q} \right)$ for some derivations $d_1 \Vdash_R p_1, \dots, d_n \Vdash_R p_n$ such that $\left(\frac{p_1, \dots, p_n}{q} \right) \in R$ is an inference rule of R .

a derivation is a proof tree (whose leaves are axioms)

Example

$$R = \left\{ \frac{}{N \longrightarrow n}, \frac{E_0 \longrightarrow n_0 \quad E_1 \longrightarrow n_1}{E_0 \oplus E_1 \longrightarrow n_0 + n_1}, \frac{E_0 \longrightarrow n_0 \quad E_1 \longrightarrow n_1}{E_0 \otimes E_1 \longrightarrow n_0 \cdot n_1} \right\}$$

$$d = \frac{\frac{1 \longrightarrow 1 \quad 2 \longrightarrow 2}{(1 \oplus 2) \longrightarrow 3} \quad \frac{3 \longrightarrow 3 \quad 4 \longrightarrow 4}{(3 \oplus 4) \longrightarrow 7}}{(1 \oplus 2) \otimes (3 \oplus 4) \longrightarrow 21}$$

$$d \Vdash_R (1 \oplus 2) \otimes (3 \oplus 4) \longrightarrow 21$$

Example

$S ::= \epsilon \mid (S) \mid S S$

$s \in \mathcal{L}$

Example

$$S ::= \epsilon \mid (S) \mid S S \qquad s \in \mathcal{L}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}}, \frac{S \in \mathcal{L}}{(S) \in \mathcal{L}}, \frac{S_0 \in \mathcal{L} \quad S_1 \in \mathcal{L}}{S_0 S_1 \in \mathcal{L}} \right\}$$

Example

$$S ::= \epsilon \mid (S) \mid S S \qquad s \in \mathcal{L}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}}, \frac{S \in \mathcal{L}}{(S) \in \mathcal{L}}, \frac{S_0 \in \mathcal{L} \quad S_1 \in \mathcal{L}}{S_0 S_1 \in \mathcal{L}} \right\}$$

$$((()((()))) \in \mathcal{L}$$

Example

$$S ::= \epsilon \mid (S) \mid S S \qquad s \in \mathcal{L}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}}, \frac{S \in \mathcal{L}}{(S) \in \mathcal{L}}, \frac{S_0 \in \mathcal{L} \quad S_1 \in \mathcal{L}}{S_0 S_1 \in \mathcal{L}} \right\}$$

$$((()(()))) \in \mathcal{L}$$

Example

$$S ::= \epsilon \mid (S) \mid S S \qquad s \in \mathcal{L}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}}, \frac{S \in \mathcal{L}}{(S) \in \mathcal{L}}, \frac{S_0 \in \mathcal{L} \quad S_1 \in \mathcal{L}}{S_0 S_1 \in \mathcal{L}} \right\}$$

$$\frac{() (()) \in \mathcal{L}}{((())()) \in \mathcal{L}}$$

Example

$$S ::= \epsilon \mid (S) \mid S S \qquad s \in \mathcal{L}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}}, \frac{S \in \mathcal{L}}{(S) \in \mathcal{L}}, \frac{S_0 \in \mathcal{L} \quad S_1 \in \mathcal{L}}{S_0 S_1 \in \mathcal{L}} \right\}$$

$$\frac{\frac{}{() (()) \in \mathcal{L}}}{((()) (())) \in \mathcal{L}}$$

Example

$$S ::= \epsilon \mid (S) \mid S S \qquad s \in \mathcal{L}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}}, \frac{S \in \mathcal{L}}{(S) \in \mathcal{L}}, \frac{S_0 \in \mathcal{L} \quad S_1 \in \mathcal{L}}{S_0 S_1 \in \mathcal{L}} \right\}$$

$$\frac{\frac{() \in \mathcal{L} \quad (()) \in \mathcal{L}}{() (()) \in \mathcal{L}}}{((()) (())) \in \mathcal{L}}$$

Example

$$S ::= \epsilon \mid (S) \mid S S \qquad s \in \mathcal{L}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}}, \frac{S \in \mathcal{L}}{(S) \in \mathcal{L}}, \frac{S_0 \in \mathcal{L} \quad S_1 \in \mathcal{L}}{S_0 S_1 \in \mathcal{L}} \right\}$$

$$\frac{\frac{\frac{}{() \in \mathcal{L}}}{()() \in \mathcal{L}}}{((()) \in \mathcal{L}}$$

Example

$$S ::= \epsilon \mid (S) \mid S S \qquad s \in \mathcal{L}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}}, \frac{S \in \mathcal{L}}{(S) \in \mathcal{L}}, \frac{S_0 \in \mathcal{L} \quad S_1 \in \mathcal{L}}{S_0 S_1 \in \mathcal{L}} \right\}$$

$$\begin{array}{c} \frac{\epsilon \in \mathcal{L}}{() \in \mathcal{L}} \qquad (()) \in \mathcal{L} \\ \hline ()(()) \in \mathcal{L} \\ \hline ((())()) \in \mathcal{L} \end{array}$$

Example

$$S ::= \epsilon \mid (S) \mid S S \qquad s \in \mathcal{L}$$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}}, \frac{S \in \mathcal{L}}{(S) \in \mathcal{L}}, \frac{S_0 \in \mathcal{L} \quad S_1 \in \mathcal{L}}{S_0 S_1 \in \mathcal{L}} \right\}$$

$$\frac{\frac{\frac{}{\epsilon \in \mathcal{L}}}{() \in \mathcal{L}} \quad (() \in \mathcal{L})}{() (() \in \mathcal{L})} \in \mathcal{L}$$

$$\frac{() (() \in \mathcal{L})}{(() (() \in \mathcal{L}))} \in \mathcal{L}$$

Example

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$$\frac{\overline{\epsilon \in \mathcal{L}}}{\begin{array}{cc} \frac{() \in \mathcal{L}}{\frac{() (()) \in \mathcal{L}}{((()) \in \mathcal{L})}} & \frac{((()) \in \mathcal{L}}{((()) (()) \in \mathcal{L})} \end{array}}$$

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$$d \Vdash_R (())(()) \in \mathcal{L}$$

Theorems

Given a logical system R , a **theorem of R** is written

$$\vdash_R q$$

$$\exists d. d \vdash_R q$$

where q is a formula such that
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$$I_R \triangleq \{ q \mid \Vdash_R q \}$$

Inline notation

$$d = \frac{\frac{\overline{1 \longrightarrow 1} \quad \overline{2 \longrightarrow 2}}{(1 \oplus 2) \longrightarrow 3} \quad \frac{\overline{3 \longrightarrow 3} \quad \overline{4 \longrightarrow 4}}{(3 \oplus 4) \longrightarrow 7}}{(1 \oplus 2) \otimes (3 \oplus 4) \longrightarrow 21}$$

$$(1 \oplus 2) \otimes (3 \oplus 4) \longrightarrow 21$$

goal oriented
derivation

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goal oriented
derivation

nothing left to prove

Backtracking

$$(1 \oplus 2) \otimes (3 \oplus 4) \longrightarrow 21$$

goal oriented
derivation
(depth-first)

Backtracking

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fail! need to backtrack to the last choice and retry

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alternatively, all possibilities can be explored in parallel
(breadth-first vs depth-first)

Logic programming

PROLOG

Prolog is a simple, yet powerful declarative programming language, based on first-order predicate logic

PROgrammation en LOGique

['70] (Univ. Marseilles) A. Colmerauer, P. Roussel, R. Kowalski
(aimed at processing natural (French) language)

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```
Every psychiatrist is a person.  
Every person he analyzes is sick.  
Jacques is a psychiatrist in Marseille.  
Is Jacques a person?  
Where is Jacques?  
Is Jacques sick?
```

```
Yes.  In Marseille.  
I don't know.
```

```
TOUT PSYCHIATRE EST UNE PERSONNE.  
CHAQUE PERSONNE QU'IL ANALYSE, EST MALADE.  
JACQUES EST UN PSYCHIATRE A *MARSEILLE.  
EST-CE QUE *JACQUES EST UNE PERSONNE?  
OU EST *JACQUES?  
EST-CE QUE *JACQUES EST MALADE?  
OUI. A MARSEILLE. JE NE SAIS PAS.
```

Algorithm

algorithm = logic + control

what
(problem description)

how
(steps to reach a solution)

Horn clauses

resolution

PROLOG
database

PROLOG
interpreter

Formulas

$X = \{x, y, \dots\}$ a set of variables

$\Sigma = \{\Sigma_n\}_n$ a signature of function symbols $c, f, g, \dots$

$\Pi = \{\Pi_n\}_n$ a signature of predicate symbols $p, q, \dots$

atomic formula

$$a = p(t_1, \dots, t_n) \quad \begin{array}{l} p \in \Pi_n \\ t_1, \dots, t_n \in T_{\Sigma, X} \end{array}$$

formula

$a_1, \dots, a_n$ a possibly empty conjunction of atomic formulas

Example

$X = \{S, S_0, S_1, \dots\}$ a set of variables

$\Sigma_0 = \{\epsilon, (,)\}$ a set of constants

$\Sigma_2 = \{- _ \}$ a binary (infix) operator

$\Pi_1 = \{- \in \mathcal{L}\}$ a unary predicate symbol

atomic formula

$$S) \in \mathcal{L}$$

formula

$$S) \in \mathcal{L}, SS)) \in \mathcal{L}$$

Example

$X = \{N, E, E_0, E_1, \dots, n, n_0, n_1, \dots\}$ a set of variables

$\Sigma_0 = \{0, 1, 2, \dots, 0, 1, 2, \dots\}$ a set of constants

$\Sigma_2 = \{- \oplus -, - \otimes -\}$ a set of binary (infix) operators

$\Pi_2 = \{- \longrightarrow -\}$ a binary (infix) predicate symbol

atomic formula

$$E \oplus 2 \longrightarrow 5$$

formula

$$E \oplus 2 \longrightarrow 5, E \otimes 7 \longrightarrow n$$

Logic programs

Horn clause

an atomic formula
(the HEAD) $h \text{ :- } r \cdot$ a formula
(the BODY)

Logic programs

Horn clause

an atomic formula
(the HEAD) $h \text{ :- } r$. a formula
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$a \text{ :- } a_1, \dots, a_n$. analogous to $\frac{a_1 \ \cdots \ a_n}{a}$

Logic programs

Horn clause

an atomic formula
(the HEAD) $h \text{ :- } r.$ a formula
(the BODY)

$a \text{ :- } a_1, \dots, a_n.$ analogous to $\frac{a_1 \cdots a_n}{a}$

a set (or list) of Horn clauses

logic program L

$$L = \left\{ \begin{array}{c} \dots \\ h \text{ :- } r. \\ \dots \end{array} \right\}$$

Applications

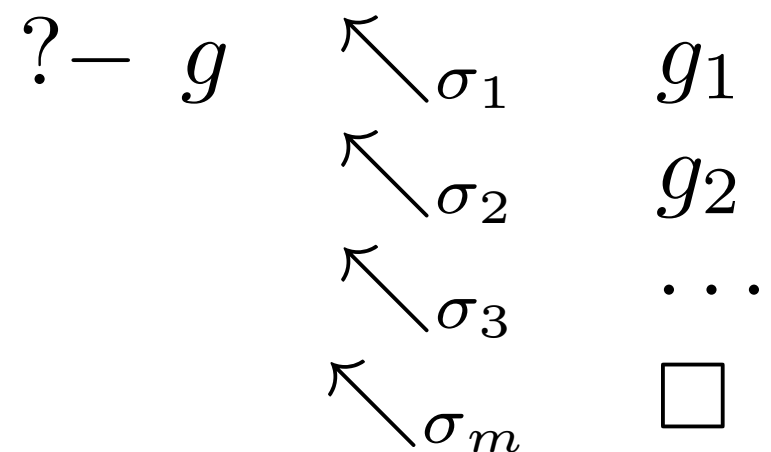
a logic program serves to answer the following question:
given a formula g that we want to prove,
what are the valid instances of g ?

$$?- \quad (1 \oplus 2) \otimes (3 \oplus 4) \longrightarrow n, \quad n \oplus E \longrightarrow 26$$

SLD resolution

Idea: iteratively reduce the initial goal g by applying one of the Horn clauses in L to one of the atomic formulas in the goal;

each application computes a most general unifier (mgu), replaces the selected formula with the body of the selected clause and applies the mgu to the new goal



then $g\sigma_1\sigma_2\dots\sigma_m$ is a theorem

Selective Linear Definite clause resolution



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The origin of the name "SLD" [\[edit \]](#)

The name "SLD resolution" was given by Maarten van Emden for the unnamed inference rule introduced by [Robert Kowalski](#).^[1] Its name is derived from SL resolution,^[2] which is both sound and refutation complete for the unrestricted clausal form of logic. "SLD" stands for "SL resolution with Definite clauses".

In both, SL and SLD, "L" stands for the fact that a resolution proof can be restricted to a linear sequence of clauses:

$$C_1, C_2, \dots, C_l$$

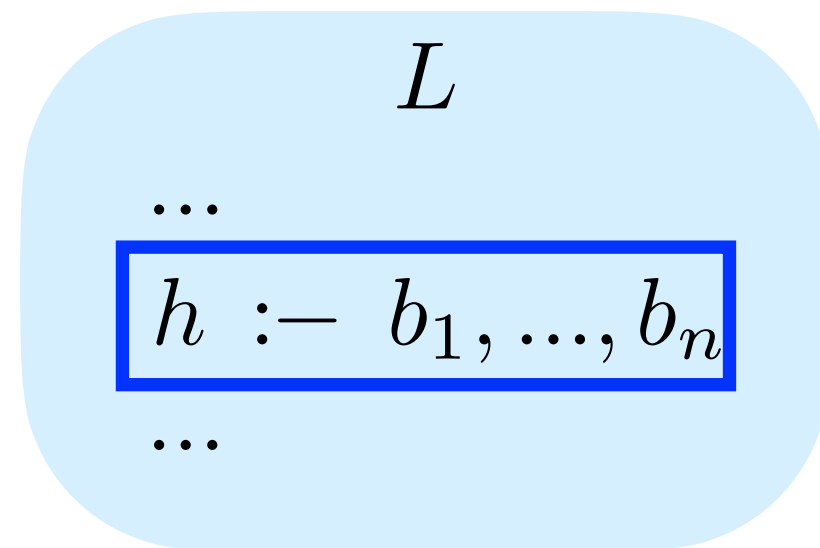
where the "top clause" C_1 is an input clause, and every other clause C_{i+1} is a resolvent one of whose parents is the previous clause C_i . The proof is a refutation if the last clause C_l is the empty clause.

In SLD, all of the clauses in the sequence are goal clauses, and the other parent is an input clause. In SL resolution, the other parent is either an input clause or an ancestor clause earlier in the sequence.

In both SL and SLD, "S" stands for the fact that the only literal resolved upon in any clause C_i is one that is uniquely selected by a selection rule or selection function. In SL resolution, the selected literal is restricted to one which has been most recently introduced into the clause. In the simplest case, such a [last-in-first-out](#) selection function can be specified by the order in which literals are written, as in [Prolog](#). However, the selection function in SLD resolution is more general than in SL resolution and in Prolog. There is no restriction on the literal that can be selected.

SLD resolution

$?- a_1, \dots, \boxed{a_i}, \dots, a_k$



repeat the following until no goal is left:

1. select a clause of the goal a_i (e.g., the first);
2. select a Horn clause $h :- b_1, \dots, b_n$ in L whose head unifies with a_i ;
3. let σ be a most general unifier ($a_i\sigma = h\sigma$);
4. replace a_i with $b_1, \dots, b_n$;
5. apply the computed substitution σ to the goal $(a_1, \dots, b_1, \dots, b_n, \dots, a_k)\sigma$

Pay attention

atomic goals can share variables: the substitution must be applied to all of them to propagate the information



$$(a_1, \dots, b_1, \dots, b_n, \dots, a_k)\sigma$$



$$a_1, \dots, (b_1, \dots, b_n)\sigma, \dots, a_k$$

Pay attention

the same clause can be reused many times:
each time its variables must be renamed (before unification)
with *fresh* identifiers to avoid clashes

repeat the following until no goal is left:

1. select a clause of the goal a_i (e.g., the first);
2. select a Horn clause $h :- b_1, \dots, b_n$ in L ;
3. let $\rho : X \rightarrow X$ rename the variables in $vars(h :- b_1, \dots, b_n)$ to fresh ones;
4. $(h :- b_1, \dots, b_n)\rho$ is called a *variant* of the original clause;
5. let σ be a most general unifier $(a_i\sigma = (h\rho)\sigma)$;
6. replace a_i with $(b_1, \dots, b_n)\rho$;
7. apply the computed substitution σ to the goal $(a_1, \dots, (b_1, \dots, b_n)\rho, \dots, a_k)\sigma$

SLD resolution

$?- a_1 , \dots , a_i , \dots , a_k .$

L

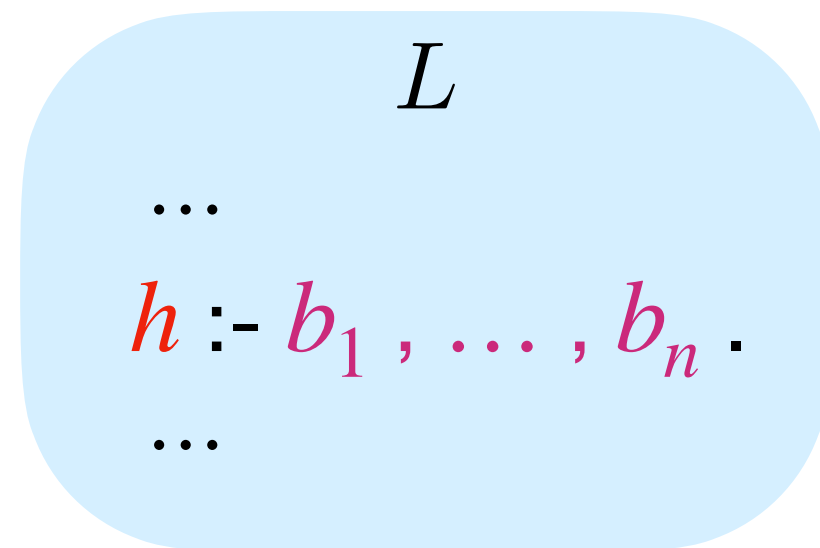
...

$h \text{ :- } b_1 , \dots , b_n .$

...

SLD resolution

$?- a_1 , \dots , a_i , \dots , a_k .$



$(h :- b_1 , \dots , b_n) \rho$
fresh renaming

SLD resolution

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L

...

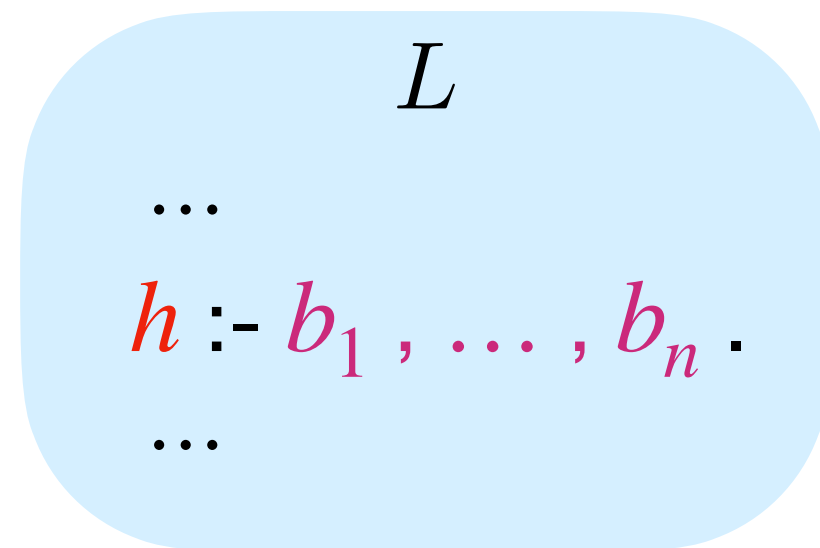
$h :- b_1 , \dots , b_n .$

...

$(h :- b_1 , \dots , b_n) \rho$
fresh renaming

$\sigma = \text{mgu} \{ a_i = h\rho \} \quad g = b_1\rho , \dots , b_n\rho$

SLD resolution



?- $a_1, \dots, a_i, \dots, a_k$.

$(h \text{ :- } b_1, \dots, b_n) \rho$ fresh renaming

$\sigma = \text{mgu}\{ a_i = h\rho \} \quad g = b_1\rho, \dots, b_n\rho$

?- $a_1\sigma, \dots, g\sigma, \dots, a_k\sigma$

Pay attention

in the computed substitutions, only the variables that appears in the goal are of some interest to us

$$\begin{array}{l} \sigma : X \rightarrow T_{\Sigma, X} \\ Y \subseteq X \end{array} \quad \sigma|_Y(x) \triangleq \begin{cases} \sigma(x) & \text{if } x \in Y \\ x & \text{otherwise} \end{cases}$$

we only record this partial information

$$a_1, \dots, a_i, \dots, a_k \quad \nwarrow_{\hat{\sigma}} \quad (a_1, \dots, (b_1, \dots, b_n)\rho, \dots, a_k)\sigma$$

$$\hat{\sigma} \triangleq \sigma|_{vars(a_1, \dots, a_k)}$$

Computed answer substitution

$$\begin{array}{ccc}
 ?- \ g & \nwarrow \hat{\sigma}_1 & g_1 \\
 & \nwarrow \hat{\sigma}_2 & g_2 \\
 & \nwarrow \hat{\sigma}_3 & \dots \\
 & \nwarrow \hat{\sigma}_m & \square
 \end{array}$$

$$\sigma = \hat{\sigma}_1 \cdot \hat{\sigma}_2 \cdot \hat{\sigma}_3 \cdots \hat{\sigma}_m$$

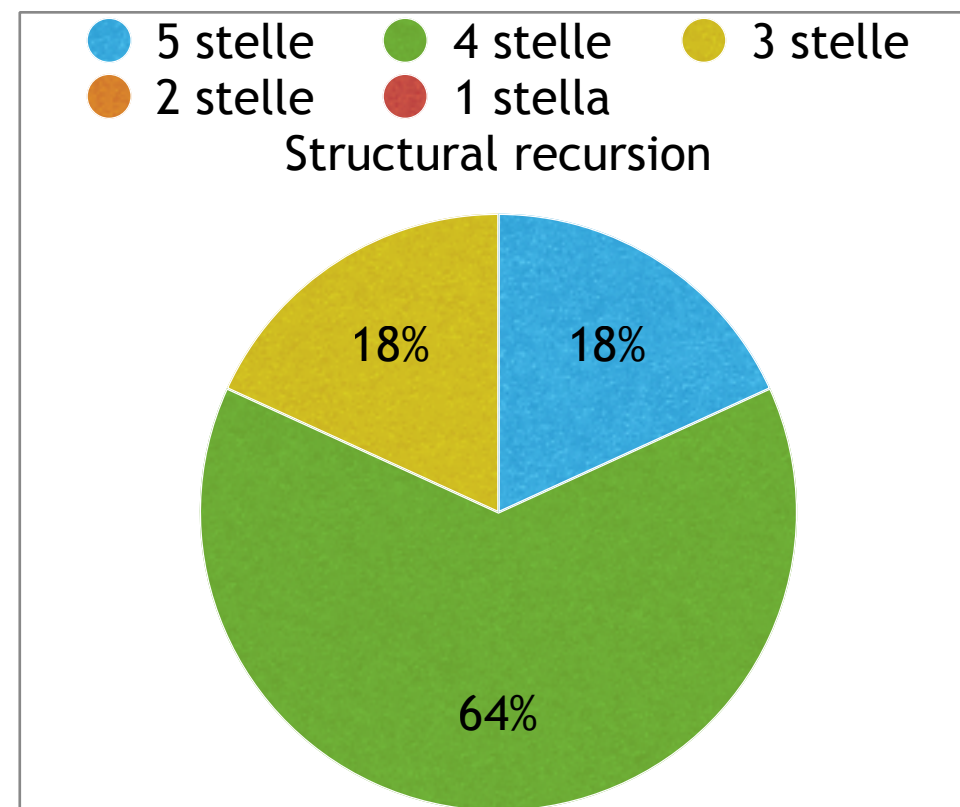
is called cas (computed answer substitution)

where

$$t(\sigma_1 \cdot \sigma_2) \stackrel{\Delta}{=} t\sigma_1\sigma_2 = \sigma_2(\sigma_1(t)) = (\sigma_2 \circ \sigma_1)(t)$$

Structural recursion

the previous definition is an example of structural recursion



(over 11 answers)

Example

$$\Sigma_0 = \{0, \dots\} \quad \Pi_3 = \{\text{sum}, \dots\}$$

$$\Sigma_1 = \{s, \dots\}$$

sum as a predicate $\text{sum}(x, y, z)$ means $x + y = z$

(a fact)

L

$\text{sum}(0, y, y).$

$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

in PROLOG
each statement
ends with dot

SLD resolution

?- sum(s(s(0)) , s(s(0)) , *n*).

L

...

sum(s(*x*) , *y* , s(*z*)) :- sum(*x* , *y* , *z*).

...

SLD resolution

?- sum(s(s(0)) , s(s(0)) , *n*).

L

...

sum(s(*x*) , *y* , s(*z*)) :- sum(*x* , *y* , *z*).

...

sum(s(*x*₁) , *y*₁ , s(*z*₁)) :- sum(*x*₁ , *y*₁ , *z*₁).

fresh renaming

SLD resolution

?- sum(s(s(0)) , s(s(0)) , *n*).

L

...

sum(s(*x*) , *y* , s(*z*)) :- sum(*x* , *y* , *z*).

...

sum(s(*x*₁) , *y*₁ , s(*z*₁)) :- sum(*x*₁ , *y*₁ , *z*₁).

fresh renaming

σ = mgu{ sum(s(s(0)) , s(s(0)) , *n*) = sum(s(*x*₁) , *y*₁ , s(*z*₁)) } *g* = sum(*x*₁ , *y*₁ , *z*₁)

SLD resolution

?- sum(s(s(0)) , s(s(0)) , n).

L

...

sum(s(x) , y , s(z)) :- sum(x , y , z).

...

sum(s(x_1) , y_1 , s(z_1)) :- sum(x_1 , y_1 , z_1).
fresh renaming

$\sigma = \text{mgu} \{ \text{sum}(s(s(0)), s(s(0)), n) = \text{sum}(s(x_1), y_1, s(z_1)) \}$ $g = \text{sum}(x_1, y_1, z_1)$

$\sigma = [x_1 = s(0) , y_1 = s(s(0)) , n = s(z_1)]$

SLD resolution

?- sum(s(s(0)) , s(s(0)) , n).

L

...

sum(s(x) , y , s(z)) :- sum(x , y , z).

...

sum(s(x_1) , y_1 , s(z_1)) :- sum(x_1 , y_1 , z_1).
fresh renaming

σ = mgu{ sum(s(s(0)) , s(s(0)) , n) = sum(s(x_1) , y_1 , s(z_1)) } g = sum(x_1 , y_1 , z_1)

σ = [x_1 = s(0) , y_1 = s(s(0)) , n = s(z_1)]

?- sum(s(0) , s(s(0)) , z_1).

$$2+2=?$$

our goal $? - \text{sum}(s(s(0)), s(s(0)), n).$

L

$\text{sum}(0, y, y).$

$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

$\text{sum}(s(s(0)), s(s(0)), n)$

$$2+2=?$$

our goal $? - \text{sum}(s(s(0)), s(s(0)), n).$

L

$\text{sum}(0, y, y).$

$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

$$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(0, y', y')\}$$

$\text{sum}(s(s(0)), s(s(0)), n)$

$$2+2=?$$

our goal $? - \text{sum}(s(s(0)), s(s(0)), n).$

L

$\text{sum}(0, y, y).$

$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(0, y', y')\}$ **fails**

$\text{sum}(s(s(0)), s(s(0)), n)$

2+2=?

our goal $? - \text{sum}(s(s(0)), s(s(0)), n).$

L

$\text{sum}(0, y, y).$
 $\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(0, y', y')\}$ **fails**

$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(s(x_1), y_1, s(z_1))\}$

$\text{sum}(s(s(0)), s(s(0)), n)$

2+2=?

our goal $? - \text{sum}(s(s(0)), s(s(0)), n).$

L

$\text{sum}(0, y, y).$
 $\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(0, y', y')\}$ **fails**

$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(s(x_1), y_1, s(z_1))\}$ **succeeds**

$\text{sum}(s(s(0)), s(s(0)), n)$

2+2=?

our goal $? - \text{sum}(s(s(0)), s(s(0)), n).$

L
 $\text{sum}(0, y, y).$
 $\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(0, y', y')\}$ **fails**

$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(s(x_1), y_1, s(z_1))\}$ **succeeds**

$$\sigma_1 = [x_1 = s(0), y_1 = s(s(0)), n = s(z_1)]$$

$\text{sum}(s(s(0)), s(s(0)), n)$

2+2=?

our goal $? - \text{sum}(s(s(0)), s(s(0)), n).$

L
 $\text{sum}(0, y, y).$
 $\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(0, y', y')\}$ **fails**

$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(s(x_1), y_1, s(z_1))\}$ **succeeds**

$$\sigma_1 = [x_1 = s(0), y_1 = s(s(0)), n = s(z_1)]$$

$$\hat{\sigma}_1 = [n = s(z_1)]$$

$\text{sum}(s(s(0)), s(s(0)), n)$

2+2=?

our goal $? - \text{sum}(s(s(0)), s(s(0)), n).$

L

$\text{sum}(0, y, y).$
 $\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(0, y', y')\}$ **fails**

$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(s(x_1), y_1, s(z_1))\}$ **succeeds**

$$\sigma_1 = [x_1 = s(0), y_1 = s(s(0)), n = s(z_1)]$$

$$\hat{\sigma}_1 = [n = s(z_1)]$$

$$\text{sum}(s(s(0)), s(s(0)), n) \xleftarrow{\hat{\sigma}_1} (\text{sum}(x_1, y_1, z_1))\sigma_1$$

$\hat{\sigma}_1 = [n = s(z_1)]$

2+2=?

our goal $? - \text{sum}(s(s(0)), s(s(0)), n).$

L

$\text{sum}(0, y, y).$
 $\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(0, y', y')\}$ **fails**

$\{\text{sum}(s(s(0)), s(s(0)), n) \stackrel{?}{=} \text{sum}(s(x_1), y_1, s(z_1))\}$ **succeeds**

$$\sigma_1 = [x_1 = s(0), y_1 = s(s(0)), n = s(z_1)]$$

$$\hat{\sigma}_1 = [n = s(z_1)]$$

$$\text{sum}(s(s(0)), s(s(0)), n) \xleftarrow{\hat{\sigma}_1} (\text{sum}(x_1, y_1, z_1))\sigma_1$$

$$= \text{sum}(s(0), s(s(0)), z_1)$$

$\hat{\sigma}_1 = [n = s(z_1)]$

$$1+2=?$$

our new
goal

$$\text{sum}(s(0), s(s(0)), z_1).$$

$$L$$

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\text{sum}(s(s(0)), s(s(0)), n) \xleftarrow{\hat{\sigma}_1} \text{sum}(s(0), s(s(0)), z_1) \quad \hat{\sigma}_1 = [n = s(z_1)]$$

$$1+2=?$$

our new
goal

$$\text{sum}(s(0), s(s(0)), z_1).$$

L

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\{\text{sum}(s(0), s(s(0)), z_1) \stackrel{?}{=} \text{sum}(0, y', y')\}$$

$$\text{sum}(s(s(0)), s(s(0)), n) \xleftarrow{\hat{\sigma}_1} \text{sum}(s(0), s(s(0)), z_1)$$

$$\hat{\sigma}_1 = [n = s(z_1)]$$

$$1+2=?$$

our new
goal

$$\text{sum}(s(0), s(s(0)), z_1).$$

L

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\{\text{sum}(s(0), s(s(0)), z_1) \stackrel{?}{=} \text{sum}(0, y', y')\}$$

fails

$$\text{sum}(s(s(0)), s(s(0)), n) \nwarrow_{\hat{\sigma}_1} \text{sum}(s(0), s(s(0)), z_1)$$

$$\hat{\sigma}_1 = [n = s(z_1)]$$

1+2=?

our new
goal

$$\text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1).$$

$$\begin{array}{l} L \\ \text{sum}(0, \textcolor{red}{y}, \textcolor{red}{y}). \\ \text{sum}(\textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) :- \text{sum}(\textcolor{red}{x}, \textcolor{red}{y}, \textcolor{red}{z}). \end{array}$$

$$\{\text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1) \stackrel{?}{=} \text{sum}(0, \textcolor{red}{y}', \textcolor{red}{y}')\} \quad \text{fails}$$

$$\{\text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1) \stackrel{?}{=} \text{sum}(\textcolor{blue}{s}(\textcolor{red}{x}_2), \textcolor{red}{y}_2, \textcolor{blue}{s}(\textcolor{red}{z}_2))\}$$

$$\text{sum}(\textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{n}) \nwarrow_{\hat{\sigma}_1} \text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1) \quad \hat{\sigma}_1 = [\textcolor{red}{n} = \textcolor{blue}{s}(\textcolor{red}{z}_1)]$$

$$1+2=?$$

our new
goal

$$\text{sum}(s(0), s(s(0)), z_1).$$

L

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\{\text{sum}(s(0), s(s(0)), z_1) \stackrel{?}{=} \text{sum}(0, y', y')\}$$

fails

$$\{\text{sum}(s(0), s(s(0)), z_1) \stackrel{?}{=} \text{sum}(s(x_2), y_2, s(z_2))\}$$

succeeds

$$\text{sum}(s(s(0)), s(s(0)), n) \nwarrow_{\hat{\sigma}_1} \text{sum}(s(0), s(s(0)), z_1)$$

$$\hat{\sigma}_1 = [n = s(z_1)]$$

1+2=?

our new
goal

$$\text{sum}(s(0), s(s(0)), z_1).$$

$$L$$

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\{\text{sum}(s(0), s(s(0)), z_1) \stackrel{?}{=} \text{sum}(0, y', y')\} \quad \text{fails}$$

$$\{\text{sum}(s(0), s(s(0)), z_1) \stackrel{?}{=} \text{sum}(s(x_2), y_2, s(z_2))\} \quad \text{succeeds}$$

$$\sigma_2 = [x_2 = 0, y_2 = s(s(0)), z_1 = s(z_2)]$$

$$\text{sum}(s(s(0)), s(s(0)), n) \xleftarrow{\hat{\sigma}_1} \text{sum}(s(0), s(s(0)), z_1) \quad \hat{\sigma}_1 = [n = s(z_1)]$$

1+2=?

our new
goal

$$\text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1).$$

$$\begin{array}{l} L \\ \text{sum}(0, \textcolor{red}{y}, \textcolor{red}{y}). \\ \text{sum}(\textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) :- \text{sum}(\textcolor{red}{x}, \textcolor{red}{y}, \textcolor{red}{z}). \end{array}$$

$$\{\text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1) \stackrel{?}{=} \text{sum}(0, \textcolor{red}{y}', \textcolor{red}{y}')\} \quad \text{fails}$$

$$\{\text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1) \stackrel{?}{=} \text{sum}(\textcolor{blue}{s}(\textcolor{red}{x}_2), \textcolor{red}{y}_2, \textcolor{blue}{s}(\textcolor{red}{z}_2))\} \quad \text{succeeds}$$

$$\sigma_2 = [\textcolor{red}{x}_2 = 0, \textcolor{red}{y}_2 = \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{red}{z}_2)]$$

$$\hat{\sigma}_2 = [\textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{red}{z}_2)]$$

$$\text{sum}(\textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{n}) \quad \nwarrow_{\hat{\sigma}_1} \quad \text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1) \quad \hat{\sigma}_1 = [\textcolor{red}{n} = \textcolor{blue}{s}(\textcolor{red}{z}_1)]$$

1+2=?

our new
goal

$$\text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1).$$

$$\begin{array}{l} L \\ \text{sum}(\textcolor{blue}{0}, \textcolor{red}{y}, \textcolor{red}{y}). \\ \text{sum}(\textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) :- \text{sum}(\textcolor{red}{x}, \textcolor{red}{y}, \textcolor{red}{z}). \end{array}$$

$$\{\text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1) \stackrel{?}{=} \text{sum}(\textcolor{blue}{0}, \textcolor{red}{y}', \textcolor{red}{y}')\} \quad \text{fails}$$

$$\{\text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1) \stackrel{?}{=} \text{sum}(\textcolor{blue}{s}(\textcolor{red}{x}_2), \textcolor{red}{y}_2, \textcolor{blue}{s}(\textcolor{red}{z}_2))\} \quad \text{succeeds}$$

$$\sigma_2 = [\textcolor{red}{x}_2 = \textcolor{blue}{0}, \textcolor{red}{y}_2 = \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{red}{z}_2)]$$

$$\hat{\sigma}_2 = [\textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{red}{z}_2)]$$

$$\begin{array}{ccc} \text{sum}(\textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{n}) & \begin{array}{l} \nwarrow \hat{\sigma}_1 \\ \nwarrow \hat{\sigma}_2 \end{array} & \begin{array}{l} \text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1) \\ (\text{sum}(\textcolor{red}{x}_2, \textcolor{red}{y}_2, \textcolor{red}{z}_2))\sigma_2 \end{array} \end{array}$$

$$\begin{array}{l} \hat{\sigma}_1 = [\textcolor{red}{n} = \textcolor{blue}{s}(\textcolor{red}{z}_1)] \\ \hat{\sigma}_2 = [\textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{red}{z}_2)] \end{array}$$

1+2=?

our new
goal

$$\text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1).$$

$$\begin{array}{l} L \\ \text{sum}(\textcolor{blue}{0}, \textcolor{red}{y}, \textcolor{red}{y}). \\ \text{sum}(\textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) :- \text{sum}(\textcolor{red}{x}, \textcolor{red}{y}, \textcolor{red}{z}). \end{array}$$

$$\{\text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1) \stackrel{?}{=} \text{sum}(\textcolor{blue}{0}, \textcolor{red}{y}', \textcolor{red}{y}')\} \quad \text{fails}$$

$$\{\text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1) \stackrel{?}{=} \text{sum}(\textcolor{blue}{s}(\textcolor{red}{x}_2), \textcolor{red}{y}_2, \textcolor{blue}{s}(\textcolor{red}{z}_2))\} \quad \text{succeeds}$$

$$\sigma_2 = [\textcolor{red}{x}_2 = \textcolor{blue}{0}, \textcolor{red}{y}_2 = \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{red}{z}_2)]$$

$$\hat{\sigma}_2 = [\textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{red}{z}_2)]$$

$$\begin{array}{l} \text{sum}(\textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{n}) \quad \nwarrow_{\hat{\sigma}_1} \quad \text{sum}(\textcolor{blue}{s}(0), \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_1) \\ \quad \nwarrow_{\hat{\sigma}_2} \quad (\text{sum}(\textcolor{red}{x}_2, \textcolor{red}{y}_2, \textcolor{red}{z}_2))\sigma_2 \\ \quad \quad = \text{sum}(\textcolor{blue}{0}, \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{z}_2) \end{array}$$

$$\begin{array}{l} \hat{\sigma}_1 = [\textcolor{red}{n} = \textcolor{blue}{s}(\textcolor{red}{z}_1)] \\ \hat{\sigma}_2 = [\textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{red}{z}_2)] \end{array}$$

$$0+2=?$$

our new
goal

$$\text{sum}(0, s(s(0)), z_2).$$

L

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\text{sum}(s(s(0)), s(s(0)), n)$$

$\nwarrow \hat{\sigma}_1$

$$\text{sum}(s(0), s(s(0)), z_1)$$

$\nwarrow \hat{\sigma}_2$

$$\text{sum}(0, s(s(0)), z_2)$$

$$\hat{\sigma}_1 = [n = s(z_1)]$$

$$\hat{\sigma}_2 = [z_1 = s(z_2)]$$

$$0+2=?$$

our new
goal

$$\text{sum}(0, s(s(0)), z_2).$$

L

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\{\text{sum}(0, s(s(0)), z_2) \stackrel{?}{=} \text{sum}(0, y_3, y_3)\}$$

$$\text{sum}(s(s(0)), s(s(0)), n)$$

$\nwarrow \hat{\sigma}_1$

$$\text{sum}(s(0), s(s(0)), z_1)$$

$\nwarrow \hat{\sigma}_2$

$$\text{sum}(0, s(s(0)), z_2)$$

$$\hat{\sigma}_1 = [n = s(z_1)]$$

$$\hat{\sigma}_2 = [z_1 = s(z_2)]$$

$$0+2=?$$

our new
goal

$$\text{sum}(0, s(s(0)), z_2).$$

L

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\{\text{sum}(0, s(s(0)), z_2) \stackrel{?}{=} \text{sum}(0, y_3, y_3)\}$$

succeeds

$$\text{sum}(s(s(0)), s(s(0)), n)$$

$\nwarrow \hat{\sigma}_1$

$$\text{sum}(s(0), s(s(0)), z_1)$$

$\nwarrow \hat{\sigma}_2$

$$\text{sum}(0, s(s(0)), z_2)$$

$$\hat{\sigma}_1 = [n = s(z_1)]$$

$$\hat{\sigma}_2 = [z_1 = s(z_2)]$$

$$0+2=?$$

our new
goal

$$\text{sum}(0, s(s(0)), z_2).$$

L

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\{\text{sum}(0, s(s(0)), z_2) \stackrel{?}{=} \text{sum}(0, y_3, y_3)\}$$

succeeds

$$\sigma_3 = [y_3 = s(s(0)), z_2 = s(s(0))]$$

$$\text{sum}(s(s(0)), s(s(0)), n)$$

$$\nwarrow_{\hat{\sigma}_1}$$

$$\text{sum}(s(0), s(s(0)), z_1)$$

$$\nwarrow_{\hat{\sigma}_2}$$

$$\text{sum}(0, s(s(0)), z_2)$$

$$\hat{\sigma}_1 = [n = s(z_1)]$$

$$\hat{\sigma}_2 = [z_1 = s(z_2)]$$

$$0+2=?$$

our new
goal

$$\text{sum}(0, s(s(0)), z_2).$$

L

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\{\text{sum}(0, s(s(0)), z_2) \stackrel{?}{=} \text{sum}(0, y_3, y_3)\}$$

succeeds

$$\sigma_3 = [y_3 = s(s(0)), z_2 = s(s(0))]$$

$$\hat{\sigma}_3 = [z_2 = s(s(0))]$$

$$\begin{array}{ccc} \text{sum}(s(s(0)), s(s(0)), n) & \nwarrow_{\hat{\sigma}_1} & \text{sum}(s(0), s(s(0)), z_1) \\ & \nwarrow_{\hat{\sigma}_2} & \text{sum}(0, s(s(0)), z_2) \end{array}$$

$$\hat{\sigma}_1 = [n = s(z_1)]$$

$$\hat{\sigma}_2 = [z_1 = s(z_2)]$$

$$0+2=?$$

our new
goal

$$\text{sum}(0, s(s(0)), z_2).$$

L

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\{\text{sum}(0, s(s(0)), z_2) \stackrel{?}{=} \text{sum}(0, y_3, y_3)\}$$

succeeds

$$\sigma_3 = [y_3 = s(s(0)), z_2 = s(s(0))]$$

$$\hat{\sigma}_3 = [z_2 = s(s(0))]$$

$$\text{sum}(s(s(0)), s(s(0)), n)$$

$$\nwarrow \hat{\sigma}_1$$

$$\text{sum}(s(0), s(s(0)), z_1)$$

$$\nwarrow \hat{\sigma}_2$$

$$\text{sum}(0, s(s(0)), z_2)$$

$$\nwarrow \hat{\sigma}_3$$



$$\hat{\sigma}_1 = [n = s(z_1)]$$

$$\hat{\sigma}_2 = [z_1 = s(z_2)]$$

$$\hat{\sigma}_3 = [z_2 = s(s(0))]$$

$$0+2=?$$

our new
goal

$$\text{sum}(0, s(s(0)), z_2).$$

L

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\{\text{sum}(0, s(s(0)), z_2) \stackrel{?}{=} \text{sum}(0, y_3, y_3)\}$$

succeeds

$$\sigma_3 = [y_3 = s(s(0)), z_2 = s(s(0))]$$

$$\hat{\sigma}_3 = [z_2 = s(s(0))]$$

$$\text{sum}(s(s(0)), s(s(0)), n)$$

$$\nwarrow \hat{\sigma}_1$$

$$\text{sum}(s(0), s(s(0)), z_1)$$

$$\nwarrow \hat{\sigma}_2$$

$$\text{sum}(0, s(s(0)), z_2)$$

$$\nwarrow \hat{\sigma}_3$$

□

$$\hat{\sigma}_1 = [n = s(z_1)]$$

$$\hat{\sigma}_2 = [z_1 = s(z_2)]$$

$$\hat{\sigma}_3 = [z_2 = s(s(0))]$$

$$\hat{\sigma}_1 \cdot \hat{\sigma}_2 \cdot \hat{\sigma}_3 = [n = s(s(s(s(0))))]$$

0+2=?

our new
goal

$$\text{sum}(0, s(s(0)), z_2).$$

L

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\{\text{sum}(0, s(s(0)), z_2) \stackrel{?}{=} \text{sum}(0, y_3, y_3)\}$$

succeeds

$$\sigma_3 = [y_3 = s(s(0)), z_2 = s(s(0))]$$

$$\hat{\sigma}_3 = [z_2 = s(s(0))]$$

$$\text{sum}(s(s(0)), s(s(0)), n)$$

$$\nwarrow \hat{\sigma}_1$$

$$\text{sum}(s(0), s(s(0)), z_1)$$

$$\hat{\sigma}_1 = [n = s(z_1)]$$

$$\nwarrow \hat{\sigma}_2$$

$$\text{sum}(0, s(s(0)), z_2)$$

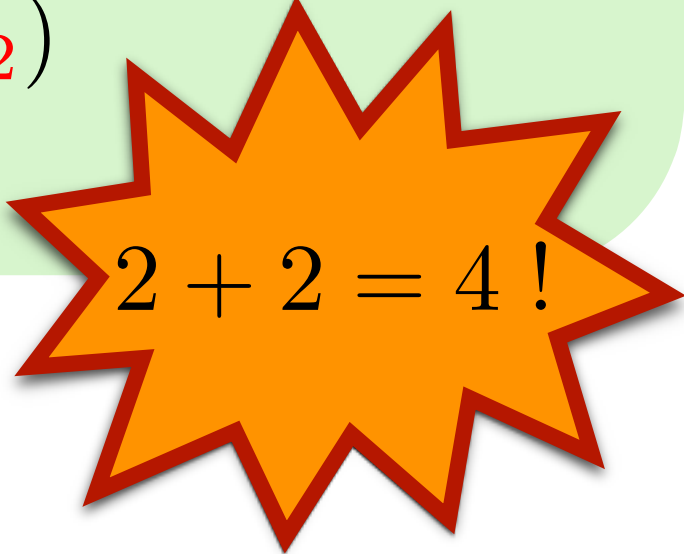
$$\hat{\sigma}_2 = [z_1 = s(z_2)]$$

$$\nwarrow \hat{\sigma}_3$$



$$\hat{\sigma}_3 = [z_2 = s(s(0))]$$

$$\hat{\sigma}_1 \cdot \hat{\sigma}_2 \cdot \hat{\sigma}_3 = [n = s(s(s(s(0))))]$$



$$2 + 2 = 4 !$$

$1+n=3?$

our goal ? — $\text{sum}(s(0), n, s(s(s(0))))$.

L

$\text{sum}(0, y, y).$

$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

$\text{sum}(s(0), n, s(s(s(0))))$

1+n=3?

our goal $? - \text{sum}(s(0), n, s(s(s(0))))$.

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$\text{sum}(s(0), n, s(s(s(0))))$

1+n=3?

our goal $? - \text{sum}(s(0), n, s(s(s(0))))$.

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$\text{sum}(s(0), n, s(s(s(0))))$

1+n=3?

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$\{\text{sum}(s(0), n, s(s(s(0)))) \stackrel{?}{=} \text{sum}(s(x_1), y_1, s(z_1))\}$

$\text{sum}(s(0), n, s(s(s(0))))$

1+n=3?

our goal $? - \text{sum}(s(0), n, s(s(s(0))))$.

L
 $\text{sum}(0, y, y).$
 $\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

$\{\text{sum}(s(0), n, s(s(s(0)))) \stackrel{?}{=} \text{sum}(0, y', y')\}$ **fails**

$\{\text{sum}(s(0), n, s(s(s(0)))) \stackrel{?}{=} \text{sum}(s(x_1), y_1, s(z_1))\}$ **succeeds**

$\text{sum}(s(0), n, s(s(s(0))))$

1+n=3?

our goal $?- \text{sum}(\textcolor{blue}{s}(0), \textcolor{red}{n}, \textcolor{blue}{s}(\textcolor{blue}{s}(\textcolor{blue}{s}(0))))$.

L
 $\text{sum}(0, \textcolor{red}{y}, \textcolor{red}{y}).$
 $\text{sum}(\textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) :- \text{sum}(\textcolor{red}{x}, \textcolor{red}{y}, \textcolor{red}{z}).$

$\{\text{sum}(\textcolor{blue}{s}(0), \textcolor{red}{n}, \textcolor{blue}{s}(\textcolor{blue}{s}(\textcolor{blue}{s}(0)))) \stackrel{?}{=} \text{sum}(0, \textcolor{red}{y}', \textcolor{red}{y}')\}$ **fails**

$\{\text{sum}(\textcolor{blue}{s}(0), \textcolor{red}{n}, \textcolor{blue}{s}(\textcolor{blue}{s}(\textcolor{blue}{s}(0)))) \stackrel{?}{=} \text{sum}(\textcolor{blue}{s}(\textcolor{red}{x}_1), \textcolor{red}{y}_1, \textcolor{blue}{s}(\textcolor{red}{z}_1))\}$ **succeeds**

$$\sigma_1 = [\textcolor{red}{x}_1 = 0, \textcolor{red}{y}_1 = \textcolor{red}{n}, \textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))]$$

$\text{sum}(\textcolor{blue}{s}(0), \textcolor{red}{n}, \textcolor{blue}{s}(\textcolor{blue}{s}(\textcolor{blue}{s}(0))))$

1+n=3?

our goal $?- \text{sum}(\textcolor{blue}{s}(0), \textcolor{red}{n}, \textcolor{blue}{s}(\textcolor{blue}{s}(\textcolor{blue}{s}(0))))$.

L

$\text{sum}(0, \textcolor{red}{y}, \textcolor{red}{y}).$
 $\text{sum}(\textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) :- \text{sum}(\textcolor{red}{x}, \textcolor{red}{y}, \textcolor{red}{z}).$

$\{\text{sum}(\textcolor{blue}{s}(0), \textcolor{red}{n}, \textcolor{blue}{s}(\textcolor{blue}{s}(\textcolor{blue}{s}(0)))) \stackrel{?}{=} \text{sum}(0, \textcolor{red}{y}', \textcolor{red}{y}')\}$ **fails**

$\{\text{sum}(\textcolor{blue}{s}(0), \textcolor{red}{n}, \textcolor{blue}{s}(\textcolor{blue}{s}(\textcolor{blue}{s}(0)))) \stackrel{?}{=} \text{sum}(\textcolor{blue}{s}(\textcolor{red}{x}_1), \textcolor{red}{y}_1, \textcolor{blue}{s}(\textcolor{red}{z}_1))\}$ **succeeds**

$$\sigma_1 = [\textcolor{red}{x}_1 = 0, \textcolor{red}{y}_1 = \textcolor{red}{n}, \textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))]$$

$$\hat{\sigma}_1 = []$$

$\text{sum}(\textcolor{blue}{s}(0), \textcolor{red}{n}, \textcolor{blue}{s}(\textcolor{blue}{s}(\textcolor{blue}{s}(0))))$

1+n=3?

our goal $? - \text{sum}(s(0), n, s(s(s(0))))$.

L

$\text{sum}(0, y, y).$
 $\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

$\{\text{sum}(s(0), n, s(s(s(0)))) \stackrel{?}{=} \text{sum}(0, y', y')\}$ **fails**

$\{\text{sum}(s(0), n, s(s(s(0)))) \stackrel{?}{=} \text{sum}(s(x_1), y_1, s(z_1))\}$ **succeeds**

$$\sigma_1 = [x_1 = 0, y_1 = n, z_1 = s(s(0))]$$

$$\hat{\sigma}_1 = []$$

$$\text{sum}(s(0), n, s(s(s(0)))) \xleftarrow{\hat{\sigma}_1} (\text{sum}(x_1, y_1, z_1))\sigma_1 \quad \hat{\sigma}_1 = []$$

1+n=3?

our goal $? - \text{sum}(s(0), n, s(s(s(0))))$.

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$\text{sum}(0, y, y).$
 $\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$

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$$\sigma_1 = [x_1 = 0, y_1 = n, z_1 = s(s(0))]$$

$$\hat{\sigma}_1 = []$$

$$\text{sum}(s(0), n, s(s(s(0)))) \xleftarrow{\hat{\sigma}_1} (\text{sum}(x_1, y_1, z_1))\sigma_1$$

$$= \text{sum}(0, n, s(s(0)))$$

$\hat{\sigma}_1 = []$

$$0+n=2?$$

our new
goal

$$\text{sum}(0, n, s(s(0)))$$

L

$$\text{sum}(0, y, y).$$

$$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z).$$

$$\text{sum}(s(0), n, s(s(s(0)))) \xleftarrow{\hat{\sigma}_1} \text{sum}(0, n, s(s(0)))$$

$$\hat{\sigma}_1 = []$$

$$0+n=2?$$

our new
goal

$$\text{sum}(0, n, s(s(0)))$$

L

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$$\{\text{sum}(0, n, s(s(0))) \stackrel{?}{=} \text{sum}(0, y_2, y_2)\}$$

$$\text{sum}(s(0), n, s(s(s(0)))) \nwarrow_{\hat{\sigma}_1} \text{sum}(0, n, s(s(0)))$$

$$\hat{\sigma}_1 = []$$

$$0+n=2?$$

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succeeds

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$$0+n=2?$$

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succeeds

$$\sigma_2 = [y_2 = s(s(0)), n = s(s(0))]$$

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$$\xleftarrow{\hat{\sigma}_2} \square$$

$$\hat{\sigma}_1 = []$$

$$\hat{\sigma}_2 = [n = s(s(0))]$$

$$0+n=2?$$

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goal

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$$\hat{\sigma}_1 = []$$

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$0+n=2?$

our new
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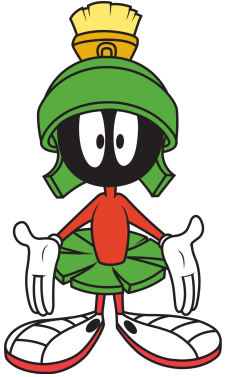
$$\hat{\sigma}_1 = []$$

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$$\hat{\sigma}_1 \cdot \hat{\sigma}_2 = [n = s(s(0))]$$

$$(1 + n = 3) \Rightarrow n = 2!$$

Jumping creatures



Assuming that:

1. All jumping creatures are green
2. All small jumping creatures are Martians
3. All green Martians are intelligent
4. Ngtrks is small and green
5. Pgvdrk is a jumping Martian

Who is intelligent?

Jumping creatures



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?— intelligent(W).

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intelligent(*W*)

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intelligent(**W**)

↙ green(**W**) , martian(**W**)

?— intelligent(**W**).

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intelligent(**W**)

↙ green(**W**) , martian(**W**)

↙ jumping(**W**) , martian(**W**)

[**W** = pgvdrk]

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intelligent(**W**)

↖ green(**W**) , martian(**W**)

↖ jumping(**W**) , martian(**W**)

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[**W** = **pgvdrk**]

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↙ martian(**pgvdrk**)

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↙ green(**W**) , martian(**W**)

↙ jumping(**W**) , martian(**W**)

↙ martian(**pgvdrk**)

↙ □ [**W** = **pgvdrk**]

intelligent(**W**)

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↙ martian(**pgvdrk**)

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intelligent(**W**)

↙ green(**W**) , martian(**W**)

↙ martian(**ngtrks**)

↙ small(**ngtrks**) , jumping(**ngtrks**)

↙ jumping(**ngtrks**) [**W** = **ngtrks**]

Jumping creatures



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2. All small jumping creatures are Martians
3. All green Martians are intelligent
4. Ngtrks is small and green
5. Pgvdrk is a jumping Martian

Who is intelligent?

```
green(X) :- jumping(X) .
martian(X) :- small(X) , jumping(X) .
intelligent(X) :- green(X) , martian(X) .
small(ngtrks) .
green(ngtrks) .
jumping(pgvdrk) .
martian(pgvdrk) .
```

?— intelligent(**W**).

intelligent(**W**)

↙ green(**W**) , martian(**W**)

↙ jumping(**W**) , martian(**W**)

↙ martian(**pgvdrk**)

↙ □ [**W** = **pgvdrk**]

intelligent(**W**)

↙ green(**W**) , martian(**W**)

↙ martian(**ngtrks**)

↙ small(**ngtrks**) , jumping(**ngtrks**)

↙ jumping(**ngtrks**) [**W** = **ngtrks**]

fail!

Some exercises (logic programming)

$\text{sum}(x, s(0), s(s(0)))$

$\text{sum}(0, y, y) .$
 $\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$\text{sum}(x, s(0), s(s(0)))$

sum($0, y, y$) .
sum($s(x), y, s(z)$) :- sum(x, y, z) .

sum($x, s(0), s(s(0))$)

$\sigma_1 = [x = s(x_1), y_1 = s(0), z_1 = s(0)]$

$\text{sum}(0, y, y)$.
 $\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z)$.

$\text{sum}(x, s(0), s(s(0)))$

$\sigma_1 = [x = s(x_1), y_1 = s(0), z_1 = s(0)]$

$\text{sum}(0, y, y) .$
 $\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$\text{sum}(x, s(0), s(s(0)))$

$\sigma_1 = [x = s(x_1), y_1 = s(0), z_1 = s(0)]$

$\nwarrow_{\hat{\sigma}_1} \text{sum}(x_1, s(0), s(0))$

$\text{sum}(0, y, y) .$
 $\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$\text{sum}(x, s(0), s(s(0)))$

$\sigma_1 = [x = s(x_1), y_1 = s(0), z_1 = s(0)]$

$\nwarrow_{\hat{\sigma}_1} \text{sum}(x_1, s(0), s(0))$

$\sigma_2 = [x_1 = 0, y_2 = s(0)]$

$\text{sum}(0, y, y) .$
 $\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$\text{sum}(x, s(0), s(s(0)))$

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$\nwarrow_{\hat{\sigma}_1} \text{sum}(x_1, s(0), s(0))$

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$\text{sum}(0, y, y) .$
 $\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

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$\sigma_1 = [x = s(x_1), y_1 = s(0), z_1 = s(0)]$

$\nwarrow_{\hat{\sigma}_1} \text{sum}(x_1, s(0), s(0))$

$\sigma_2 = [x_1 = 0, y_2 = s(0)]$

$\nwarrow_{\hat{\sigma}_2} \square$

$\text{sum}(0, y, y) .$
 $\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$\text{sum}(x, s(0), s(s(0)))$

$\sigma_1 = [x = s(x_1), y_1 = s(0), z_1 = s(0)]$

$\nwarrow_{\hat{\sigma}_1} \text{sum}(x_1, s(0), s(0))$

$\sigma_2 = [x_1 = 0, y_2 = s(0)]$

$\nwarrow_{\hat{\sigma}_2} \square$

$x = s(0)$

$\text{sum}(0, y, y) .$
 $\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$\text{sum}(x, s(0), s(s(0)))$

$\sigma_1 = [x = s(x_1), y_1 = s(0), z_1 = s(0)]$

$\nwarrow_{\hat{\sigma}_1} \text{sum}(x_1, s(0), s(0))$

$\sigma_2 = [x_1 = 0, y_2 = s(0)]$

$\nwarrow_{\hat{\sigma}_2} \square$

$x = s(0)$

Alternatively:

$\text{sum}(0, y, y) .$
 $\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$\text{sum}(x, s(0), s(s(0)))$

$\sigma_1 = [x = s(x_1), y_1 = s(0), z_1 = s(0)]$

$\nwarrow_{\hat{\sigma}_1} \text{sum}(x_1, s(0), s(0))$

Alternatively:

$\text{sum}(0, y, y) .$
 $\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

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$\sigma_1 = [x = s(x_1), y_1 = s(0), z_1 = s(0)]$

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Alternatively:

$\sigma_2 = [x_1 = s(x_2), y_2 = s(0), z_2 = 0]$

$\text{sum}(0, y, y) .$
 $\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$\text{sum}(x, s(0), s(s(0)))$

$\nwarrow_{\hat{\sigma}_1} \text{sum}(x_1, s(0), s(0))$

$\sigma_1 = [x = s(x_1), y_1 = s(0), z_1 = s(0)]$

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$\sigma_2 = [x_1 = s(x_2), y_2 = s(0), z_2 = 0]$

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 $\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$\text{sum}(x, s(0), s(s(0)))$

$\sigma_1 = [x = s(x_1), y_1 = s(0), z_1 = s(0)]$

$\nwarrow_{\hat{\sigma}_1} \text{sum}(x_1, s(0), s(0))$

Alternatively:

$\sigma_2 = [x_1 = s(x_2), y_2 = s(0), z_2 = 0]$

$\nwarrow_{\hat{\sigma}_2} \text{sum}(x_2, s(0), 0)$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

sum(x , s(0), s(s(0)))

$\sigma_1 = [x = s(x_1), y_1 = s(0), z_1 = s(0)]$

$\nwarrow_{\hat{\sigma}_1}$ sum(x_1 , s(0), s(0))

Alternatively:

$\sigma_2 = [x_1 = s(x_2), y_2 = s(0), z_2 = 0]$

$\nwarrow_{\hat{\sigma}_2}$ sum(x_2 , s(0), 0)

Fail!

[**Ex. 1**] Let $\Sigma_0 = \{0\}$ and $\Sigma_1 = \{\mathbf{s}\}$. Extend the logic program that defines the predicate $\mathbf{sum} \in \Pi_3$ (seen in classroom) to define:

1. a predicate $\mathbf{prod} \in \Pi_3$ for computing the product of two numbers;
2. a predicate $\mathbf{pow} \in \Pi_3$ for computing the power;
3. a predicate $\mathbf{div} \in \Pi_2$ for telling if the first argument divides the second.

1. a predicate $\text{prod} \in \Pi_3$ for computing the product of two numbers;

$\text{sum}(\textcolor{blue}{0}, \textcolor{red}{y}, \textcolor{red}{y})$.

$$0 + y = y$$

$\text{sum}(\textcolor{blue}{s}(\textcolor{red}{x}), \textcolor{red}{y}, \textcolor{blue}{s}(\textcolor{red}{z})) \text{ :- } \text{sum}(\textcolor{red}{x}, \textcolor{red}{y}, \textcolor{red}{z})$.

$$(x + 1) + y = (x + y) + 1$$

1. a predicate $\text{prod} \in \Pi_3$ for computing the product of two numbers;

$\text{sum}(0, y, y) .$

$$0 + y = y$$

$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z) .$

$$(x + 1) + y = (x + y) + 1$$

1

$\text{prod}(0, y, 0) .$

$\text{prod}(s(x), y, z) :- \text{prod}(x, y, w) , \text{sum}(w, y, z) .$

1. a predicate $\text{prod} \in \Pi_3$ for computing the product of two numbers;

$\text{sum}(0, y, y) .$

$$0 + y = y$$

$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z) .$

$$(x + 1) + y = (x + y) + 1$$

$$0 \cdot y = 0$$

1

$\text{prod}(0, y, 0) .$

$\text{prod}(s(x), y, z) :- \text{prod}(x, y, w) , \text{sum}(w, y, z) .$

1. a predicate $\text{prod} \in \Pi_3$ for computing the product of two numbers;

$\text{sum}(0, y, y) .$

$$0 + y = y$$

$\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$$(x + 1) + y = (x + y) + 1$$

$$0 \cdot y = 0$$

1

$\text{prod}(0, y, 0) .$

$\text{prod}(s(x), y, z) \text{ :- } \text{prod}(x, y, w) , \text{sum}(w, y, z) .$

$$(x + 1) \cdot y = (x \cdot y) + y$$

$\text{prod}(s(s(0)), y, s(s(0)))$

sum(0,y,y) .
sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

prod(s(s(0)), y, s(s(0)))

sum(0,y,y) .
sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

prod(s(s(0)), y, s(s(0)))

$\sigma_1 = [x_1 = s(0), y_1 = y, z_1 = s(s(0))]$

$\text{sum}(0, y, y) .$
 $\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$\text{prod}(0, y, 0) .$
 $\text{prod}(s(x), y, z) \text{ :- } \text{prod}(x, y, w) , \text{sum}(w, y, z) .$

$\text{prod}(s(s(0)), y, s(s(0))) \quad \sigma_1 = [x_1 = s(0), y_1 = y, z_1 = s(s(0))]$
 $\nwarrow_{\hat{\sigma}_1} \text{prod}(s(0), y, w_1) , \text{sum}(w_1, y, s(s(0)))$


```
sum(0,y,y) .
sum(s(x),y,s(z)) :- sum(x,y,z) .
```

```
prod(0,y,0) .
prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .
```

$\text{prod}(s(s(0)), y, s(s(0)))$ $\sigma_1 = [x_1 = s(0), y_1 = y, z_1 = s(s(0))]$
 $\nwarrow_{\hat{\sigma}_1} \text{prod}(s(0), y, w_1), \text{sum}(w_1, y, s(s(0)))$
 $\sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\nwarrow_{\hat{\sigma}_2} \text{prod}(0, y, w_2), \text{sum}(w_2, y, w_1), \text{sum}(w_1, y, s(s(0)))$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

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 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

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 $\nwarrow_{\hat{\sigma}_1}$ prod(s(0), y, w₁) , sum(w₁, y, s(s(0)))
 $\nwarrow_{\hat{\sigma}_2}$ prod(0, y, w₂) , sum(w₂, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\sigma_3 = [y_3 = y, w_2 = 0]$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

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 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

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 $\nwarrow_{\hat{\sigma}_2}$ prod(0, y, w₂) , sum(w₂, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
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 sum(s(x),y,s(z)) :- sum(x,y,z) .

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prod(s(s(0)), y, s(s(0))) $\sigma_1 = [x_1 = s(0), y_1 = y, z_1 = s(s(0))]$
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 $\nwarrow_{\hat{\sigma}_2}$ prod(0, y, w₂) , sum(w₂, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\nwarrow_{\hat{\sigma}_3}$ sum(0, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_3 = [y_3 = y, w_2 = 0]$

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prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

prod(s(s(0)), y, s(s(0))) $\sigma_1 = [x_1 = s(0), y_1 = y, z_1 = s(s(0))]$
 $\nwarrow_{\hat{\sigma}_1}$ prod(s(0), y, w₁) , sum(w₁, y, s(s(0)))
 $\nwarrow_{\hat{\sigma}_2}$ prod(0, y, w₂) , sum(w₂, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\nwarrow_{\hat{\sigma}_3}$ sum(0, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_3 = [y_3 = y, w_2 = 0]$
 $\nwarrow_{\hat{\sigma}_4}$ sum(0, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_4 = [w_1 = y, y_4 = y]$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

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 $\nwarrow_{\hat{\sigma}_1}$ prod(s(0), y, w₁) , sum(w₁, y, s(s(0)))
 $\nwarrow_{\hat{\sigma}_2}$ prod(0, y, w₂) , sum(w₂, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\nwarrow_{\hat{\sigma}_3}$ sum(0, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_3 = [y_3 = y, w_2 = 0]$
 $\nwarrow_{\hat{\sigma}_4}$ sum(0, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_4 = [w_1 = y, y_4 = y]$

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 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

$\text{prod}(\text{s}(\text{s}(0)), y, \text{s}(\text{s}(0)))$ $\sigma_1 = [x_1 = \text{s}(0), y_1 = y, z_1 = \text{s}(\text{s}(0))]$
 $\nwarrow_{\hat{\sigma}_1} \text{prod}(\text{s}(0), y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0)))$
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 $\nwarrow_{\hat{\sigma}_4} \text{sum}(y, y, \text{s}(\text{s}(0)))$ $\sigma_4 = [w_1 = y, y_4 = y]$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

prod(s(s(0)), y, s(s(0))) $\sigma_1 = [x_1 = s(0), y_1 = y, z_1 = s(s(0))]$
 $\nwarrow_{\hat{\sigma}_1}$ prod(s(0), y, w₁) , sum(w₁, y, s(s(0)))
 $\nwarrow_{\hat{\sigma}_2}$ prod(0, y, w₂) , sum(w₂, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\nwarrow_{\hat{\sigma}_3}$ sum(0, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_3 = [y_3 = y, w_2 = 0]$
 $\nwarrow_{\hat{\sigma}_4}$ sum(y, y, s(s(0))) $\sigma_4 = [w_1 = y, y_4 = y]$
 $\nwarrow_{\hat{\sigma}_5}$ sum(y, y, s(s(0))) $\sigma_5 = [y = s(x_5), y_5 = s(x_5), z_5 = s(0)]$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

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 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

$\text{prod}(\text{s}(\text{s}(0)), y, \text{s}(\text{s}(0)))$ $\sigma_1 = [x_1 = \text{s}(0), y_1 = y, z_1 = \text{s}(\text{s}(0))]$
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 $\nwarrow_{\hat{\sigma}_3} \text{sum}(0, y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0)))$ $\sigma_3 = [y_3 = y, w_2 = 0]$
 $\nwarrow_{\hat{\sigma}_4} \text{sum}(y, y, \text{s}(\text{s}(0)))$ $\sigma_4 = [w_1 = y, y_4 = y]$
 $\nwarrow_{\hat{\sigma}_5} \text{sum}(y, y, \text{s}(\text{s}(0)))$ $\sigma_5 = [y = \text{s}(x_5), y_5 = \text{s}(x_5), z_5 = \text{s}(0)]$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

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 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

$\text{prod}(\text{s}(\text{s}(0)), y, \text{s}(\text{s}(0))) \quad \sigma_1 = [x_1 = \text{s}(0), y_1 = y, z_1 = \text{s}(\text{s}(0))]$
 $\nwarrow_{\hat{\sigma}_1} \text{prod}(\text{s}(0), y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0))) \quad \sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\nwarrow_{\hat{\sigma}_2} \text{prod}(0, y, w_2), \text{sum}(w_2, y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0))) \quad \sigma_3 = [y_3 = y, w_2 = 0]$
 $\nwarrow_{\hat{\sigma}_3} \text{sum}(0, y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0))) \quad \sigma_4 = [w_1 = y, y_4 = y]$
 $\nwarrow_{\hat{\sigma}_4} \text{sum}(y, y, \text{s}(\text{s}(0))) \quad \sigma_5 = [y = \text{s}(x_5), y_5 = \text{s}(x_5), z_5 = \text{s}(0)]$
 $\nwarrow_{\hat{\sigma}_5} \text{sum}(x_5, \text{s}(x_5), \text{s}(0))$

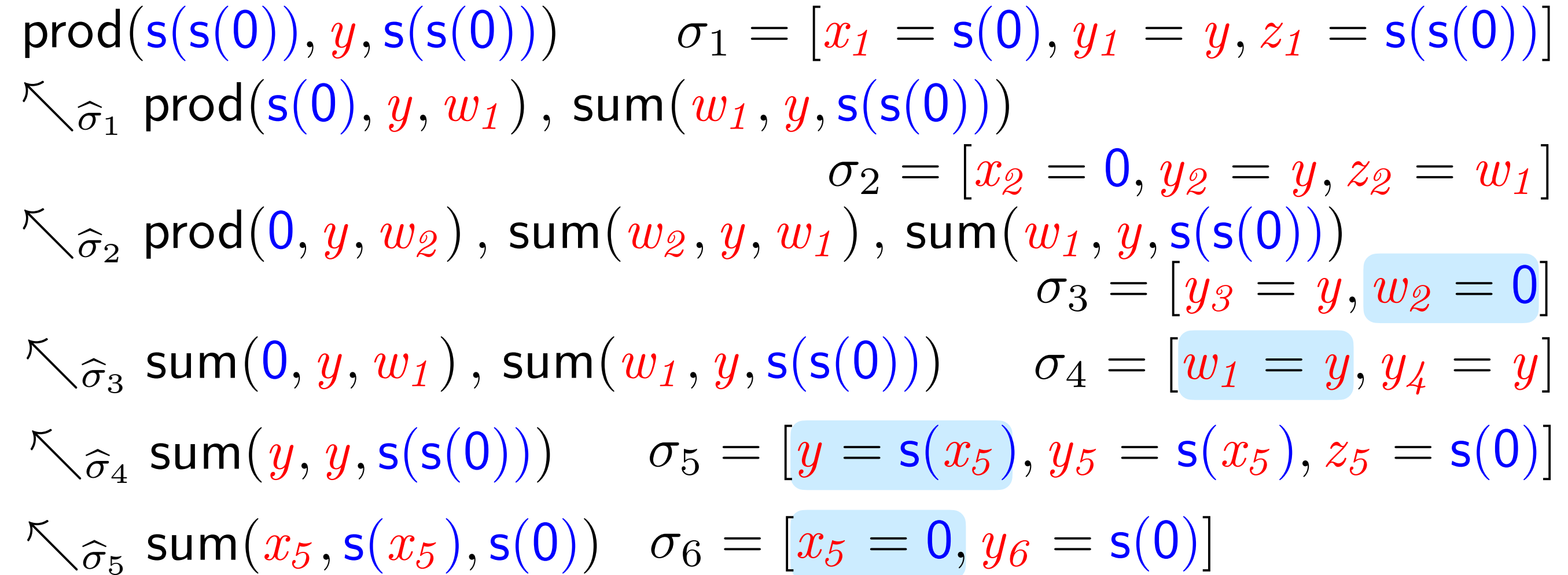
sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

prod(s(s(0)), y, s(s(0))) $\sigma_1 = [x_1 = s(0), y_1 = y, z_1 = s(s(0))]$
 $\nwarrow_{\hat{\sigma}_1}$ prod(s(0), y, w₁) , sum(w₁, y, s(s(0)))
 $\nwarrow_{\hat{\sigma}_2}$ prod(0, y, w₂) , sum(w₂, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\nwarrow_{\hat{\sigma}_3}$ sum(0, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_3 = [y_3 = y, w_2 = 0]$
 $\nwarrow_{\hat{\sigma}_4}$ sum(y, y, s(s(0))) $\sigma_4 = [w_1 = y, y_4 = y]$
 $\nwarrow_{\hat{\sigma}_5}$ sum(x₅, s(x₅), s(0)) $\sigma_5 = [y = s(x_5), y_5 = s(x_5), z_5 = s(0)]$
 $\nwarrow_{\hat{\sigma}_6}$ sum(x₅, s(x₅), s(0)) $\sigma_6 = [x_5 = 0, y_6 = s(0)]$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .



sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

$\text{prod}(\text{s}(\text{s}(0)), y, \text{s}(\text{s}(0))) \quad \sigma_1 = [x_1 = \text{s}(0), y_1 = y, z_1 = \text{s}(\text{s}(0))]$
 $\swarrow_{\hat{\sigma}_1} \text{prod}(\text{s}(0), y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0))) \quad \sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\swarrow_{\hat{\sigma}_2} \text{prod}(0, y, w_2), \text{sum}(w_2, y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0))) \quad \sigma_3 = [y_3 = y, w_2 = 0]$
 $\swarrow_{\hat{\sigma}_3} \text{sum}(0, y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0))) \quad \sigma_4 = [w_1 = y, y_4 = y]$
 $\swarrow_{\hat{\sigma}_4} \text{sum}(y, y, \text{s}(\text{s}(0))) \quad \sigma_5 = [y = \text{s}(x_5), y_5 = \text{s}(x_5), z_5 = \text{s}(0)]$
 $\swarrow_{\hat{\sigma}_5} \text{sum}(x_5, \text{s}(x_5), \text{s}(0)) \quad \sigma_6 = [x_5 = 0, y_6 = \text{s}(0)]$
 $\swarrow_{\hat{\sigma}_6} \square$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

$\text{prod}(s(s(0)), y, s(s(0))) \quad \sigma_1 = [x_1 = s(0), y_1 = y, z_1 = s(s(0))]$
 $\nwarrow_{\hat{\sigma}_1} \text{prod}(s(0), y, w_1), \text{sum}(w_1, y, s(s(0))) \quad \sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\nwarrow_{\hat{\sigma}_2} \text{prod}(0, y, w_2), \text{sum}(w_2, y, w_1), \text{sum}(w_1, y, s(s(0))) \quad \sigma_3 = [y_3 = y, w_2 = 0]$
 $\nwarrow_{\hat{\sigma}_3} \text{sum}(0, y, w_1), \text{sum}(w_1, y, s(s(0))) \quad \sigma_4 = [w_1 = y, y_4 = y]$
 $\nwarrow_{\hat{\sigma}_4} \text{sum}(y, y, s(s(0))) \quad \sigma_5 = [y = s(x_5), y_5 = s(x_5), z_5 = s(0)]$
 $\nwarrow_{\hat{\sigma}_5} \text{sum}(x_5, s(x_5), s(0)) \quad \sigma_6 = [x_5 = 0, y_6 = s(0)]$
 $\nwarrow_{\hat{\sigma}_6} \square \quad y = s(0)$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

$\text{prod}(s(s(0)), y, s(s(0))) \quad \sigma_1 = [x_1 = s(0), y_1 = y, z_1 = s(s(0))]$
 $\swarrow_{\hat{\sigma}_1} \text{prod}(s(0), y, w_1), \text{sum}(w_1, y, s(s(0)))$
 $\swarrow_{\hat{\sigma}_2} \text{prod}(0, y, w_2), \text{sum}(w_2, y, w_1), \text{sum}(w_1, y, s(s(0))) \quad \sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\swarrow_{\hat{\sigma}_3} \text{sum}(0, y, w_1), \text{sum}(w_1, y, s(s(0))) \quad \sigma_3 = [y_3 = y, w_2 = 0]$
 $\swarrow_{\hat{\sigma}_4} \text{sum}(y, y, s(s(0))) \quad \sigma_4 = [w_1 = y, y_4 = y]$
 $\swarrow_{\hat{\sigma}_5} \text{sum}(x_5, s(x_5), s(0)) \quad \sigma_5 = [y = s(x_5), y_5 = s(x_5), z_5 = s(0)]$
 $\swarrow_{\hat{\sigma}_6} \square \quad \sigma_6 = [x_5 = 0, y_6 = s(0)]$
 $y = s(0)$

Alternatively:

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

prod(s(s(0)), y, s(s(0))) $\sigma_1 = [x_1 = s(0), y_1 = y, z_1 = s(s(0))]$
 $\nwarrow_{\hat{\sigma}_1}$ prod(s(0), y, w₁) , sum(w₁, y, s(s(0)))
 $\nwarrow_{\hat{\sigma}_2}$ prod(0, y, w₂) , sum(w₂, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\nwarrow_{\hat{\sigma}_3}$ sum(0, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_3 = [y_3 = y, w_2 = 0]$
 $\nwarrow_{\hat{\sigma}_4}$ sum(y, y, s(s(0))) $\sigma_4 = [w_1 = y, y_4 = y]$
 $\nwarrow_{\hat{\sigma}_5}$ sum(x₅, s(x₅), s(0)) $\sigma_5 = [y = s(x_5), y_5 = s(x_5), z_5 = s(0)]$
 $\nwarrow_{\hat{\sigma}_6}$ sum(x₅, s(x₅), s(0)) $\sigma_6 = [x_5 = s(x_6), y_6 = s(s(x_6)), z_6 = 0]$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

$\text{prod}(\text{s}(\text{s}(0)), y, \text{s}(\text{s}(0))) \quad \sigma_1 = [x_1 = \text{s}(0), y_1 = y, z_1 = \text{s}(\text{s}(0))]$
 $\nwarrow_{\hat{\sigma}_1} \text{prod}(\text{s}(0), y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0))) \quad \sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\nwarrow_{\hat{\sigma}_2} \text{prod}(0, y, w_2), \text{sum}(w_2, y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0))) \quad \sigma_3 = [y_3 = y, w_2 = 0]$
 $\nwarrow_{\hat{\sigma}_3} \text{sum}(0, y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0))) \quad \sigma_4 = [w_1 = y, y_4 = y]$
 $\nwarrow_{\hat{\sigma}_4} \text{sum}(y, y, \text{s}(\text{s}(0))) \quad \sigma_5 = [y = \text{s}(x_5), y_5 = \text{s}(x_5), z_5 = \text{s}(0)]$
 $\nwarrow_{\hat{\sigma}_5} \text{sum}(x_5, \text{s}(x_5), \text{s}(0)) \quad \sigma_6 = [x_5 = \text{s}(x_6), y_6 = \text{s}(\text{s}(x_6)), z_6 = 0]$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

prod(s(s(0)), y, s(s(0))) $\sigma_1 = [x_1 = s(0), y_1 = y, z_1 = s(s(0))]$
 $\nwarrow_{\hat{\sigma}_1}$ prod(s(0), y, w₁) , sum(w₁, y, s(s(0)))
 $\nwarrow_{\hat{\sigma}_2}$ prod(0, y, w₂) , sum(w₂, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\nwarrow_{\hat{\sigma}_3}$ sum(0, y, w₁) , sum(w₁, y, s(s(0))) $\sigma_3 = [y_3 = y, w_2 = 0]$
 $\nwarrow_{\hat{\sigma}_4}$ sum(y, y, s(s(0))) $\sigma_4 = [w_1 = y, y_4 = y]$
 $\nwarrow_{\hat{\sigma}_5}$ sum(x₅, s(x₅), s(0)) $\sigma_5 = [y = s(x_5), y_5 = s(x_5), z_5 = s(0)]$
 $\nwarrow_{\hat{\sigma}_6}$ sum(x₆, s(s(x₆)), 0) $\sigma_6 = [x_5 = s(x_6), y_6 = s(s(x_6)), z_6 = 0]$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

$\text{prod}(\text{s}(\text{s}(0)), y, \text{s}(\text{s}(0))) \quad \sigma_1 = [x_1 = \text{s}(0), y_1 = y, z_1 = \text{s}(\text{s}(0))]$
 $\swarrow_{\hat{\sigma}_1} \text{prod}(\text{s}(0), y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0))) \quad \sigma_2 = [x_2 = 0, y_2 = y, z_2 = w_1]$
 $\swarrow_{\hat{\sigma}_2} \text{prod}(0, y, w_2), \text{sum}(w_2, y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0))) \quad \sigma_3 = [y_3 = y, w_2 = 0]$
 $\swarrow_{\hat{\sigma}_3} \text{sum}(0, y, w_1), \text{sum}(w_1, y, \text{s}(\text{s}(0))) \quad \sigma_4 = [w_1 = y, y_4 = y]$
 $\swarrow_{\hat{\sigma}_4} \text{sum}(y, y, \text{s}(\text{s}(0))) \quad \sigma_5 = [y = \text{s}(x_5), y_5 = \text{s}(x_5), z_5 = \text{s}(0)]$
 $\swarrow_{\hat{\sigma}_5} \text{sum}(x_5, \text{s}(x_5), \text{s}(0)) \quad \sigma_6 = [x_5 = \text{s}(x_6), y_6 = \text{s}(\text{s}(x_6)), z_6 = 0]$
 $\swarrow_{\hat{\sigma}_6} \text{sum}(x_6, \text{s}(\text{s}(x_6)), 0)$

Fail!

2. a predicate $\text{pow} \in \Pi_3$ for computing the power;

$\text{sum}(0, y, y)$.

$\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z)$.

$$0 + y = y$$

$$(x + 1) + y = (x + y) + 1$$

$\text{prod}(0, y, 0)$.

$\text{prod}(s(x), y, z) \text{ :- } \text{prod}(x, y, w) , \text{sum}(w, y, z)$.

$$0 \cdot y = 0$$

$$(x + 1) \cdot y = (x \cdot y) + y$$

2. a predicate $\text{pow} \in \Pi_3$ for computing the power;

$\text{sum}(0, y, y) .$

$$0 + y = y$$

$\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$$(x + 1) + y = (x + y) + 1$$

$\text{prod}(0, y, 0) .$

$$0 \cdot y = 0$$

$\text{prod}(s(x), y, z) \text{ :- } \text{prod}(x, y, w) , \text{sum}(w, y, z) .$

$$(x + 1) \cdot y = (x \cdot y) + y$$

2

$\text{pow}(0, s(y), 0) .$

$\text{pow}(s(x), 0, s(0)) .$

$\text{pow}(s(x), s(y), z) \text{ :- } \text{pow}(s(x), y, w) , \text{prod}(w, s(x), z) .$

2. a predicate $\text{pow} \in \Pi_3$ for computing the power;

$\text{sum}(0, y, y) .$

$$0 + y = y$$

$\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$$(x + 1) + y = (x + y) + 1$$

$\text{prod}(0, y, 0) .$

$$0 \cdot y = 0$$

$\text{prod}(s(x), y, z) \text{ :- } \text{prod}(x, y, w) , \text{sum}(w, y, z) .$

$$(x + 1) \cdot y = (x \cdot y) + y$$

$$0^{(y+1)} = 0$$

2

$\text{pow}(0, s(y), 0) .$

$\text{pow}(s(x), 0, s(0)) .$

$\text{pow}(s(x), s(y), z) \text{ :- } \text{pow}(s(x), y, w) , \text{prod}(w, s(x), z) .$

2. a predicate $\text{pow} \in \Pi_3$ for computing the power;

$\text{sum}(0, y, y) .$

$$0 + y = y$$

$\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z) .$

$$(x + 1) + y = (x + y) + 1$$

$\text{prod}(0, y, 0) .$

$$0 \cdot y = 0$$

$\text{prod}(s(x), y, z) \text{ :- } \text{prod}(x, y, w) , \text{sum}(w, y, z) .$

$$(x + 1) \cdot y = (x \cdot y) + y$$

$$0^{(y+1)} = 0 \quad (x + 1)^0 = 1$$

2

$\text{pow}(0, s(y), 0) .$

$\text{pow}(s(x), 0, s(0)) .$

$\text{pow}(s(x), s(y), z) \text{ :- } \text{pow}(s(x), y, w) , \text{prod}(w, s(x), z) .$

2. a predicate $\text{pow} \in \Pi_3$ for computing the power;

$\text{sum}(0, y, y)$.

$$0 + y = y$$

$\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z)$.

$$(x + 1) + y = (x + y) + 1$$

$\text{prod}(0, y, 0)$.

$$0 \cdot y = 0$$

$\text{prod}(s(x), y, z) \text{ :- } \text{prod}(x, y, w) , \text{sum}(w, y, z)$.

$$(x + 1) \cdot y = (x \cdot y) + y$$

$$0^{(y+1)} = 0 \quad (x + 1)^0 = 1$$

2

$\text{pow}(0, s(y), 0)$.

$\text{pow}(s(x), 0, s(0))$.

$\text{pow}(s(x), s(y), z) \text{ :- } \text{pow}(s(x), y, w) , \text{prod}(w, s(x), z)$.

$$(x + 1)^{y+1} = (x + 1)^y \cdot (x + 1)$$

3. a predicate $\text{div} \in \Pi_2$ for telling if the first argument divides the second.

$\text{sum}(0, y, y)$.

$$0 + y = y$$

$\text{sum}(s(x), y, s(z)) \text{ :- } \text{sum}(x, y, z)$.

$$(x + 1) + y = (x + y) + 1$$

$\text{prod}(0, y, 0)$.

$$0 \cdot y = 0$$

$\text{prod}(s(x), y, z) \text{ :- } \text{prod}(x, y, w) , \text{sum}(w, y, z)$.

$$(x + 1) \cdot y = (x \cdot y) + y$$

$\text{pow}(0, s(y), 0)$.

$$0^{(y+1)} = 0$$

$\text{pow}(s(x), 0, s(0))$.

$$(x + 1)^0 = 1$$

$\text{pow}(s(x), s(y), z) \text{ :- } \text{pow}(s(x), y, w) , \text{prod}(w, s(x), z)$.

$$(x + 1)^{y+1} = (x + 1)^y \cdot (x + 1)$$

3. a predicate $\text{div} \in \Pi_2$ for telling if the first argument divides the second.

$\text{sum}(0, y, y) .$

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$\text{sum}(s(x), y, s(z)) :- \text{sum}(x, y, z) .$

$$(x + 1) + y = (x + y) + 1$$

$\text{prod}(0, y, 0) .$

$$0 \cdot y = 0$$

$\text{prod}(s(x), y, z) :- \text{prod}(x, y, w) , \text{sum}(w, y, z) .$

$$(x + 1) \cdot y = (x \cdot y) + y$$

$\text{pow}(0, s(y), 0) .$

$$0^{(y+1)} = 0$$

$\text{pow}(s(x), 0, s(0)) .$

$$(x + 1)^0 = 1$$

$\text{pow}(s(x), s(y), z) :- \text{pow}(s(x), y, w) , \text{prod}(w, s(x), z) .$

$$(x + 1)^{y+1} = (x + 1)^y \cdot (x + 1)$$

3 $\text{div}(s(y), z) :- \text{prod}(x, s(y), z) .$

$$\text{div}(z, s(s(0)))$$

sum(0,y,y) .

sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .

prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

div(z, s(s(0)))

sum(0,y,y) .
sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

div(z , s(s(0)))

$\sigma_1 = [z = s(y_1), z_1 = s(s(0))]$

sum(0,y,y) .
sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

div(z , s(s(0)))

$\sigma_1 = [z = s(y_1), z_1 = s(s(0))]$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

div(z , s(s(0)))
 $\nwarrow_{\hat{\sigma}_1}$ prod(x_1 , s(y_1), s(s(0)))

$\sigma_1 = [z = s(y_1), z_1 = s(s(0))]$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

$$\begin{array}{lcl} \text{div}(\textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_1 = [\textcolor{red}{z} = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] & \\ \swarrow_{\hat{\sigma}_1} \text{prod}(\textcolor{red}{x}_1, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_2 = [\textcolor{red}{x}_1 = \textcolor{blue}{s}(\textcolor{red}{x}_2), \textcolor{red}{y}_2 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_2 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] & \end{array}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

$$\begin{array}{lcl} \text{div}(\textcolor{red}{z}, \textcolor{blue}{s(s(0))}) & & \sigma_1 = [\textcolor{red}{z} = \textcolor{blue}{s(y_1)}, \textcolor{red}{z_1} = \textcolor{blue}{s(s(0))}] \\ \swarrow_{\hat{\sigma}_1} \text{prod}(\textcolor{red}{x_1}, \textcolor{blue}{s(y_1)}, \textcolor{blue}{s(s(0))}) & & \sigma_2 = [\textcolor{red}{x_1} = \textcolor{blue}{s(x_2)}, \textcolor{red}{y_2} = \textcolor{blue}{s(y_1)}, \textcolor{red}{z_2} = \textcolor{blue}{s(s(0))}] \end{array}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

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 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

div(z , s(s(0))) $\sigma_1 = [z = s(y_1), z_1 = s(s(0))]$
 $\nwarrow_{\hat{\sigma}_1}$ prod(x_1 , s(y_1), s(s(0)))
 $\sigma_2 = [x_1 = s(x_2), y_2 = s(y_1), z_2 = s(s(0))]$
 $\nwarrow_{\hat{\sigma}_2}$ prod(x_2 , s(y_1), w_2) , sum(w_2 , s(y_1), s(s(0)))

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

$$\begin{aligned}
 & \text{div}(\textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_1 &= [\textcolor{blue}{z} = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 & \swarrow_{\hat{\sigma}_1} \text{prod}(\textcolor{red}{x}_1, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_2 &= [\textcolor{red}{x}_1 = \textcolor{blue}{s}(\textcolor{red}{x}_2), \textcolor{red}{y}_2 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_2 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 & \swarrow_{\hat{\sigma}_2} \text{prod}(\textcolor{red}{x}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{w}_2), \text{sum}(\textcolor{red}{w}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_3 &= [\textcolor{red}{x}_2 = 0, \textcolor{red}{y}_3 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{w}_2 = 0]
 \end{aligned}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

$$\begin{aligned}
 & \text{div}(\textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_1 &= [\textcolor{blue}{z} = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 & \nwarrow_{\hat{\sigma}_1} \text{prod}(\textcolor{red}{x}_1, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_2 &= [\textcolor{blue}{x}_1 = \textcolor{blue}{s}(\textcolor{red}{x}_2), \textcolor{red}{y}_2 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_2 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 & \nwarrow_{\hat{\sigma}_2} \text{prod}(\textcolor{red}{x}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{w}_2), \text{sum}(\textcolor{red}{w}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_3 &= [\textcolor{blue}{x}_2 = 0, \textcolor{red}{y}_3 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{w}_2 = 0]
 \end{aligned}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

$$\begin{aligned}
 & \text{div}(\textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_1 &= [\textcolor{red}{z} = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 & \nwarrow_{\hat{\sigma}_1} \text{prod}(\textcolor{red}{x}_1, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_2 &= [\textcolor{red}{x}_1 = \textcolor{blue}{s}(\textcolor{red}{x}_2), \textcolor{red}{y}_2 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_2 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 & \nwarrow_{\hat{\sigma}_2} \text{prod}(\textcolor{red}{x}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{w}_2), \text{sum}(\textcolor{red}{w}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_3 &= [\textcolor{red}{x}_2 = 0, \textcolor{red}{y}_3 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{w}_2 = 0] \\
 & \nwarrow_{\hat{\sigma}_3} \text{sum}(0, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0)))
 \end{aligned}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

$$\begin{array}{lcl}
 \text{div}(\textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_1 = [\textcolor{blue}{z} = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 \swarrow_{\hat{\sigma}_1} \text{prod}(\textcolor{red}{x}_1, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_2 = [\textcolor{blue}{x}_1 = \textcolor{blue}{s}(\textcolor{red}{x}_2), \textcolor{red}{y}_2 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_2 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 \swarrow_{\hat{\sigma}_2} \text{prod}(\textcolor{red}{x}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{w}_2), \text{sum}(\textcolor{red}{w}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_3 = [\textcolor{red}{x}_2 = 0, \textcolor{red}{y}_3 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{w}_2 = 0] \\
 \swarrow_{\hat{\sigma}_3} \text{sum}(0, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_4 = [\textcolor{red}{y}_4 = \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{y}_1 = \textcolor{blue}{s}(0)]
 \end{array}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

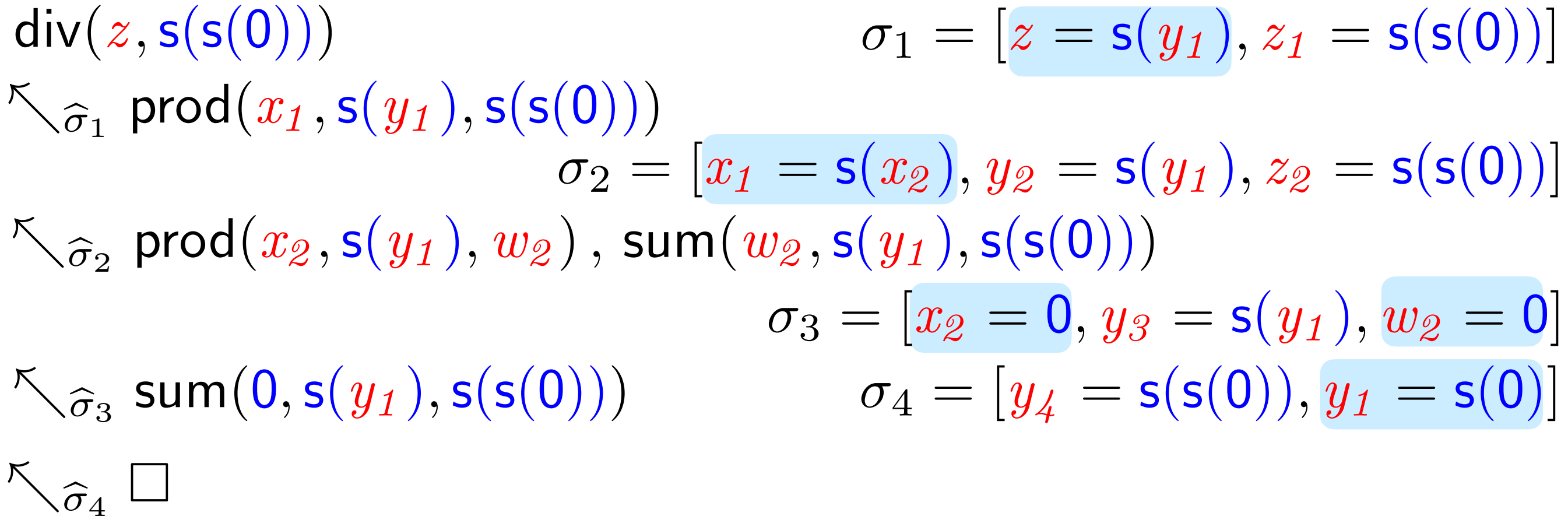
div(s(y),z) :- prod(x,s(y),z) .

$$\begin{array}{lcl}
 \text{div}(\textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_1 = [\textcolor{blue}{z} = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 \swarrow_{\hat{\sigma}_1} \text{prod}(\textcolor{red}{x}_1, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_2 = [\textcolor{red}{x}_1 = \textcolor{blue}{s}(\textcolor{red}{x}_2), \textcolor{red}{y}_2 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_2 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 \swarrow_{\hat{\sigma}_2} \text{prod}(\textcolor{red}{x}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{w}_2), \text{sum}(\textcolor{red}{w}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_3 = [\textcolor{red}{x}_2 = 0, \textcolor{red}{y}_3 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{w}_2 = 0] \\
 \swarrow_{\hat{\sigma}_3} \text{sum}(0, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_4 = [\textcolor{red}{y}_4 = \textcolor{blue}{s}(\textcolor{blue}{s}(0)), \textcolor{red}{y}_1 = \textcolor{blue}{s}(0)]
 \end{array}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

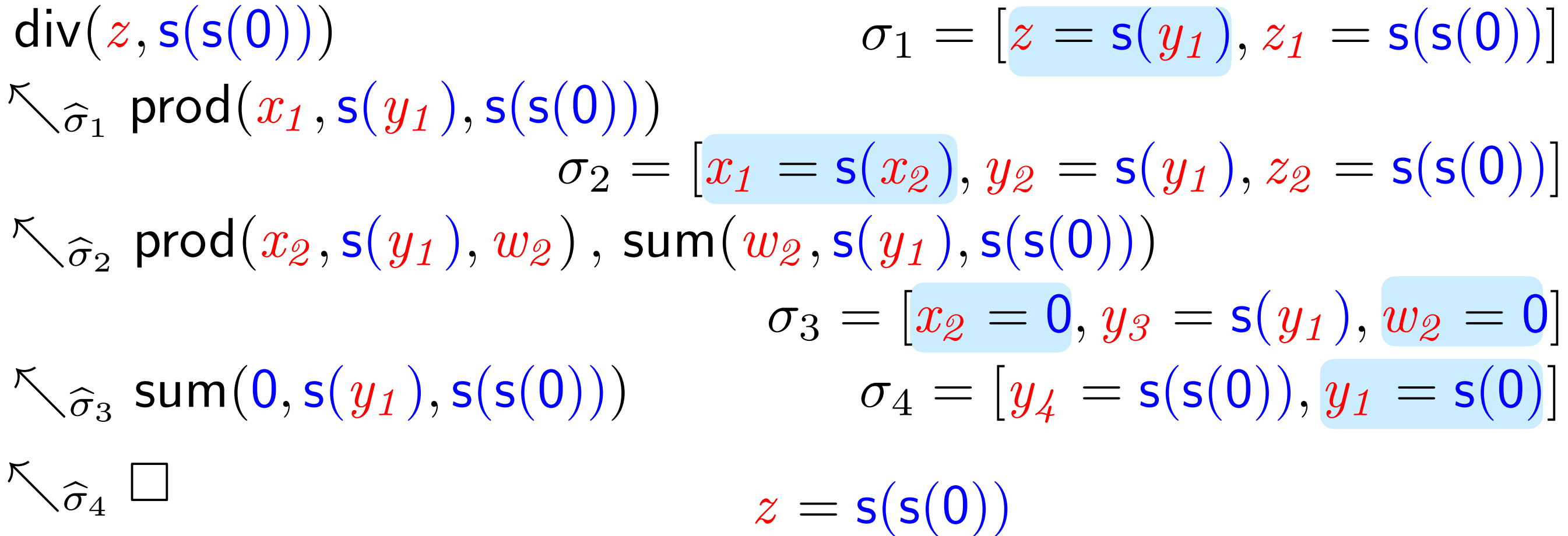
div(s(y),z) :- prod(x,s(y),z) .



sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

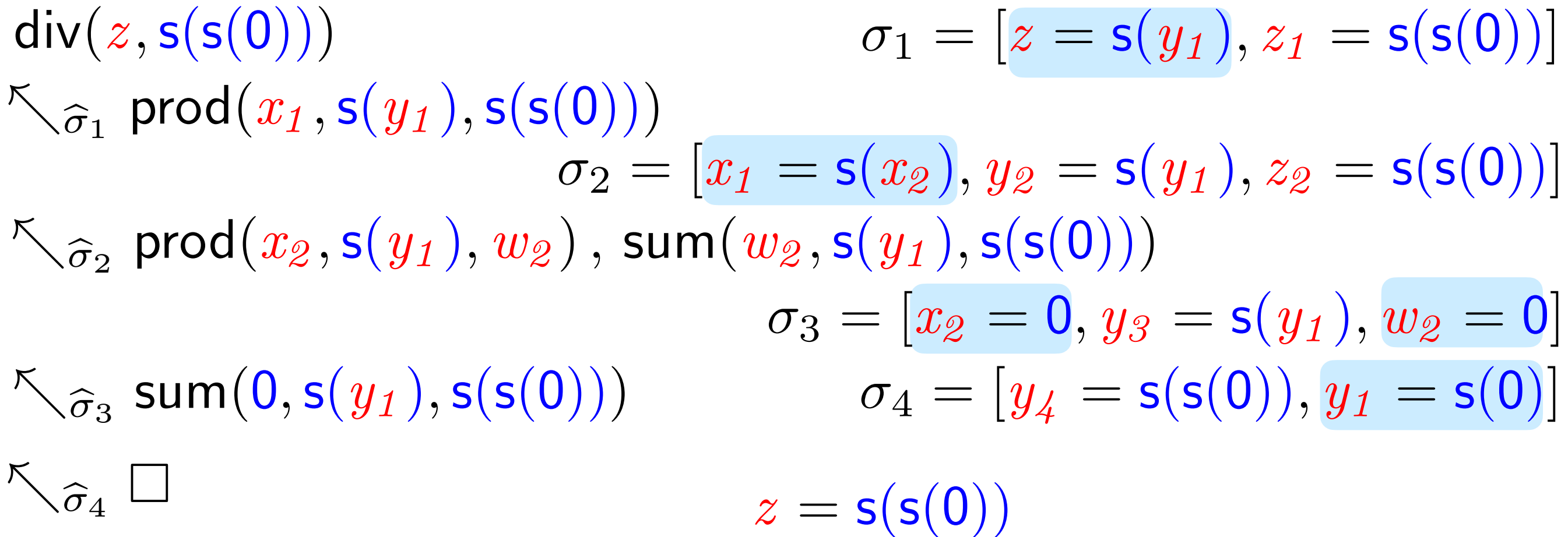
div(s(y),z) :- prod(x,s(y),z) .



sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .



Alternatively:

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

div(z , s(s(0))) $\sigma_1 = [z = s(y_1), z_1 = s(s(0))]$
 $\nwarrow_{\hat{\sigma}_1}$ prod(x_1 , s(y_1), s(s(0)))
 $\sigma_2 = [x_1 = s(x_2), y_2 = s(y_1), z_2 = s(s(0))]$
 $\nwarrow_{\hat{\sigma}_2}$ prod(x_2 , s(y_1), w_2) , sum(w_2 , s(y_1), s(s(0)))

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

$$\begin{aligned}
 & \text{div}(z, s(s(0))) & \sigma_1 &= [z = s(y_1), z_1 = s(s(0))] \\
 \nwarrow_{\hat{\sigma}_1} & \text{prod}(x_1, s(y_1), s(s(0))) & \sigma_2 &= [x_1 = s(x_2), y_2 = s(y_1), z_2 = s(s(0))] \\
 \nwarrow_{\hat{\sigma}_2} & \text{prod}(x_2, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_3 &= [x_2 = s(x_3), y_3 = s(y_1), z_3 = w_2]
 \end{aligned}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

$$\begin{aligned}
 & \text{div}(\textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_1 &= [\textcolor{red}{z} = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 & \swarrow_{\hat{\sigma}_1} \text{prod}(\textcolor{red}{x}_1, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_2 &= [\textcolor{red}{x}_1 = \textcolor{blue}{s}(\textcolor{red}{x}_2), \textcolor{red}{y}_2 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_2 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 & \swarrow_{\hat{\sigma}_2} \text{prod}(\textcolor{red}{x}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{w}_2), \text{sum}(\textcolor{red}{w}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_3 &= [\textcolor{red}{x}_2 = \textcolor{blue}{s}(\textcolor{red}{x}_3), \textcolor{red}{y}_3 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_3 = \textcolor{red}{w}_2]
 \end{aligned}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

$$\begin{aligned}
 & \text{div}(\textcolor{red}{z}, \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_1 = [\textcolor{blue}{z} = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_1 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 & \swarrow_{\hat{\sigma}_1} \text{prod}(\textcolor{red}{x}_1, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_2 = [\textcolor{blue}{x}_1 = \textcolor{blue}{s}(\textcolor{red}{x}_2), \textcolor{red}{y}_2 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_2 = \textcolor{blue}{s}(\textcolor{blue}{s}(0))] \\
 & \swarrow_{\hat{\sigma}_2} \text{prod}(\textcolor{red}{x}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{w}_2), \text{sum}(\textcolor{red}{w}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0))) & \sigma_3 = [\textcolor{blue}{x}_2 = \textcolor{blue}{s}(\textcolor{red}{x}_3), \textcolor{red}{y}_3 = \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{z}_3 = \textcolor{red}{w}_2] \\
 & \swarrow_{\hat{\sigma}_3} \text{prod}(\textcolor{red}{x}_3, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{w}_3), \text{sum}(\textcolor{red}{w}_3, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{red}{w}_2), \text{sum}(\textcolor{red}{w}_2, \textcolor{blue}{s}(\textcolor{red}{y}_1), \textcolor{blue}{s}(\textcolor{blue}{s}(0)))
 \end{aligned}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

$$\begin{aligned}
 & \text{div}(z, s(s(0))) & \sigma_1 &= [z = s(y_1), z_1 = s(s(0))] \\
 & \swarrow_{\hat{\sigma}_1} \text{prod}(x_1, s(y_1), s(s(0))) & \sigma_2 &= [x_1 = s(x_2), y_2 = s(y_1), z_2 = s(s(0))] \\
 & \swarrow_{\hat{\sigma}_2} \text{prod}(x_2, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_3 &= [x_2 = s(x_3), y_3 = s(y_1), z_3 = w_2] \\
 & \swarrow_{\hat{\sigma}_3} \text{prod}(x_3, s(y_1), w_3), \text{sum}(w_3, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_4 &= [x_3 = 0, y_4 = s(y_1), w_3 = 0]
 \end{aligned}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

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 & \swarrow_{\hat{\sigma}_2} \text{prod}(x_2, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_3 &= [x_2 = s(x_3), y_3 = s(y_1), z_3 = w_2] \\
 & \swarrow_{\hat{\sigma}_3} \text{prod}(x_3, s(y_1), w_3), \text{sum}(w_3, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_4 &= [x_3 = 0, y_4 = s(y_1), w_3 = 0]
 \end{aligned}$$

sum(0,y,y) .
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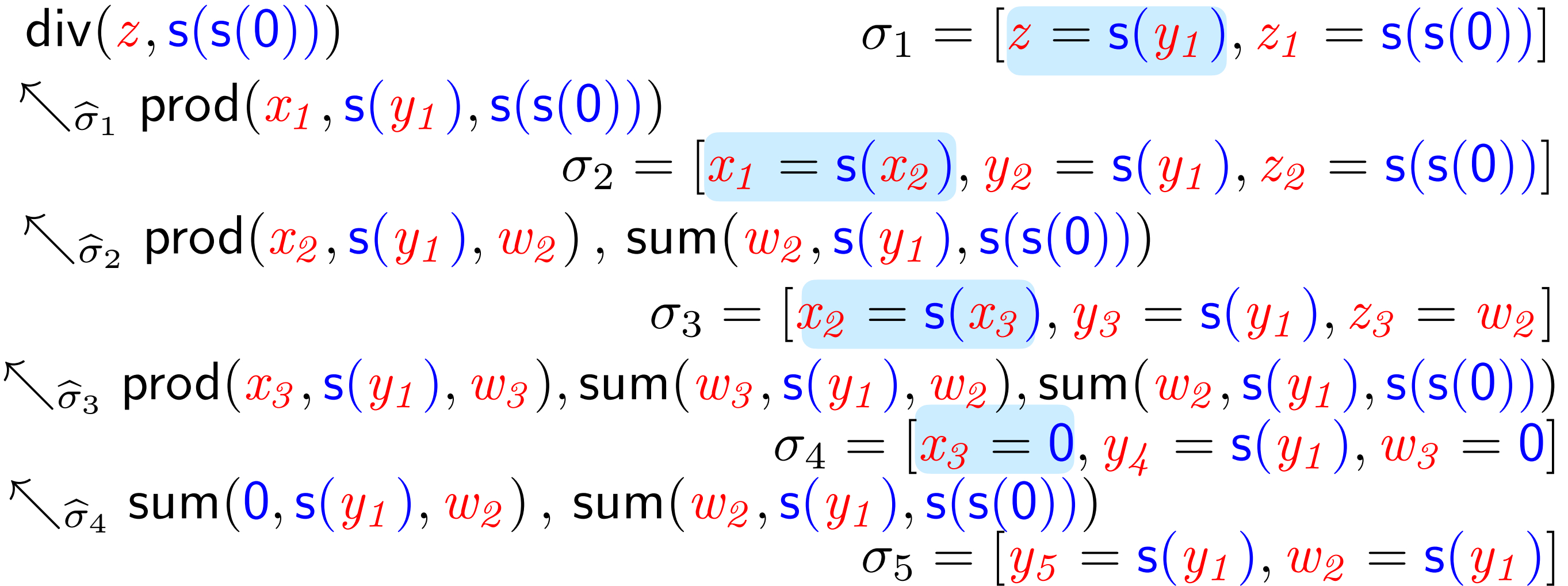
div(s(y),z) :- prod(x,s(y),z) .

$$\begin{aligned}
 & \text{div}(z, s(s(0))) & \sigma_1 &= [z = s(y_1), z_1 = s(s(0))] \\
 & \swarrow_{\hat{\sigma}_1} \text{prod}(x_1, s(y_1), s(s(0))) & \sigma_2 &= [x_1 = s(x_2), y_2 = s(y_1), z_2 = s(s(0))] \\
 & \swarrow_{\hat{\sigma}_2} \text{prod}(x_2, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_3 &= [x_2 = s(x_3), y_3 = s(y_1), z_3 = w_2] \\
 & \swarrow_{\hat{\sigma}_3} \text{prod}(x_3, s(y_1), w_3), \text{sum}(w_3, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_4 &= [x_3 = 0, y_4 = s(y_1), w_3 = 0] \\
 & \swarrow_{\hat{\sigma}_4} \text{sum}(0, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0)))
 \end{aligned}$$

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sum(0,y,y) .
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 & \text{div}(z, s(s(0))) & \sigma_1 &= [z = s(y_1), z_1 = s(s(0))] \\
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 & \swarrow_{\hat{\sigma}_2} \text{prod}(x_2, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_3 &= [x_2 = s(x_3), y_3 = s(y_1), z_3 = w_2] \\
 & \swarrow_{\hat{\sigma}_3} \text{prod}(x_3, s(y_1), w_3), \text{sum}(w_3, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_4 &= [x_3 = 0, y_4 = s(y_1), w_3 = 0] \\
 & \swarrow_{\hat{\sigma}_4} \text{sum}(0, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_5 &= [y_5 = s(y_1), w_2 = s(y_1)]
 \end{aligned}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

div(s(y),z) :- prod(x,s(y),z) .

$$\begin{array}{rcl}
 \text{div}(z, s(s(0))) & \sigma_1 = [z = s(y_1), z_1 = s(s(0))] & \\
 \swarrow_{\hat{\sigma}_1} \text{prod}(x_1, s(y_1), s(s(0))) & \sigma_2 = [x_1 = s(x_2), y_2 = s(y_1), z_2 = s(s(0))] & \\
 \swarrow_{\hat{\sigma}_2} \text{prod}(x_2, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_3 = [x_2 = s(x_3), y_3 = s(y_1), z_3 = w_2] & \\
 \swarrow_{\hat{\sigma}_3} \text{prod}(x_3, s(y_1), w_3), \text{sum}(w_3, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_4 = [x_3 = 0, y_4 = s(y_1), w_3 = 0] & \\
 \swarrow_{\hat{\sigma}_4} \text{sum}(0, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_5 = [y_5 = s(y_1), w_2 = s(y_1)] & \\
 \swarrow_{\hat{\sigma}_5} \text{sum}(s(y_1), s(y_1), s(s(0))) & &
 \end{array}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

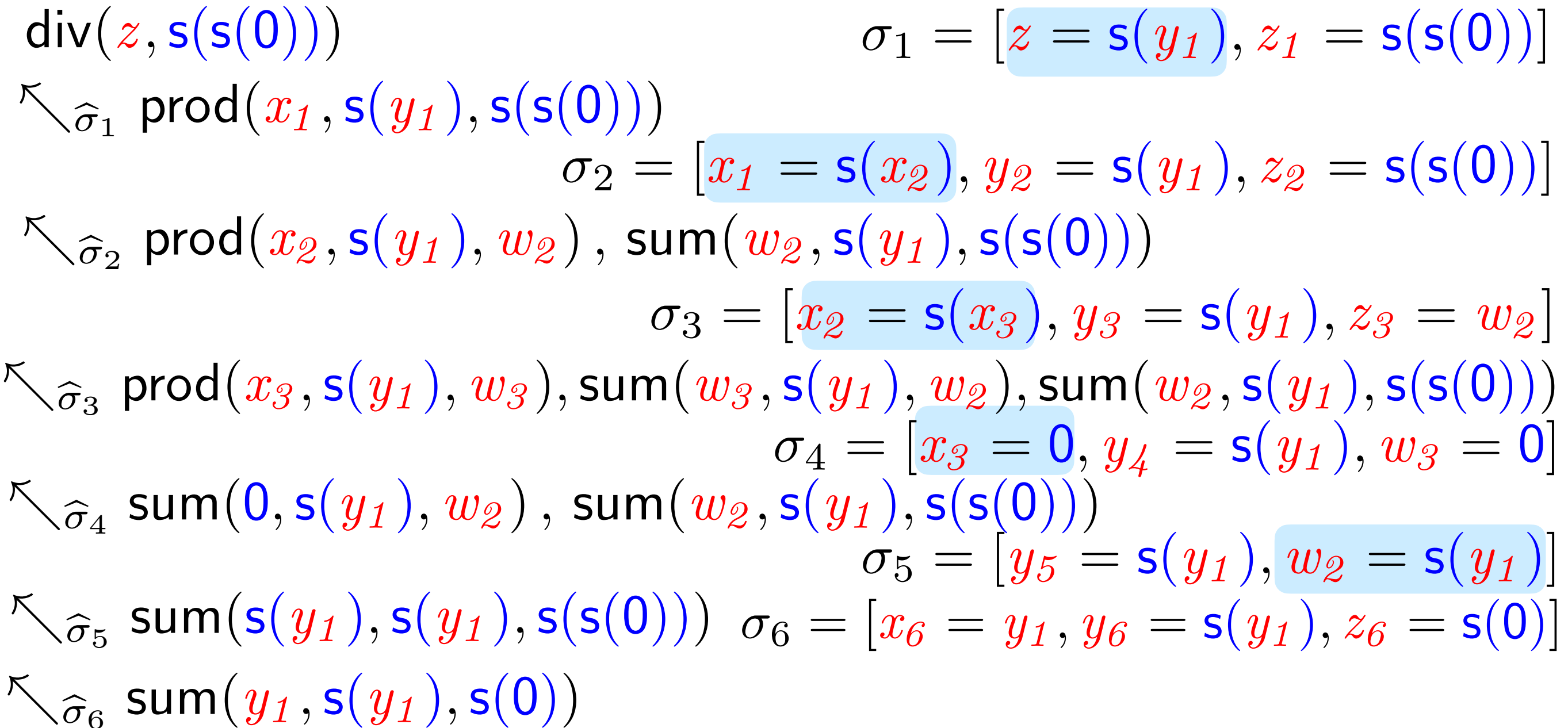
div(s(y),z) :- prod(x,s(y),z) .

$$\begin{array}{rcl}
 \text{div}(z, s(s(0))) & \sigma_1 = [z = s(y_1), z_1 = s(s(0))] & \\
 \swarrow_{\hat{\sigma}_1} \text{prod}(x_1, s(y_1), s(s(0))) & \sigma_2 = [x_1 = s(x_2), y_2 = s(y_1), z_2 = s(s(0))] & \\
 \swarrow_{\hat{\sigma}_2} \text{prod}(x_2, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_3 = [x_2 = s(x_3), y_3 = s(y_1), z_3 = w_2] & \\
 \swarrow_{\hat{\sigma}_3} \text{prod}(x_3, s(y_1), w_3), \text{sum}(w_3, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_4 = [x_3 = 0, y_4 = s(y_1), w_3 = 0] & \\
 \swarrow_{\hat{\sigma}_4} \text{sum}(0, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_5 = [y_5 = s(y_1), w_2 = s(y_1)] & \\
 \swarrow_{\hat{\sigma}_5} \text{sum}(s(y_1), s(y_1), s(s(0))) & \sigma_6 = [x_6 = y_1, y_6 = s(y_1), z_6 = s(0)] &
 \end{array}$$

sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

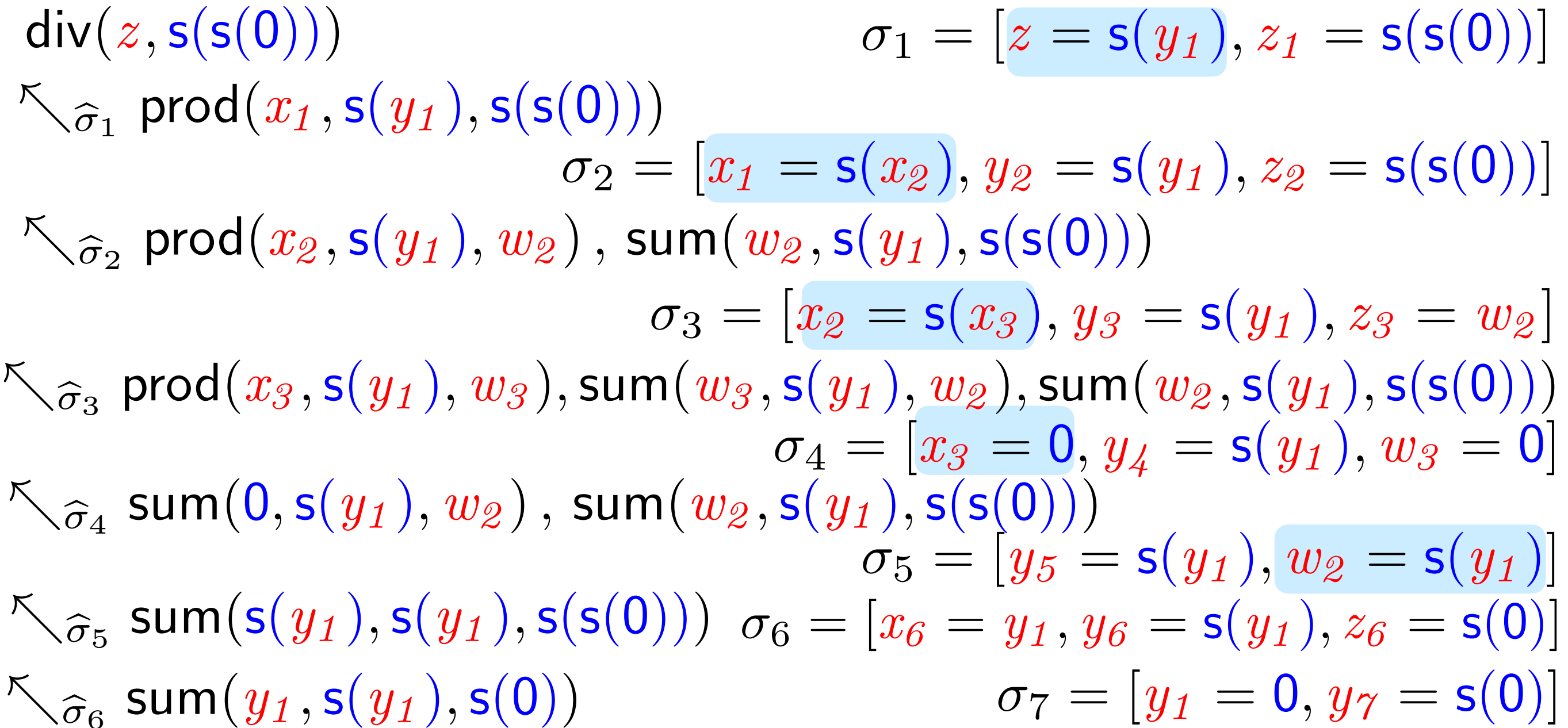
div(s(y),z) :- prod(x,s(y),z) .



sum(0,y,y) .
 sum(s(x),y,s(z)) :- sum(x,y,z) .

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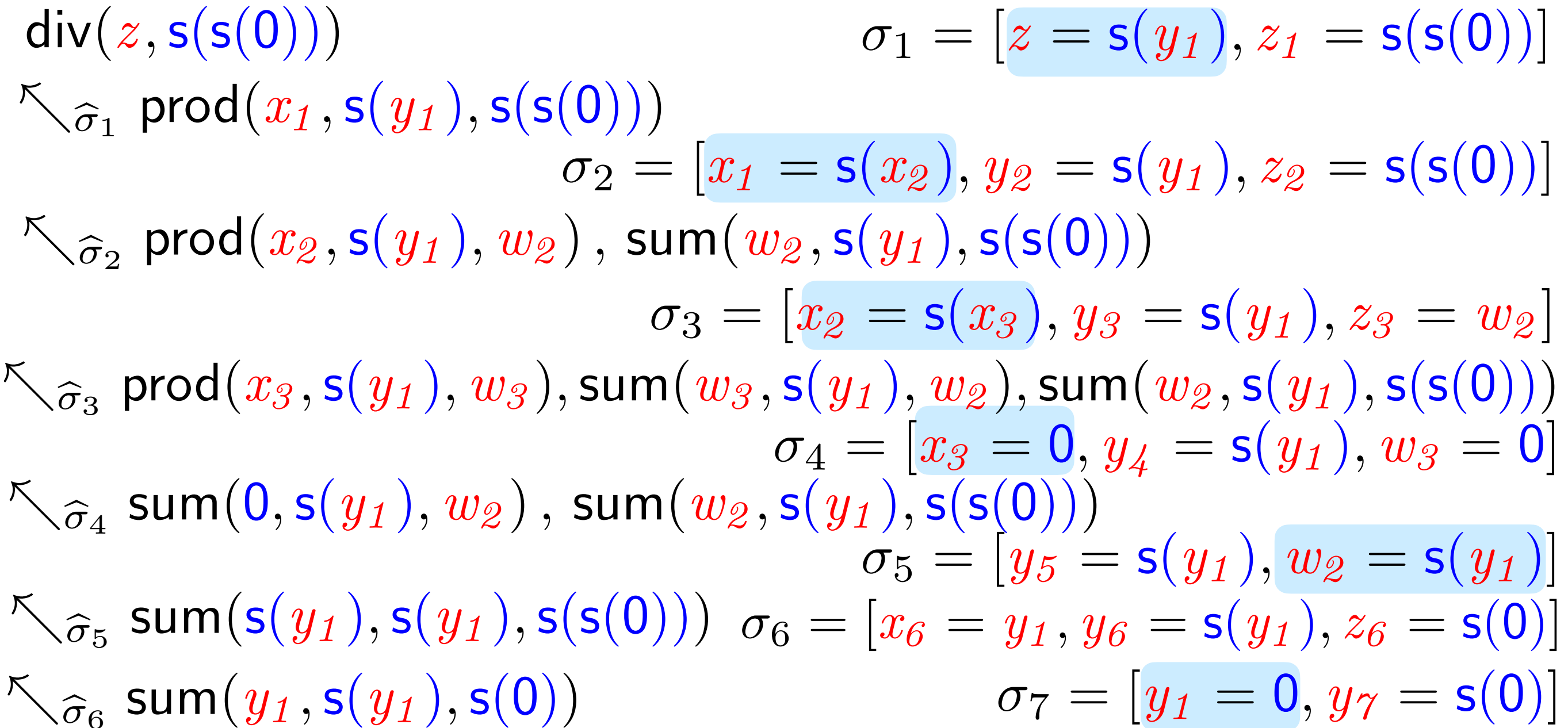
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sum(0,y,y) .
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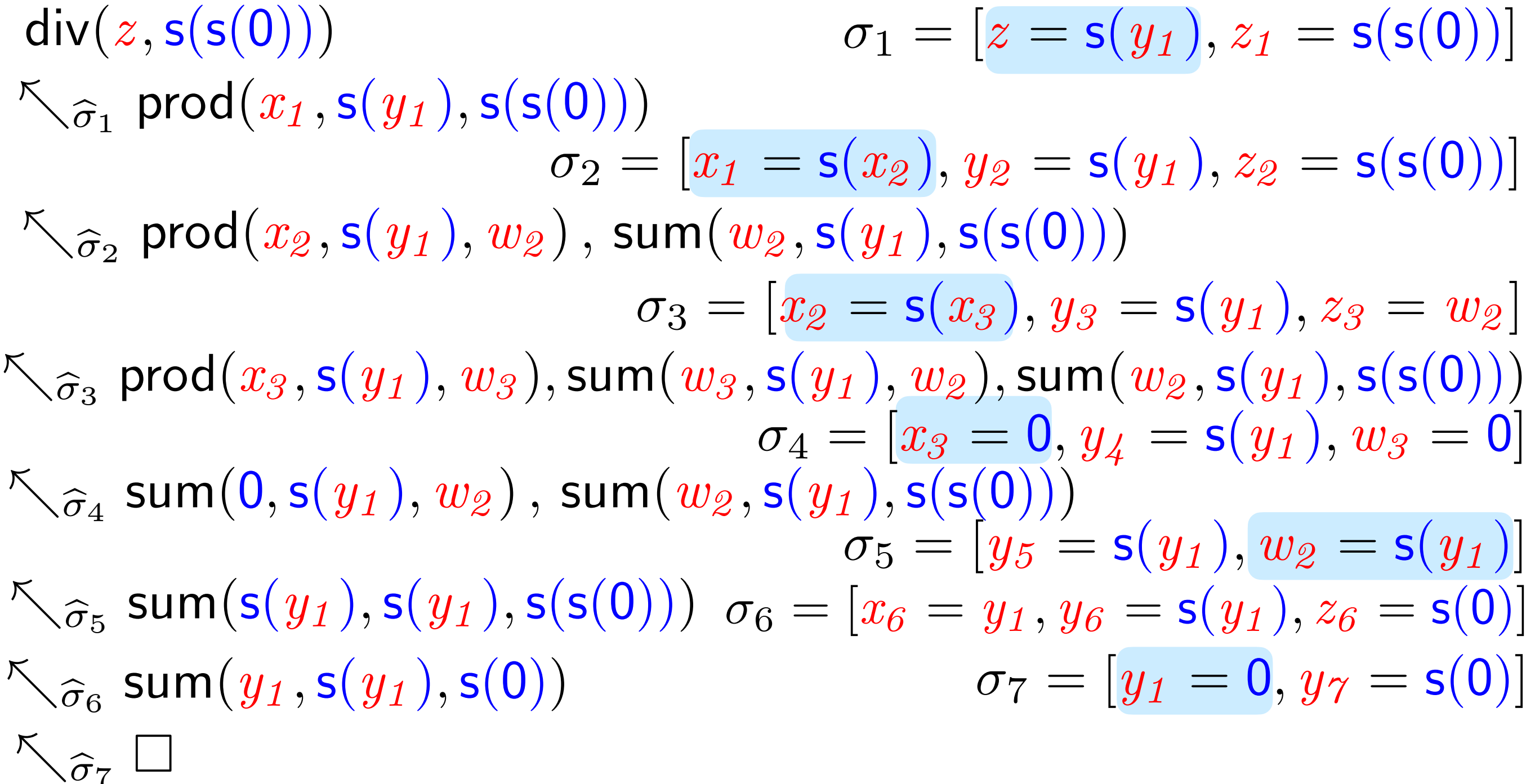
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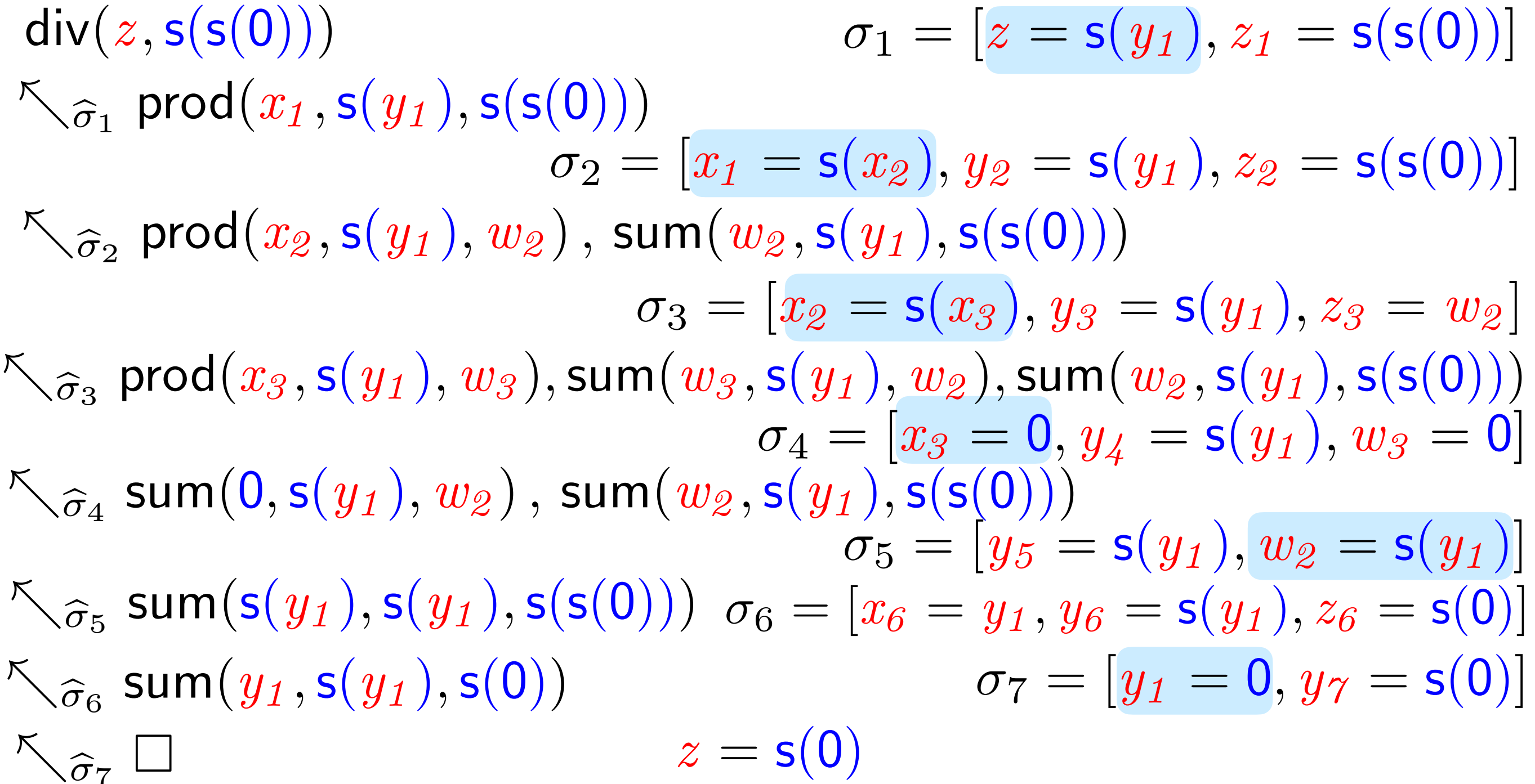
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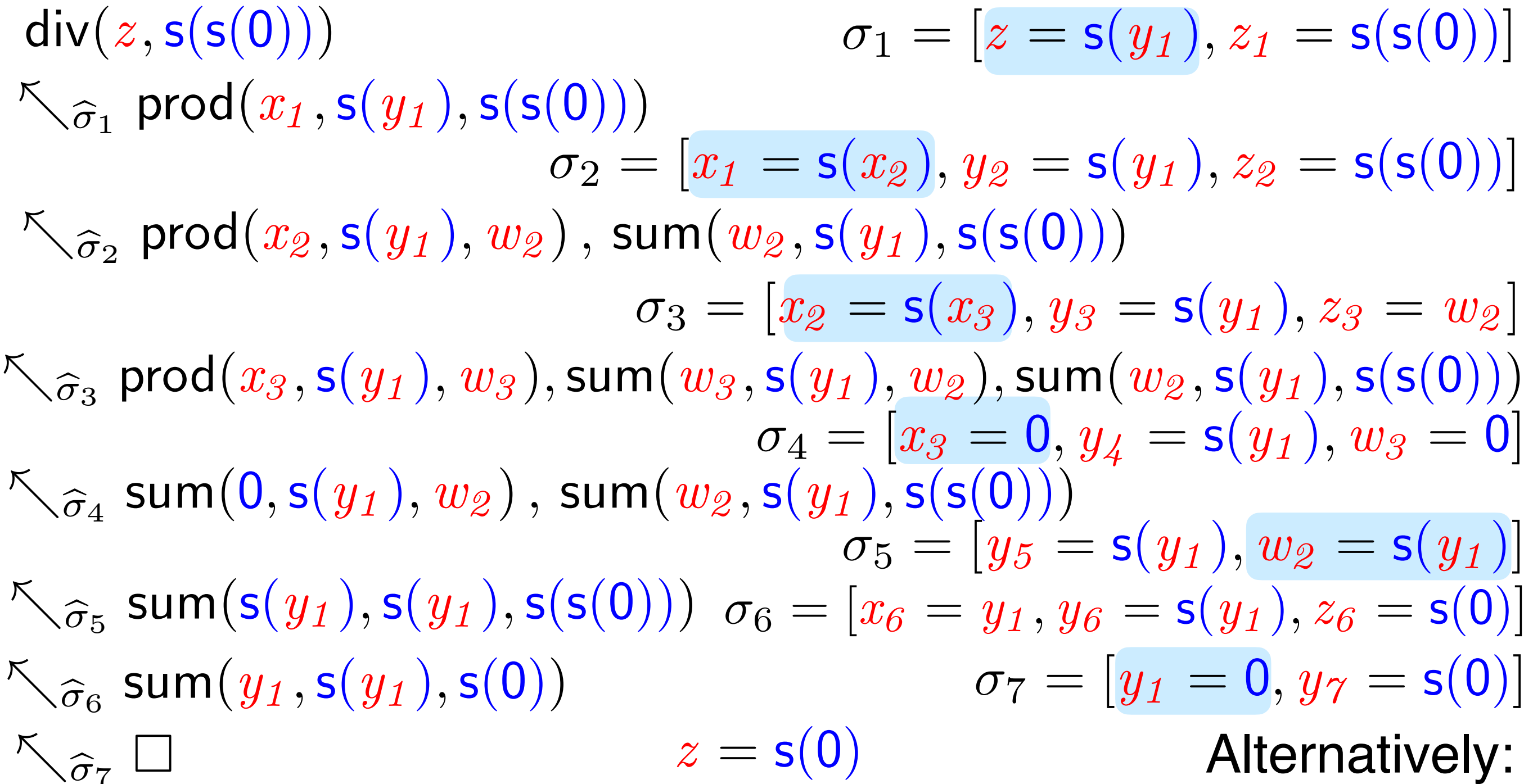
div(s(y),z) :- prod(x,s(y),z) .



sum(0,y,y) .
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prod(0,y,0) .
 prod(s(x),y,z) :- prod(x,y,w) , sum(w,y,z) .

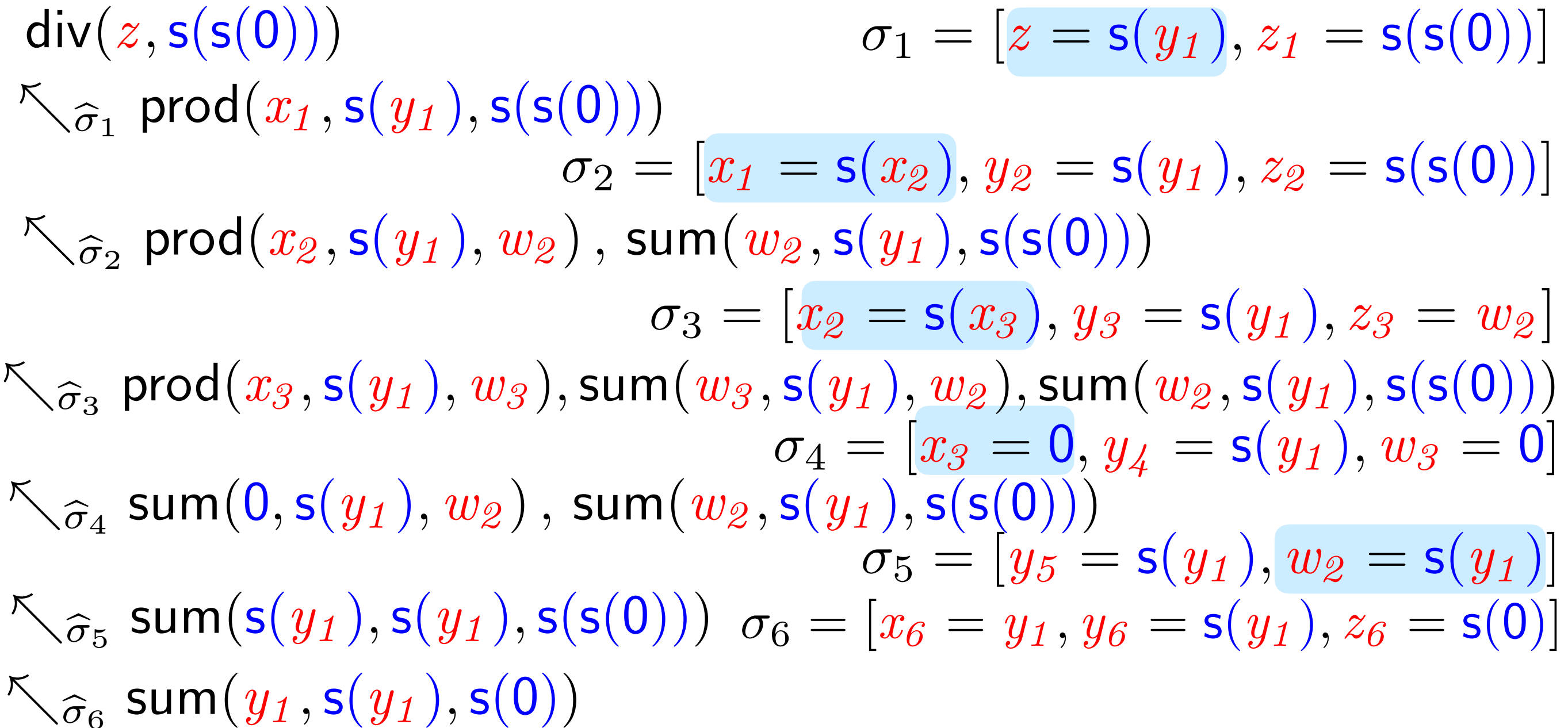
div(s(y),z) :- prod(x,s(y),z) .



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$$\begin{array}{rcl}
 \text{div}(z, s(s(0))) & \sigma_1 = [z = s(y_1), z_1 = s(s(0))] & \\
 \swarrow_{\hat{\sigma}_1} \text{prod}(x_1, s(y_1), s(s(0))) & \sigma_2 = [x_1 = s(x_2), y_2 = s(y_1), z_2 = s(s(0))] & \\
 \swarrow_{\hat{\sigma}_2} \text{prod}(x_2, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_3 = [x_2 = s(x_3), y_3 = s(y_1), z_3 = w_2] & \\
 \swarrow_{\hat{\sigma}_3} \text{prod}(x_3, s(y_1), w_3), \text{sum}(w_3, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_4 = [x_3 = 0, y_4 = s(y_1), w_3 = 0] & \\
 \swarrow_{\hat{\sigma}_4} \text{sum}(0, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_5 = [y_5 = s(y_1), w_2 = s(y_1)] & \\
 \swarrow_{\hat{\sigma}_5} \text{sum}(s(y_1), s(y_1), s(s(0))) & \sigma_6 = [x_6 = y_1, y_6 = s(y_1), z_6 = s(0)] & \\
 \swarrow_{\hat{\sigma}_6} \text{sum}(y_1, s(y_1), s(0)) & \sigma_7 = [y_1 = s(x_7), y_7 = s(s(x_7)), z_7 = 0] &
 \end{array}$$

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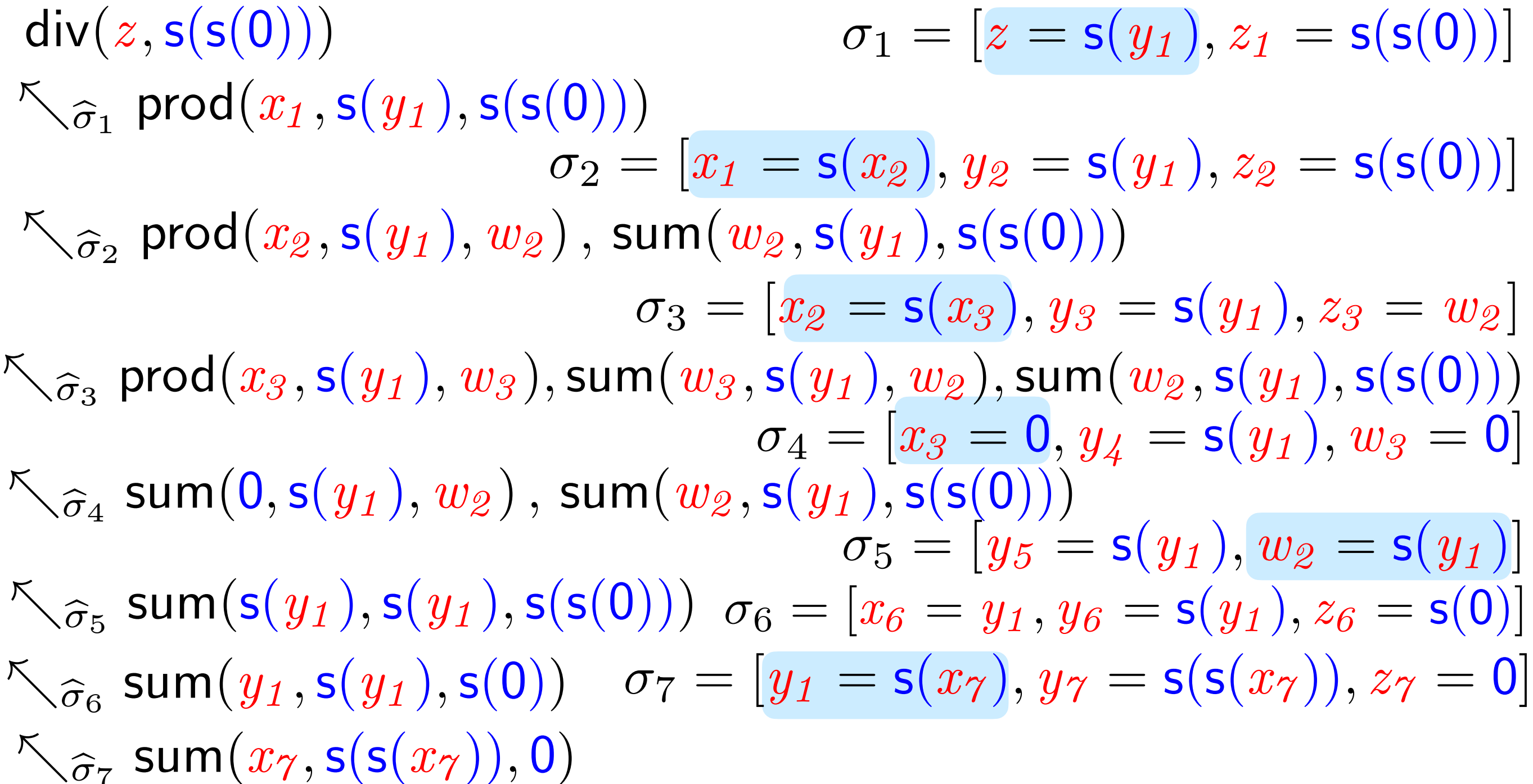
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 \swarrow_{\hat{\sigma}_2} \text{prod}(x_2, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_3 = [x_2 = s(x_3), y_3 = s(y_1), z_3 = w_2] & \\
 \swarrow_{\hat{\sigma}_3} \text{prod}(x_3, s(y_1), w_3), \text{sum}(w_3, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_4 = [x_3 = 0, y_4 = s(y_1), w_3 = 0] & \\
 \swarrow_{\hat{\sigma}_4} \text{sum}(0, s(y_1), w_2), \text{sum}(w_2, s(y_1), s(s(0))) & \sigma_5 = [y_5 = s(y_1), w_2 = s(y_1)] & \\
 \swarrow_{\hat{\sigma}_5} \text{sum}(s(y_1), s(y_1), s(s(0))) & \sigma_6 = [x_6 = y_1, y_6 = s(y_1), z_6 = s(0)] & \\
 \swarrow_{\hat{\sigma}_6} \text{sum}(y_1, s(y_1), s(0)) & \sigma_7 = [y_1 = s(x_7), y_7 = s(s(x_7)), z_7 = 0] &
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