MPP 2025/26 (0077A, 9CFU)

Models for Programming Paradigms

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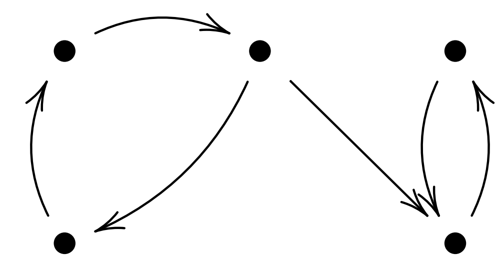
22b - Mu-calculus

μ-calculus

mu-calculus

models



syntax

$$
\begin{array}{lll}
\psi \;::=\; \mathbf{tt} \mid \mathbf{ff} \mid \psi_0 \wedge \psi_1 \mid \psi_0 \vee \psi_1 & & \text{classical ops} \\
\quad\;\;\mid\; p \mid \neg p & \text{atomic propositions} & \\
\quad\;\;\mid\; \Diamond\psi & \text{there is a next state where } \psi \text{ holds} & \\
\quad\;\;\mid\; \Box\psi & \psi \text{ holds at every next state} & \\
\quad\;\;\mid\; x & \text{predicate variable, for recursive def} & \\
\quad\;\;\mid\; \mu x.\ \psi & \text{LEAST FIXPOINT of } x =_{\mathsf{min}} \psi & \\
\quad\;\;\mid\; \nu x.\ \psi & \text{GREATEST FIXPOINT of } x =_{\mathsf{max}} \psi &
\end{array}
$$

mu-calculus: semantics

$$G = (V, \to)$$

$[\![\psi]\!]\rho$ set of nodes where ψ holds

$\rho : P \cup X \to \wp(V)$

assignment

$(\wp(V), \subseteq)$ is a complete lattice

for monotone functions: least / greatest fixpoint exist

for this reason negation is not present in the syntax

the formulas we consider: *positive normal form*

otherwise:

even number of negations before each variable occurrence

Positive normal form

$$\neg\Diamond\psi \equiv \Box\neg\psi \qquad \neg\Box\psi \equiv \Diamond\neg\psi$$

$$\neg\mu x.\ \psi \equiv \nu x.\ \neg\psi[{}^{\neg x}/_{x}] \qquad \neg\nu x.\ \psi \equiv \mu x.\ \neg\psi[{}^{\neg x}/_{x}]$$

$$\neg\neg\psi \equiv \psi$$

plus usual De Morgan's laws

mu-calculus: semantics

$$[\![\cdot]\!] : \mathcal{F} \to (P \cup X \to \wp(V)) \to \wp(V)$$

defined by structural induction

$$[\![\mathbf{tt}]\!]\rho \triangleq V \qquad [\![\psi_0 \wedge \psi_1]\!]\rho \triangleq [\![\psi_0]\!]\rho \cap [\![\psi_1]\!]\rho$$

$$[\![\mathbf{ff}]\!]\rho \triangleq \emptyset \qquad [\![\psi_0 \vee \psi_1]\!]\rho \triangleq [\![\psi_0]\!]\rho \cup [\![\psi_1]\!]\rho$$

$$[\![p]\!]\rho \triangleq \rho(p) \qquad [\![\Diamond\psi]\!]\rho \triangleq \{v \mid \exists w \in [\![\psi]\!]\rho.\ v \to w\}$$

$$[\![\neg p]\!]\rho \triangleq V \setminus \rho(p) \qquad [\![\Box\psi]\!]\rho \triangleq \{v \mid \forall w.\ v \to w \Rightarrow w \in [\![\psi]\!]\rho\}$$

$$[\![\mu x.\ \psi]\!]\rho \triangleq \mathit{fix}\ \lambda S.[\![\psi]\!]\rho[^S/_x]$$

$$[\![x]\!]\rho \triangleq \rho(x)$$

$$[\![\nu x.\ \psi]\!]\rho \triangleq \mathit{FIX}\ \lambda S.[\![\psi]\!]\rho[^S/_x]$$

mu-calculus: fixpoint

V finite

$f : \wp(V) \to \wp(V)$ monotone (hence continuous)

we can compute the least fixpoint by

$$\mathit{fix}\ f = \bigcup_{n \in \mathbb{N}} f^n(\emptyset)$$

we can compute the greatest fixpoint by

$$\mathit{FIX}\ f = \bigcap_{n \in \mathbb{N}} f^n(V)$$

Examples

$$\llbracket \Diamond \mathbf{tt} \rrbracket \rho \triangleq \{v \mid \exists w \in \llbracket \mathbf{tt} \rrbracket \rho.\ v \to w\}$$
$$= \{v \mid \exists w \in V.\ v \to w\}$$

$\boxed{\Diamond \mathbf{tt}}$

non deadlocked states

$$\llbracket \Box \mathbf{ff} \rrbracket \rho \triangleq \{v \mid \forall w.\ v \to w \Rightarrow w \in \llbracket \mathbf{ff} \rrbracket \rho\}$$
$$= \{v \mid \forall w.\ v \to w \Rightarrow w \in \emptyset\}$$
$$= \{v \mid v \not\to\}$$

$\Box \mathbf{ff}$

deadlocks

Examples

$$
\begin{aligned}
[\![\Diamond \mathbf{ff}]\!]\rho &\triangleq \{v \mid \exists w \in [\![\mathbf{ff}]\!]\rho.\ v \to w\} \\
&= \{v \mid \exists w \in \emptyset.\ v \to w\} \\
&= \emptyset
\end{aligned}
$$

$\Diamond \mathbf{ff}$

false

$$
\begin{aligned}
[\![\Box \mathbf{tt}]\!]\rho &\triangleq \{v \mid \forall w.\ v \to w \Rightarrow w \in [\![\mathbf{tt}]\!]\rho\} \\
&= \{v \mid \forall w.\ v \to w \Rightarrow w \in V\} \\
&= V
\end{aligned}
$$

$\Box \mathbf{tt}$

true

Examples

$$\llbracket \mu x.\ x \rrbracket \rho \triangleq \mathit{fix}\ \lambda S.\ \llbracket x \rrbracket \rho[{}^{S}/{}_{x}]$$
$$= \mathit{fix}\ \lambda S.\ S$$
$$= \emptyset$$

$\boxed{\begin{array}{c} \mu x.\ x \\ \text{false} \end{array}}$

$$\llbracket \nu x.\ x \rrbracket \rho \triangleq \mathit{FIX}\ \lambda S.\ \llbracket x \rrbracket \rho[{}^{S}/{}_{x}]$$
$$= \mathit{FIX}\ \lambda S.\ S$$
$$= V$$

$\boxed{\begin{array}{c} \nu x.\ x \\ \text{true} \end{array}}$

Examples

$$
\begin{aligned}
[\![\mu x.\ \Diamond x]\!]\rho &\triangleq \mathit{fix}\ \lambda S.\ [\![\Diamond x]\!]\rho[^S/_x] \\
&= \mathit{fix}\ \lambda S.\ \{v \mid \exists w \in [\![x]\!]\rho[^S/_x].\ v \to w\} \\
&= \mathit{fix}\ \lambda S.\ \{v \mid \exists w \in S.\ v \to w\}
\end{aligned}
$$

$$
\begin{aligned}
S_0 &= \emptyset \\
S_1 &= \{v \mid \exists w \in S_0.\ v \to w\} \\
&= \{v \mid \exists w \in \emptyset.\ v \to w\} \\
&= \emptyset
\end{aligned}
$$

$\mu x.\ \Diamond x$

false

Examples

how to represent?

EO ψ $\qquad\qquad$ $\Diamond\psi$

AO ψ $\qquad\qquad$ $\Box\psi$

EF p

$$p \vee \ldots$$
$$p \vee \Diamond(p \vee \cdots)$$
$$p \vee \Diamond(p \vee \Diamond(p \vee \cdots))$$
$$x = p \vee \Diamond x$$
$$\mu x.\ p \vee \Diamond x$$
$$\nu x.\ p \vee \Diamond x$$

?

EF p ?

Example

$$
\begin{aligned}
[\![\nu x.\ p \vee \Diamond x]\!]\rho &\triangleq FIX\ \ \lambda S.\ [\![p \vee \Diamond x]\!]\rho[{}^{S}/_{x}] \\
&= FIX\ \ \lambda S.\ [\![p]\!]\rho[{}^{S}/_{x}] \cup [\![\Diamond x]\!]\rho[{}^{S}/_{x}] \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in [\![x]\!]\rho[{}^{S}/_{x}].\ v \to w\} \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}
\end{aligned}
$$

$$
\begin{aligned}
S_0 &= V \\
S_1 &= \rho(p) \cup \{v \mid \exists w \in S_0.\ v \to w\} \\
&= \rho(p) \cup \{v \mid \exists w \in V.\ v \to w\} \\
&= \rho(p) \cup \{v \text{ can move}\}
\end{aligned}
$$

EF p ?

Example

$$
\begin{aligned}
[\![\nu x.\ p \vee \Diamond x]\!]\rho &\triangleq FIX\ \ \lambda S.\ [\![p \vee \Diamond x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ [\![p]\!]\rho[^S/_x] \cup [\![\Diamond x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in [\![x]\!]\rho[^S/_x].\ v \to w\} \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}
\end{aligned}
$$

$$
\begin{aligned}
S_0 &= V \\
S_1 &= \rho(p) \cup \{v \text{ can move}\} \\
S_2 &= \rho(p) \cup \{v \mid \exists w \in S_1.\ v \to w\} \\
&= \rho(p) \cup \{v \mid \exists w \in \rho(p).\ v \to w\} \cup \{v \text{ can make 2 moves}\}
\end{aligned}
$$

EF p ?

Examples

$$
\begin{aligned}
[\![\nu x.\ p \vee \Diamond x]\!]\rho &\triangleq FIX\ \ \lambda S.\ [\![p \vee \Diamond x]\!]\rho[{}^{S}/_{x}] \\
&= FIX\ \ \lambda S.\ [\![p]\!]\rho[{}^{S}/_{x}] \cup [\![\Diamond x]\!]\rho[{}^{S}/_{x}] \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in [\![x]\!]\rho[{}^{S}/_{x}].\ v \to w\} \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}
\end{aligned}
$$

$$S_0 = V$$

$$S_1 = \rho(p) \cup \{v \text{ can move}\}$$

$$S_2 = \rho(p) \cup \{v \mid \exists w \in \rho(p).\ v \to w\} \cup \{v \text{ can make 2 moves}\}$$

$$
\begin{aligned}
S_n = &\{v \text{ can reach a state in } \rho(p) \text{ in less than n moves}\} \\
&\cup \{v \text{ can make n moves}\}
\end{aligned}
$$

EF p ?

Example

$$
\begin{aligned}
[\![\nu x.\ p \vee \Diamond x]\!]\rho &\triangleq FIX\ \ \lambda S.\ [\![p \vee \Diamond x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ [\![p]\!]\rho[^S/_x] \cup [\![\Diamond x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in [\![x]\!]\rho[^S/_x].\ v \to w\} \\
&= FIX\ \ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}
\end{aligned}
$$

$S_n = \{v$ can reach a state in $\rho(p)$ in less than n moves$\}$

$\cup\{v$ can make n moves$\}$

$\bigcap_{n \in \mathbb{N}} S_n = \{v$ can reach a state in $\rho(p)$ $\boxed{\text{or has an infinite path}}\}$

EF p ?

Example

$$
\begin{aligned}
[\![\mu x.\ p \vee \Diamond x]\!]\rho &\triangleq \mathit{fix}\ \lambda S.\ [\![p \vee \Diamond x]\!]\rho[{}^{S}/_{x}] \\
&= \mathit{fix}\ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}
\end{aligned}
$$

$$S_0 = \emptyset$$

$$
\begin{aligned}
S_1 &= \rho(p) \cup \{v \mid \exists w \in S_0.\ v \to w\} \\
&= \rho(p) \cup \{v \mid \exists w \in \emptyset.\ v \to w\} \\
&= \rho(p)
\end{aligned}
$$

EF p ?

Example

$$[\![\mu x.\ p \vee \Diamond x]\!]\rho \triangleq \mathit{fix}\ \lambda S.\ [\![p \vee \Diamond x]\!]\rho[{}^{S}/_{x}]$$

$$= \mathit{fix}\ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}$$

$$S_0 = \emptyset$$

$$S_1 = \rho(p)$$

$$S_2 = \rho(p) \cup \{v \mid \exists w \in S_1.\ v \to w\}$$

$$= \rho(p) \cup \{v \mid \exists w \in \rho(p).\ v \to w\}$$

$$= \{v \text{ can reach a state in } \rho(p) \text{ in less than 2 moves}\}$$

EF p ?

Example

$$[\![\mu x.\ p \vee \Diamond x]\!]\rho \triangleq \mathit{fix}\ \lambda S.\ [\![p \vee \Diamond x]\!]\rho[{}^{S}/_{x}]$$
$$= \mathit{fix}\ \lambda S.\ \rho(p) \cup \{v \mid \exists w \in S.\ v \to w\}$$

$$S_0 = \emptyset$$

$$S_1 = \rho(p)$$

$$S_2 = \{v \text{ can reach a state in } \rho(p) \text{ in less than 2 moves}\}$$

$$S_n = \{v \text{ can reach a state in } \rho(p) \text{ in less than } n \text{ moves}\}$$

$$\bigcup_{n \in \mathbb{N}} S_n = \{v \text{ can reach a state in } \rho(p)\}$$

$\mu x.\ p \vee \Diamond x$

EF p

Example

which formula for
"some deadlock is reachable"?

$\Box \mathbf{ff}$

deadlocks

$\mu x.\ \Box \mathbf{ff} \vee \Diamond x$

$\mu x.\ p \vee \Diamond x$

EF p

which formula for
"deadlock free"?

$$\neg(\mu x.\ \Box \mathbf{ff} \vee \Diamond x) = \nu x.\ \neg(\Box \mathbf{ff} \vee \Diamond \neg x)$$
$$= \nu x.\ \neg(\Box \mathbf{ff}) \wedge \neg(\Diamond \neg x)$$
$$= \nu x.\ \Diamond \mathbf{tt} \wedge \Box x$$

Example

$$
\begin{aligned}
[\![\nu x.\ p \wedge \Box x]\!]\rho &\triangleq FIX\ \ \lambda S.\ [\![p \wedge \Box x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ [\![p]\!]\rho[^S/_x] \cap [\![\Box x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ \rho(p) \cap \{v \mid \forall w.\ v \rightarrow w \Rightarrow w \in [\![x]\!]\rho[^S/_x]\} \\
&= FIX\ \ \lambda S.\ \rho(p) \cap \{v \mid \forall w.\ v \rightarrow w \Rightarrow w \in S\}
\end{aligned}
$$

$$
\begin{aligned}
S_0 &= V \\
S_1 &= \rho(p) \cap \{v \mid \forall w.\ v \rightarrow w \Rightarrow w \in S_0\} \\
&= \rho(p) \cap \{v \mid \forall w.\ v \rightarrow w \Rightarrow w \in V\} \\
&= \rho(p) \cap V \\
&= \rho(p)
\end{aligned}
$$

Example

$$
\begin{aligned}
[\![\nu x.\ p \wedge \Box x]\!]\rho &\triangleq FIX\ \ \lambda S.\ [\![p \wedge \Box x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ [\![p]\!]\rho[^S/_x] \cap [\![\Box x]\!]\rho[^S/_x] \\
&= FIX\ \ \lambda S.\ \rho(p) \cap \{v \mid \forall w.\ v \rightarrow w \Rightarrow w \in [\![x]\!]\rho[^S/_x]\} \\
&= FIX\ \ \lambda S.\ \rho(p) \cap \{v \mid \forall w.\ v \rightarrow w \Rightarrow w \in S\}
\end{aligned}
$$

$$S_0 = V$$

$$S_1 = \rho(p)$$

$$
\begin{aligned}
S_2 &= \rho(p) \cap \{v \mid \forall w.\ v \rightarrow w \Rightarrow w \in S_1\} \\
&= \rho(p) \cap \{v \mid \forall w.\ v \rightarrow w \Rightarrow w \in \rho(p)\} \\
&= \{v \text{ s.t. all nodes reachable in less than 2 moves are in } \rho(p)\}
\end{aligned}
$$

Example

$$[\![\nu x.\ p \wedge \Box x]\!]\rho \triangleq FIX\ \ \lambda S.\ [\![p \wedge \Box x]\!]\rho[^S/_x]$$

$$= FIX\ \ \lambda S.\ [\![p]\!]\rho[^S/_x] \cap [\![\Box x]\!]\rho[^S/_x]$$

$$= FIX\ \ \lambda S.\ \rho(p) \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in [\![x]\!]\rho[^S/_x]\}$$

$$= FIX\ \ \lambda S.\ \rho(p) \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S\}$$

$$\boxed{\begin{array}{c} \nu x.\ p \wedge \Box x \\ \\ \mathsf{AG}\ p \end{array}}$$

$$S_0 = V$$

$$S_1 = \rho(p)$$

$$S_2 = \{v \text{ s.t. all nodes reachable in less than 2 moves are in } \rho(p)\}$$

$$S_n = \{v \text{ s.t. all nodes reachable in less than } n \text{ moves are in } \rho(p)\}$$

$$\bigcap_{n \in \mathbb{N}} S_n = \{v \text{ can only reach states in } \rho(p)\}$$

Invariants & possibly

Invariants $Inv(\psi) \triangleq \nu x.\ \psi \wedge \Box x$

Possibly $Pos(\psi) \triangleq \mu x.\ \psi \vee \Diamond x$

Example

which temporal formula?

$\mu x.\ q \vee (p \wedge \Diamond x)$ E$(p$ U $q)$ (CTL)

$\mu x.\ q \vee (p \wedge \Diamond x \wedge \Box x)$ A$(p$ U $q)$ (LTL / CTL)

$\nu x.\ \underbrace{\mu y.\ (p \wedge \Diamond x) \vee \Diamond y}$ E G F p (CTL*)

after a finite number of steps you reach a state where

1) *p* holds

2) there is a next step where the property holds recursively

Alternation depth

Fixed points *alternate* if a least fixed point influences the greatest fixed point, and vice-versa.

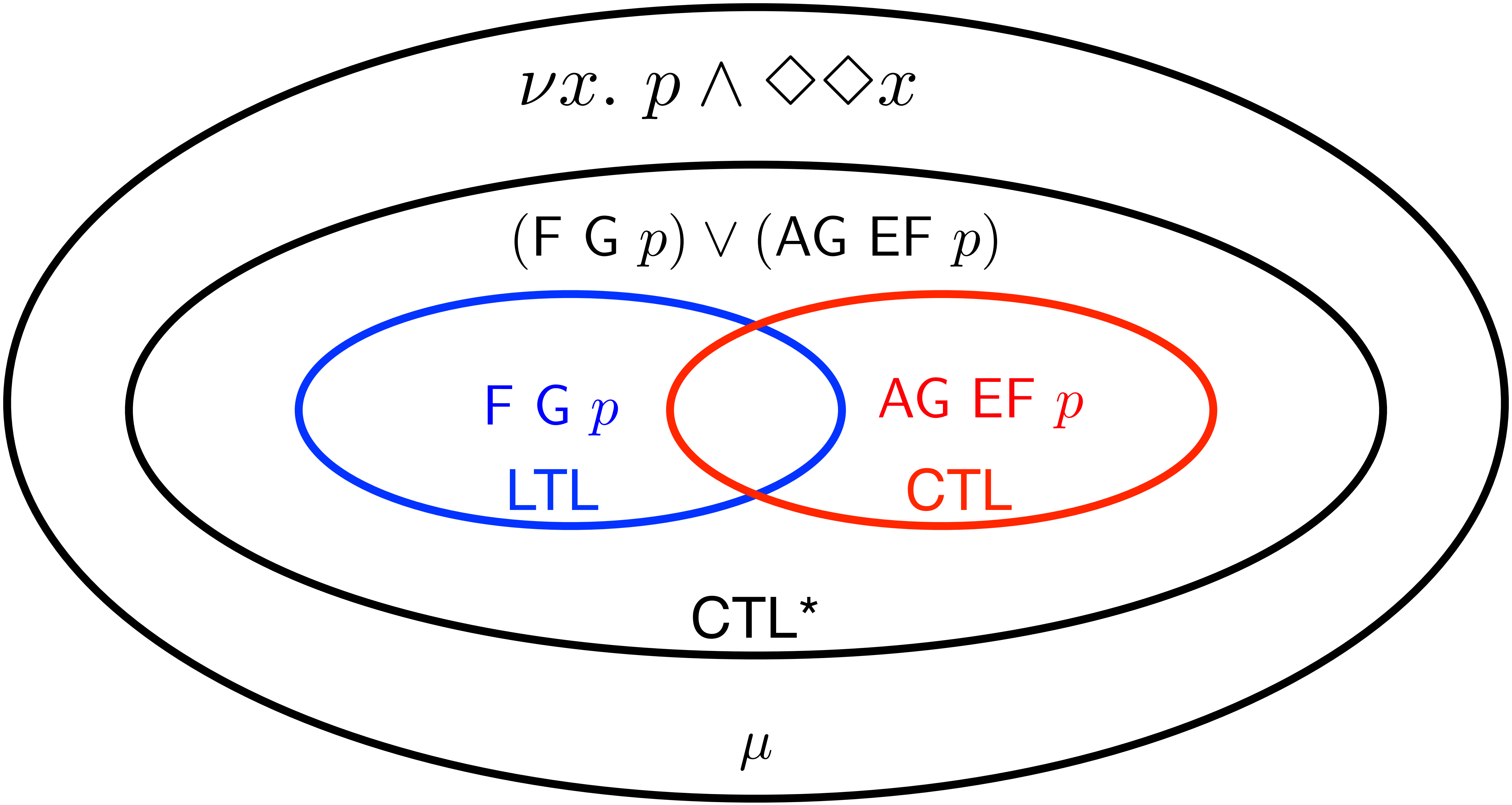
The *alternation depth* counts the number of alternations.

The alternation hierarchy of the μ-calculus is strict: strictly more properties are expressible as the alternation depth increases.

CTL can be encoded in μ-calculus with alternation depth 1

CTL* and LTL can be encoded in μ-calculus with alternation depth at most 2

Expressiveness



mu-calculus with labels

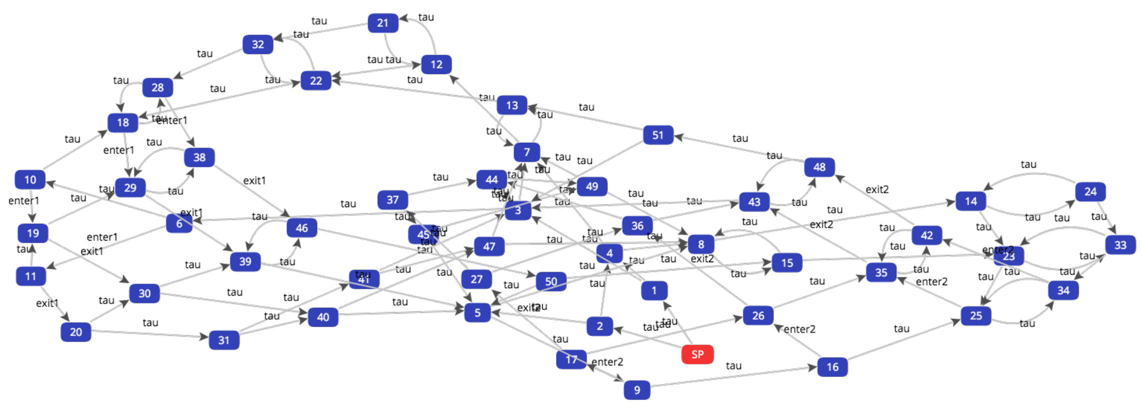
$$
\begin{array}{lll}
\psi & ::= & \cdots \\
 & \mid & \Diamond_L \psi \\
 & \mid & \Box_L \psi
\end{array}
$$

set of labels

$$[\![\Diamond_L \psi]\!]\rho \triangleq \{v \mid \boxed{\exists \mu \in L.}\ \exists w \in [\![\psi]\!]\rho.\ v \boxed{\xrightarrow{\mu}} w\}$$

$$[\![\Box_L \psi]\!]\rho \triangleq \{v \mid \boxed{\forall \mu \in L.}\ \forall w.\ v \boxed{\xrightarrow{\mu}} w \Rightarrow w \in [\![\psi]\!]\rho\}$$

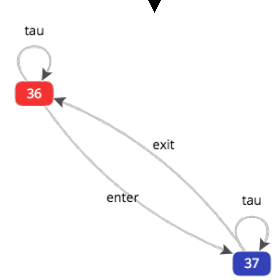
Space reduction



$$\overset{?}{\models} \psi$$

identify
bisimilar
states

$$\Updownarrow$$



$$\overset{?}{\models} \psi$$

mu-calculus exercises

Exercise: dogs

Three dogs live in a house with two couches and a front garden. Let $couch_{i,j}$ represent the predicate "the dog i sits on couch j" and $garden_i$ represent the predicate "the dog i plays in the front garden".

1. Write an LTL formula expressing the fact that whenever dog 1 plays in the garden then he keeps playing until he sits on some couch (but he may also play forever).

2. Write a CTL formula expressing the fact that dog 2 eventually plays in the garden whenever couch 1 is occupied by another dog.

3. Write a μ-calculus formula expressing the fact that no more than one couch is occupied at any time by dog 3.

Exercise: dogs

$couch_{i,j}$ — i sits on j

$garden_i$ — i plays in the garden

LTL

$$\mathsf{G}\ (garden_1 \Rightarrow ((\mathsf{G}\ garden_1) \vee (garden_1\ \mathsf{U}\ (couch_{1,1} \vee couch_{1,2}))))$$

CTL

$$\mathsf{AG}\ ((couch_{1,1} \vee couch_{3,1}) \Rightarrow \mathsf{AF}\ garden_2)$$

μ-calculus

$$\nu x.\ ((\neg couch_{3,1} \vee \neg couch_{3,2}) \wedge \Box x)$$

1. Write an LTL formula expressing the fact that whenever dog 1 plays in the garden then he keeps playing until he sits on some couch (but he may also play forever).
2. Write a CTL formula expressing the fact that dog 2 eventually plays in the garden whenever couch 1 is occupied by another dog.
3. Write a μ-calculus formula expressing the fact that no more than one couch is occupied at any time by dog 3.

Ex. mu-calculus fix

Given the μ-calculus formula $\Phi = \mu x.((p \wedge \Box x) \vee (\neg p \wedge \Diamond x))$ write its denotational semantics $[\![\Phi]\!]\,\rho$ and evaluate it on the LTS below (where $V = \{s_1, s_2, s_3, s_4\}$ and $P = \{p\}$).

$$
\begin{tikzcd}
s_1 \arrow[rr] \arrow[drr] & & s_2 \\
p \quad s_4 & & s_3 \arrow[ll] \arrow[u]
\end{tikzcd}
$$

Ex. mu-calculus fix

$$\rho(p) = \{s_4\} \qquad\qquad \overline{\rho(p)} = \{s_1, s_2, s_3\}$$

$$\llbracket \mu x.((p \wedge \Box x) \vee (\neg p \wedge \Diamond x)) \rrbracket \rho$$

$$\triangleq \mathit{fix}\ \lambda S.\ \llbracket (p \wedge \Box x) \vee (\neg p \wedge \Diamond x) \rrbracket \rho[{}^{S}/_{x}]$$

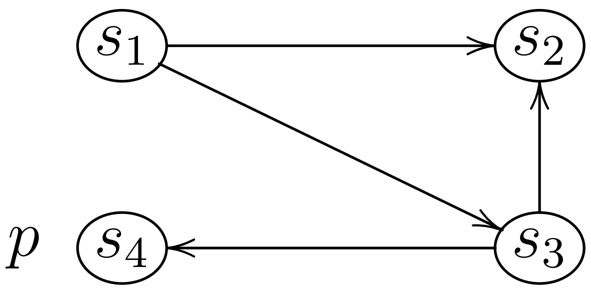
$$= \mathit{fix}\ \lambda S.\ \llbracket p \wedge \Box x \rrbracket \rho[{}^{S}/_{x}] \cup \llbracket \neg p \wedge \Diamond x \rrbracket \rho[{}^{S}/_{x}]$$

$$\begin{aligned} = \mathit{fix}\ \lambda S.\quad & (\llbracket p \rrbracket \rho[{}^{S}/_{x}] \cap \llbracket \Box x \rrbracket \rho[{}^{S}/_{x}]) \cup \\ & (\llbracket \neg p \rrbracket \rho[{}^{S}/_{x}] \cap \llbracket \Diamond x \rrbracket \rho[{}^{S}/_{x}]) \end{aligned}$$

$$\begin{aligned} = \mathit{fix}\ \lambda S.\quad & (\rho(p) \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in \llbracket x \rrbracket \rho[{}^{S}/_{x}]\}) \cup \\ & (\overline{\rho(p)} \cap \{v \mid \exists w \in \llbracket x \rrbracket \rho[{}^{S}/_{x}].\ v \to w\}) \end{aligned}$$

$$\begin{aligned} = \mathit{fix}\ \lambda S.\quad & (\{s_4\} \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S\}) \cup \\ & (\{s_1, s_2, s_3\} \cap \{v \mid \exists w \in S.\ v \to w\}) \end{aligned}$$

Ex. mu-calculus fix



$$\mathit{fix}\ \lambda S.\quad \begin{array}{l}(\{s_4\} \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S\}) \cup \\ (\{s_1, s_2, s_3\} \cap \{v \mid \exists w \in S.\ v \to w\})\end{array}$$

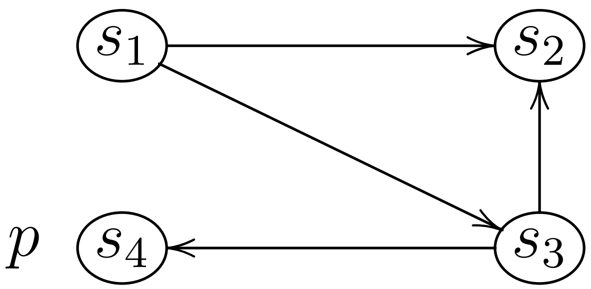
$$S_0 = \emptyset$$

$$\begin{aligned}
S_1 = \quad & (\{s_4\} \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S_0\}) \cup \\
& (\{s_1, s_2, s_3\} \cap \{v \mid \exists w \in S_0.\ v \to w\}) \\
= \quad & (\{s_4\} \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in \emptyset\}) \cup \\
& (\{s_1, s_2, s_3\} \cap \{v \mid \exists w \in \emptyset.\ v \to w\}) \\
= \quad & (\{s_4\} \cap \{s_2, s_4\}) \cup \\
& (\{s_1, s_2, s_3\} \cap \emptyset) \\
= \quad & \{s_4\}
\end{aligned}$$

deadlock

emptyset

Ex. mu-calculus fix



$$\mathit{fix}\ \lambda S.\quad (\{s_4\} \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S\}) \cup$$
$$(\{s_1, s_2, s_3\} \cap \{v \mid \exists w \in S.\ v \to w\})$$

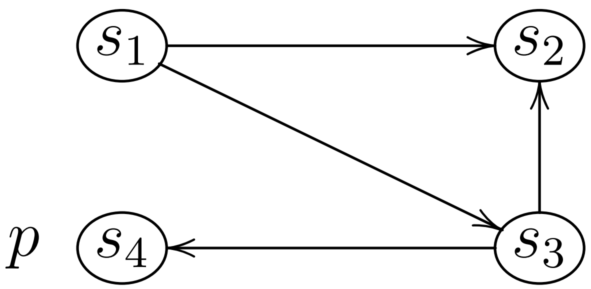
$$S_0 = \emptyset \qquad S_1 = \{s_4\}$$

deadlock or
can reach only s_4

$$\begin{aligned} S_2 = \quad & (\{s_4\} \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S_1\}) \cup \\ & (\{s_1, s_2, s_3\} \cap \{v \mid \exists w \in S_1.\ v \to w\}) \\ = \quad & (\{s_4\} \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in \{s_4\}\}) \cup \\ & (\{s_1, s_2, s_3\} \cap \{v \mid \exists w \in \{s_4\}.\ v \to w\}) \\ = \quad & (\{s_4\} \cap \{s_2, s_4\}) \cup \\ & (\{s_1, s_2, s_3\} \cap \{s_3\}) \\ = \quad & \{s_3, s_4\} \end{aligned}$$

has a transition to s_4

Ex. mu-calculus fix



$$\mathit{fix}\ \lambda S.\quad (\{s_4\} \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S\}) \cup (\{s_1, s_2, s_3\} \cap \{v \mid \exists w \in S.\ v \to w\})$$

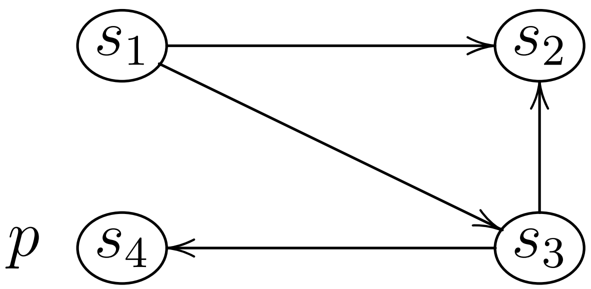
$$S_0 = \emptyset \qquad S_1 = \{s_4\} \qquad S_2 = \{s_3, s_4\}$$

deadlock or
can reach only s_3, s_4

$$\begin{aligned} S_3 = \quad & (\{s_4\} \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S_2\}) \cup \\ & (\{s_1, s_2, s_3\} \cap \{v \mid \exists w \in S_2.\ v \to w\}) \\ = \quad & (\{s_4\} \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in \{s_3, s_4\}\}) \cup \\ & (\{s_1, s_2, s_3\} \cap \{v \mid \exists w \in \{s_3, s_4\}.\ v \to w\}) \\ = \quad & (\{s_4\} \cap \{s_2, s_4\}) \cup \\ & (\{s_1, s_2, s_3\} \cap \{s_1, s_3\}) \\ = \quad & \{s_1, s_3, s_4\} \end{aligned}$$

has a transition to s_3 or s_4

Ex. mu-calculus fix



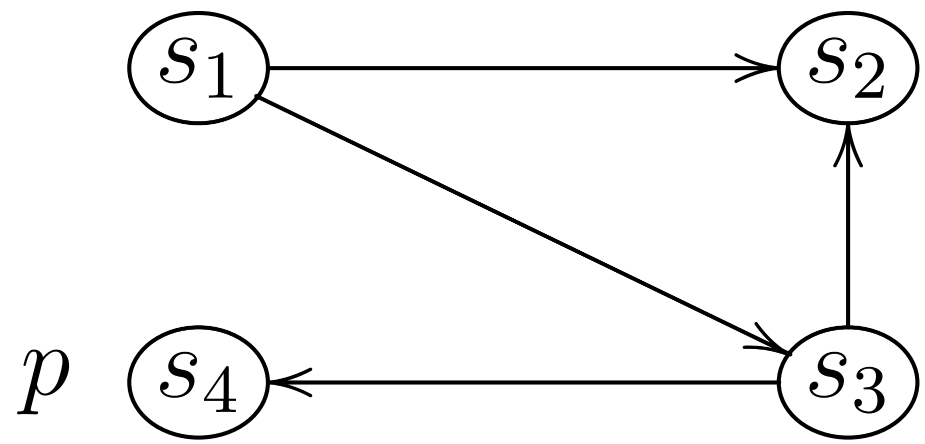
$$\mathit{fix}\ \lambda S.\quad \begin{array}{l}(\{s_4\} \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S\}) \cup \\ (\{s_1, s_2, s_3\} \cap \{v \mid \exists w \in S.\ v \to w\})\end{array}$$

$$S_0 = \emptyset \qquad S_1 = \{s_4\} \qquad S_2 = \{s_3, s_4\} \qquad S_3 = \{s_1, s_3, s_4\}$$

$$\begin{array}{rl}
S_4 = & (\{s_4\} \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in S_3\}) \cup \\
& (\{s_1, s_2, s_3\} \cap \{v \mid \exists w \in S_3.\ v \to w\}) \\[1em]
= & (\{s_4\} \cap \{v \mid \forall w.\ v \to w \Rightarrow w \in \{s_1, s_3, s_4\}\}) \cup \\
& (\{s_1, s_2, s_3\} \cap \{v \mid \exists w \in \{s_1, s_3, s_4\}.\ v \to w\}) \\[1em]
= & (\{s_4\} \cap \{s_2, s_4\}) \cup \\
& (\{s_1, s_2, s_3\} \cap \{s_1, s_3\}) \\[1em]
= & \{s_1, s_3, s_4\} = S_3
\end{array}$$

fixpoint reached!

Ex. mu-calculus fix



$$[\![\mu x.((p \wedge \Box x) \vee (\neg p \wedge \Diamond x))]\!]\rho \;=\; \{s_1, s_3, s_4\}$$