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Models for Programming Paradigms

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14 - HOFL Denotational Semantics

Interpretation Domains

Interpretation Domains

$$D_{int} \triangleq \mathbb{Z}_\bot$$

$$D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_\bot$$

to distinguish:
pair of divergent terms
from divergent pair

$$D_{\tau_1 \to \tau_2} \triangleq [D_{\tau_1} \to D_{\tau_2}]_\bot$$

to distinguish:
takes arg and diverge
from divergence without taking arg

Example

$$D_{int * int} \triangleq (\mathbb{Z}_\bot \times \mathbb{Z}_\bot)_\bot$$

$\mathbf{rec}\ p.\ p$	$(\mathbf{rec}\ x.\ x, \mathbf{rec}\ y.\ y)$
$\bot_{D_{int * int}}$	$(\bot_{D_{int}}, \bot_{D_{int}})$

Example

$$D_{int \to int} \triangleq [\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot$$

$\mathbf{rec}\ f.\ f$	$\lambda x.\ \mathbf{rec}\ y.\ y$
$\bot_{D_{int \to int}}$	$\lambda d.\ \bot_{D_{int}}$

Interpretation Domains

$$D_{int} \triangleq \mathbb{Z}_\perp \quad D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_\perp \quad D_{\tau_1 \to \tau_2} \triangleq [D_{\tau_1} \to D_{\tau_2}]_\perp$$

Equivalently: $D_\tau \triangleq (V_\tau)_\perp$

$$V_{int} \triangleq \mathbb{Z}$$

$$V_{\tau_1 * \tau_2} \triangleq D_{\tau_1} \times D_{\tau_2} = (V_{\tau_1})_\perp \times (V_{\tau_2})_\perp$$

$$V_{\tau_1 \to \tau_2} \triangleq [D_{\tau_1} \to D_{\tau_2}] = [(V_{\tau_1})_\perp \to (V_{\tau_2})_\perp]$$

Interpretation Function

$t : \tau$ $\quad\quad$ $[\![t]\!]\rho \in D_\tau$

environment $\quad \rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $\quad x : \tau \Rightarrow \rho(x) \in D_\tau$

we define the interpretation function by structural recursion

Denotational Semantics

Constants

$$\underbrace{[\![\underbrace{n}_{int}]\!]\rho}_{D_{int} = \mathbb{Z}_\perp} \triangleq \underbrace{\lfloor \underbrace{n}_{\mathbb{Z}} \rfloor}_{\mathbb{Z}_\perp}$$

Variables

$$\underbrace{[\![\underbrace{x}_{\tau}]\!]\rho}_{D_\tau} \triangleq \underbrace{\rho(x)}_{D_\tau}$$

$$x : \tau \Rightarrow \rho(x) \in D_\tau$$

Arithmetic ops

to prove: $\underline{\mathsf{op}}_\perp$ is monotone and continuous

$$\mathsf{op} \in \{+, - \times\}$$

$$\underbrace{[\![\underbrace{\underbrace{t_1}_{int} \ \mathsf{op} \ \underbrace{t_2}_{int}}_{int}]\!]\rho}_{D_{int} = \mathbb{Z}_\perp} \triangleq \underbrace{\underbrace{[\![\underbrace{t_1}_{int}]\!]\rho}_{D_{int} = \mathbb{Z}_\perp} \ \underline{\mathsf{op}}_\perp \ \underbrace{[\![\underbrace{t_2}_{int}]\!]\rho}_{D_{int} = \mathbb{Z}_\perp}}_{D_{int} = \mathbb{Z}_\perp}$$

$$\underline{\mathsf{op}}_\perp : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$v_1 \ \underline{\mathsf{op}}_\perp \ v_2 \triangleq \begin{cases} \lfloor n_1 \ \underline{\mathsf{op}} \ n_2 \rfloor & \text{if } v_1 = \lfloor n_1 \rfloor \text{ and } v_2 = \lfloor n_2 \rfloor \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise } (v_1 = \perp_{\mathbb{Z}_\perp} \text{ or } v_2 = \perp_{\mathbb{Z}_\perp}) \end{cases}$$

called *strict* extension

Conditionals

to prove: Cond_τ is monotone and continuous

$$\underbrace{[\![\underbrace{\mathbf{if}\ \underbrace{t}_{int}\ \mathbf{then}\ \underbrace{t_1}_{\tau}\ \mathbf{else}\ \underbrace{t_2}_{\tau}}_{\tau}]\!]\rho}_{D_\tau} \triangleq \underbrace{\mathsf{Cond}_\tau(\ \underbrace{[\![\underbrace{t}_{int}]\!]\rho}_{D_{int}=\mathbb{Z}_\perp}\ ,\ \underbrace{[\![\underbrace{t_1}_{\tau}]\!]\rho}_{D_\tau}\ ,\ \underbrace{[\![\underbrace{t_2}_{\tau}]\!]\rho}_{D_\tau}\ )}_{D_\tau}$$

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

Pairing

$$D_{\tau_1 * \tau_2} \triangleq (D_{\tau_1} \times D_{\tau_2})_\perp$$

$$\underbrace{[\![\underbrace{(\underbrace{t_1}_{\tau_1} \ , \ \underbrace{t_2}_{\tau_2})}_{\tau_1 * \tau_2}]\!] \rho}_{D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_\perp} \triangleq \underbrace{\lfloor \ \underbrace{(\ \underbrace{[\![\underbrace{t_1}_{\tau_1}]\!] \rho}_{D_{\tau_1}} \ , \ \underbrace{[\![\underbrace{t_2}_{\tau_2}]\!] \rho}_{D_{\tau_2}} \ )}_{D_{\tau_1} \times D_{\tau_2}} \ \rfloor}_{D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_\perp}$$

Projections

Equivalently:

$$[\![\mathbf{fst}(\ t\ )]\!]\rho \triangleq \mathbf{let}\ d \Leftarrow [\![t]\!]\rho.\ \pi_1(d)$$

$$\underbrace{[\![\underbrace{\mathbf{fst}(\ \underbrace{t}_{\tau_1 * \tau_2}\ )}_{\tau_1}]\!]\rho}_{D_{\tau_1}} \triangleq \underbrace{\underbrace{\underbrace{\pi_1}_{D_{\tau_1} \times D_{\tau_2} \to D_{\tau_1}}{}^{*}}_{(D_{\tau_1} \times D_{\tau_2})_\perp \to D_{\tau_1}}\ (\ \underbrace{[\![\underbrace{t}_{\tau_1 * \tau_2}]\!]\rho}_{D_{\tau_1 * \tau_2} = (D_{\tau_1} \times D_{\tau_2})_\perp}\ )}_{D_{\tau_1}}$$

$$[\![\mathbf{snd}(\ t\ )]\!]\rho \triangleq \pi_2^*\ (\ [\![t]\!]\rho\ )$$

Abstraction

$$D_{\tau_1 \to \tau_2} \triangleq [D_{\tau_1} \to D_{\tau_2}]_\perp$$

$$\underbrace{[\![\underbrace{\lambda \underbrace{x}_{\tau_1}.\ \underbrace{t}_{\tau_2}}_{\tau_1 \to \tau_2}]\!]}_{D_{\tau_1 \to \tau_2} = [D_{\tau_1} \to D_{\tau_2}]_\perp}\rho \triangleq \underbrace{\lfloor\ \underbrace{\lambda \underbrace{d}_{D_{\tau_1}}.\ \underbrace{[\![\underbrace{t}_{\tau_2}]\!]\rho[^d/_x]}_{D_{\tau_2}}}_{[D_{\tau_1} \to D_{\tau_2}]}\ \rfloor}_{[D_{\tau_1} \to D_{\tau_2}]_\perp}$$

Application (lazy)

Equivalently:

$$[\![\ t\ t_0\ ]\!]\rho \triangleq (\lambda\varphi.\ \varphi([\![t_0]\!]\rho))^*\ ([\![t]\!]\rho)$$

$$\underbrace{[\![\ \underbrace{\underbrace{t}_{\tau_0 \to \tau}\ \underbrace{t_0}_{\tau_0}}_{\tau}\ ]\!]\rho}_{D_\tau} \triangleq \underbrace{\textbf{let}\ \underbrace{\varphi}_{V_{\tau_0\to\tau} = [D_{\tau_0} \to D_\tau]} \Leftarrow \underbrace{[\![\underbrace{t}_{\tau_0 \to \tau}]\!]\rho}_{D_{\tau_0\to\tau} = (V_{\tau_0\to\tau})_\perp}.\ \underbrace{\underbrace{\varphi}_{[D_{\tau_0} \to D_\tau]}(\underbrace{[\![\underbrace{t_0}_{\tau_0}]\!]\rho}_{D_{\tau_0}})}_{D_\tau}}_{D_\tau}$$

Recursion

$$\underbrace{[\![\underbrace{\mathbf{rec}\ \underbrace{x}_{\tau}.\ \underbrace{t}_{\tau}}_{\tau}]\!]\rho}_{D_\tau} \triangleq \underbrace{[\![\underbrace{t}_{\tau}]\!]\rho[\underbrace{{}^{[\![\mathbf{rec}\ x.\ t]\!]\rho}}_{D_\tau}/\underbrace{x}_{\tau}]}_{D_\tau}$$

Recursion

$$\underbrace{[\![\underbrace{\mathbf{rec}\ \underbrace{x}_{\tau}.\ \underbrace{t}_{\tau}}_{\tau}]\!]\rho}_{D_\tau} \triangleq \underbrace{\underbrace{\mathit{fix}}_{[[D_\tau \to D_\tau] \to D_\tau]}\ \underbrace{\lambda \underbrace{d}_{D_\tau}.\ \underbrace{[\![\underbrace{t}_{\tau}]\!]\rho[{}^{\underbrace{d}_{D_\tau}}/_{\underbrace{x}_{\tau}}]}_{D_\tau}}_{[D_\tau \to D_\tau]}}_{D_\tau}$$

Recap

$$[\![n]\!]\rho \triangleq \lfloor n \rfloor$$

$$[\![x]\!]\rho \triangleq \rho(x)$$

$$[\![t_1 \ \mathsf{op}\ t_2]\!]\rho \triangleq [\![t_1]\!]\rho \ \underline{\mathsf{op}}_\perp \ [\![t_2]\!]\rho$$

$$[\![\mathbf{if}\ t\ \mathbf{then}\ t_1\ \mathbf{else}\ t_2]\!]\rho \triangleq \mathsf{Cond}_\tau(\ [\![t]\!]\rho\ ,\ [\![t_1]\!]\rho\ ,\ [\![t_2]\!]\rho\ )$$

$$[\![(t_1\ ,\ t_2)]\!]\rho \triangleq \lfloor\ (\ [\![t_1]\!]\rho\ ,\ [\![t_2]\!]\rho\ )\ \rfloor$$

$$[\![\mathbf{fst}(\ t\ )]\!]\rho \triangleq \pi_1^*\ (\ [\![t]\!]\rho\ )$$

$$[\![\mathbf{snd}(\ t\ )]\!]\rho \triangleq \pi_2^*\ (\ [\![t]\!]\rho\ )$$

$$[\![\lambda x.\ t]\!]\rho \triangleq \lfloor\ \lambda d.\ [\![t]\!]\rho[^d/_x]\ \rfloor$$

$$[\![\ t\ t_0\ ]\!]\rho \triangleq \mathbf{let}\ \varphi \Leftarrow [\![t]\!]\rho.\ \varphi([\![t_0]\!]\rho)$$

$$[\![\mathbf{rec}\ x.\ t]\!]\rho \triangleq \mathit{fix}\ \lambda d.\ [\![t]\!]\rho[^d/_x]$$

Example

$$f \stackrel{\text{def}}{=} \lambda x : int .\, 3$$

$$[\![\lambda x.\ t]\!]\rho \triangleq \lfloor\ \lambda d.\ [\![t]\!]\rho[^d/_x]\ \rfloor \qquad [\![n]\!]\rho \triangleq \lfloor n \rfloor$$

$$[\![f]\!]\rho = [\![\lambda x.\, 3]\!]\rho = \lfloor \lambda d.\, [\![3]\!]\rho[^d/_x] \rfloor = \lfloor \lambda d.\, \lfloor 3 \rfloor \rfloor$$

Example

$$g \stackrel{\text{def}}{=} \lambda x : int.\ \textbf{if}\ x\ \textbf{then}\ 3\ \textbf{else}\ 3$$

$$[\![\lambda x.\ t]\!]\rho \triangleq \lfloor\ \lambda d.\ [\![t]\!]\rho[^d/_x]\ \rfloor$$

$$\begin{aligned} [\![g]\!]\rho &= [\![\lambda x.\ \textbf{if}\ x\ \textbf{then}\ 3\ \textbf{else}\ 3]\!]\rho \\ &= \lfloor \lambda d.\ [\![\textbf{if}\ x\ \textbf{then}\ 3\ \textbf{else}\ 3]\!]\rho[^d/_x] \rfloor \\ &= \lfloor \lambda d.\ Cond(d, \lfloor 3 \rfloor, \lfloor 3 \rfloor) \rfloor \\ &= \lfloor \lambda d.\ \textbf{let}\ x \Leftarrow d.\ \lfloor 3 \rfloor \rfloor \end{aligned}$$

$$[\![f]\!]\rho \neq [\![g]\!]\rho$$

$$\lfloor \lambda d.\ \lfloor 3 \rfloor \rfloor$$

Example

$$h \stackrel{\text{def}}{=} \textbf{rec}\, y : int \to int\,.\ \lambda x : int\,.\ 3$$

$$\begin{aligned}
[\![h]\!]\rho &= [\![\textbf{rec}\ y.\ \lambda x.\ 3]\!]\rho \\
&= \text{fix}\ \lambda d_y.\ [\![\lambda x.\ 3]\!]\rho[^{d_y}/_y] \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ [\![3]\!]\rho[^{d_y}/_y, ^{d_x}/_x] \rfloor \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor
\end{aligned}$$

$$[\![\textbf{rec}\ x.\ t]\!]\rho \triangleq \mathit{fix}\ \lambda d.\ [\![t]\!]\rho[^d/_x]$$

$$[\![\lambda x.\ t]\!]\rho \triangleq \lfloor\ \lambda d.\ [\![t]\!]\rho[^d/_x]\ \rfloor$$

$$\Gamma_h = \lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor$$

$$\begin{aligned}
d_0 &= \Gamma_h^0(\bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot}) = \bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot} \\
d_1 &= \Gamma_h(d_0) = (\lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor)\bot = \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor \\
d_2 &= \Gamma_h(d_1) = (\lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor)\lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor = \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor = d_1
\end{aligned}$$

Example

$$h \stackrel{\text{def}}{=} \mathbf{rec}\, y : int \to int\,.\, \lambda x : int\,.\, 3$$

$$\begin{aligned}
[\![h]\!]\rho &= [\![\mathbf{rec}\ y.\ \lambda x.\ 3]\!]\rho \\
&= \text{fix}\ \lambda d_y.\ [\![\lambda x.\ 3]\!]\rho[^{d_y}/_y] \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ [\![3]\!]\rho[^{d_y}/_y, ^{d_x}/_x] \rfloor \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor
\end{aligned}$$

$$\Gamma_h = \lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor$$

$$d_0 = \Gamma_h^0(\bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot}) = \bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot}$$

$$d_1 = \Gamma_h(d_0) = (\lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor)\bot = \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor$$

Maximal element in $[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot$
we could already stop here

Example

$$h \stackrel{\text{def}}{=} \mathbf{rec}\, y : int \to int\,.\ \lambda x : int\,.\ 3$$

$$\begin{aligned}
[\![h]\!]\rho &= [\![\mathbf{rec}\ y.\ \lambda x.\ 3]\!]\rho \\
&= \text{fix}\ \lambda d_y.\ [\![\lambda x.\ 3]\!]\rho[^{d_y}/_y] \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ [\![3]\!]\rho[^{d_y}/_y,{}^{d_x}/_x] \rfloor \\
&= \text{fix}\ \lambda d_y.\ \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor
\end{aligned}$$

$$[\![h]\!]\rho = \lfloor \lambda d_x.\ \lfloor 3 \rfloor \rfloor = [\![f]\!]\rho$$

Example

$$x : \tau$$

$$\begin{aligned}[\![\mathbf{rec}\ x.\ x]\!]\rho &= \mathit{fix}\ \lambda d_x.\ [\![x]\!]\rho[^{d_x}/_x] \\ &= \mathit{fix}\ \lambda d_x.\ d_x\end{aligned}$$

$$d_0 = \bot_{D_\tau}$$

$$d_1 = (\lambda d_x.d_x)\ d_0 = d_0 = \bot_{D_\tau}$$

$$[\![\mathbf{rec}\ x.\ x]\!]\rho = \bot_{D_\tau}$$

$$x : int \to int \qquad [\![\mathbf{rec}\ x.\ x]\!]\rho = \bot_{[\mathbb{Z}_\bot \to \mathbb{Z}_\bot]_\bot}$$

$$x : int * int \qquad [\![\mathbf{rec}\ x.\ x]\!]\rho = \bot_{(\mathbb{Z}_\bot \times \mathbb{Z}_\bot)_\bot}$$

Example

$$y : \tau_1 \qquad z : \tau_2$$

$$\begin{aligned}
[\![\lambda y.\ \mathbf{rec}\ z.\ z]\!]\rho &= \lfloor \lambda d_y.\ [\![\mathbf{rec}\ z.\ z]\!]\rho[^{d_y}/_y] \rfloor \\
&= \lfloor \lambda d_y.\ \perp_{D_{\tau_2}} \rfloor \\
&= \lfloor \perp_{[D_{\tau_1} \to D_{\tau_2}]} \rfloor \\
&= \lfloor \perp_{V_{\tau_1 \to \tau_2}} \rfloor \\
&\neq \perp_{D_{\tau_1 \to \tau_2}} \quad = \perp_{(V_{\tau_1 \to \tau_2})_\perp}
\end{aligned}$$

$x : int \to int$ $\quad [\![\mathbf{rec}\ x.\ x]\!]\rho = \perp_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]_\perp}$ diverges

$y : int, z : int$ $\quad [\![\lambda y.\ \mathbf{rec}\ z.\ z]\!]\rho = \lfloor \perp_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]} \rfloor$ waits arg and diverges

Exercise

$$x : int * int \;,\; y : int \;,\; z : int$$

$$[\![\ \mathbf{rec}\ x.\ x\ ]\!]\rho \quad \stackrel{?}{=} \quad [\![\ (\ \mathbf{rec}\ y.\ y\ ,\ \mathbf{rec}\ z.\ z\ )\ ]\!]\rho$$

diverges

a pair
of diverging computations

$$\bot_{D_{int*int}} \qquad \lfloor(\bot_{D_{int}}, \bot_{D_{int}})\rfloor$$

Lazy vs Eager

Eager Application

returns $\perp$ when $[\![t]\!]\rho = \perp$

lazy $[\![\ t\ t_0\ ]\!]\rho \triangleq \mathbf{let}\ \varphi \Leftarrow [\![t]\!]\rho.\ \varphi([\![t_0]\!]\rho)$

eager $[\![\ t\ t_0\ ]\!]\rho \triangleq \mathbf{let}\ \varphi \Leftarrow [\![t]\!]\rho.\ \mathbf{let}\ d \Leftarrow [\![t_0]\!]\rho.\ \varphi(\lfloor d \rfloor)$

returns $\perp$ when $[\![t]\!]\rho = \perp$ or $[\![t_0]\!]\rho = \perp$

Well-given definitions: continuity theorems

Well-definedness

We must guarantee that all functions we have used are monotone and continuous,
so that Kleene's fix point theory is applicable

$\pi_1 \;\; \pi_2 \;\; (\cdot)^*$ already considered
$\mathbf{let}$

$apply \;\; \mathit{fix} \;\; \underline{\mathsf{op}}_\perp \;\; \mathsf{Cond}_\tau \;\; \lambda$ to be checked

TH. $(D, \sqsubseteq_D)$ $(E, \sqsubseteq_E)$ $(F, \sqsubseteq_F)$ CPO $f : D \times E \to F$

$f_d : E \to F$
$f_d \triangleq \lambda e.\ f(d, e)$

$f_e : D \to F$
$f_e \triangleq \lambda d.\ f(d, e)$

f is continuous iff $\forall d \in D.\ f_d$ are continuous
$\forall e \in E.\ f_e$ are continuous

proof. $\Rightarrow)$ assume f is continuous

take a generic $d \in D$

we want to prove f_d is continuous

take a chain $\{e_i\}_{i \in \mathbb{N}}$ in E

we prove $f_d \left(\bigsqcup_{i \in \mathbb{N}} e_i \right) = \bigsqcup_{i \in \mathbb{N}} f_d(e_i)$

$e \in E$
f_e
(omitted)

(see next slide)

(continue) $f_d\left(\bigsqcup_{i\in\mathbb{N}} e_i\right) = \bigsqcup_{i\in\mathbb{N}} f_d(e_i)$

$$f_d\left(\bigsqcup_{i\in\mathbb{N}} e_i\right) = f\left(d, \bigsqcup_{i\in\mathbb{N}} e_i\right)$$ by def of f_d

$$= f\left(\bigsqcup_{i\in\mathbb{N}} d, \bigsqcup_{i\in\mathbb{N}} e_i\right)$$ by lub of constant chain

$$= f\left(\bigsqcup_{i\in\mathbb{N}} (d, e_i)\right)$$ by lub of pairs

$$= \bigsqcup_{i\in\mathbb{N}} f(d, e_i)$$ by continuity of f

$$= \bigsqcup_{i\in\mathbb{N}} f_d(e_i)$$ by def of f_d

TH. $(D, \sqsubseteq_D)$ $(E, \sqsubseteq_E)$ $(F, \sqsubseteq_F)$ CPO $f : D \times E \to F$

$f_d : E \to F$

$f_d \triangleq \lambda e.\ f(d,e)$

$f_e : D \to F$

$f_e \triangleq \lambda d.\ f(d,e)$

f is continuous iff $\forall d \in D.\ f_d$ are continuous
$\forall e \in E.\ f_e$ are continuous

$\Leftarrow)$ assume f_d, f_e are continuous for all d, e

we want to prove f is continuous

take a chain $\{(d_k, e_k)\}_{k \in \mathbb{N}}$ in $D \times E$

we prove $f\left(\bigsqcup_{k\in\mathbb{N}} (d_k, e_k)\right) = \bigsqcup_{k\in\mathbb{N}} f(d_k, e_k)$

(see next slide)

(continue) $f\left(\bigsqcup_{k\in\mathbb{N}}(d_k, e_k)\right) = \bigsqcup_{k\in\mathbb{N}} f(d_k, e_k)$

$f(\sqcup_k(d_k, e_k))$	$= f(\sqcup_i d_i, \sqcup_j e_j)$	by def of lub of pairs
	$= f_d(\sqcup_j e_j)$	by def of f_d with $d \triangleq \sqcup_i d_i$
	$= \sqcup_j f_d(e_j)$	by continuity of f_d
	$= \sqcup_j f(d, e_j)$	by def of f_d
	$= \sqcup_j f_{e_j}(d)$	by def of f_{e_j}
	$= \sqcup_j f_{e_j}(\sqcup_i d_i)$	by def of $d \triangleq \sqcup_i d_i$
	$= \sqcup_j \sqcup_i f_{e_j}(d_i)$	by continuity of f_{e_j}
	$= \sqcup_j \sqcup_i f(d_i, e_j)$	by def of f_{e_j}
	$= \sqcup_k f(d_k, e_k)$	by switch lemma (applicable?)

(continue) $f\left(\bigsqcup_{k\in\mathbb{N}}(d_k, e_k)\right) = \bigsqcup_{k\in\mathbb{N}} f(d_k, e_k)$

if $i \leq n \wedge j \leq m$ then $f(d_i, e_j) \sqsubseteq f(d_n, e_m)$? ✓

$$\Downarrow$$

$$d_i \sqsubseteq_D d_n \wedge e_j \sqsubseteq_E e_m$$

$$f(d_i, e_j) = f_{d_i}(e_j) \sqsubseteq f_{d_i}(e_m) = f(d_i, e_m) = f_{e_m}(d_i) \sqsubseteq f_{e_m}(d_n) = f(d_n, e_m)$$

f_{d_i} monotone

f_{e_m} monotone

$= \sqcup_j \sqcup_i f(d_i, e_j)$ ✓

$= \sqcup_k f(d_k, e_k)$ by switch lemma (applicable?)

Apply

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO

$apply : [D \to E] \times D \to E$
$apply(f, d) \triangleq f(d)$

TH. $apply$ is monotone

(try to prove on your own)

TH. $apply$ is continuous

proof. from a previous theorem, we prove continuity on each parameter separately $apply_f$ $apply_d$

1. for any $f \in [D \to E]$ $apply_f \triangleq \lambda d.\ f(d)$ is continuous
2. for any $d \in D$ $apply_d \triangleq \lambda f.\ f(d)$ is continuous

(see next slides)

1. for any $f \in [D \to E]$ $\quad apply_f \triangleq \lambda d.\ f(d)$ is continuous

take $f \in [D \to E]$ and a chain $\{d_i\}_{i\in\mathbb{N}}$ in D

we want to prove $\quad apply_f \left(\bigsqcup_i d_i \right) = \bigsqcup_i apply_f(d_i)$

$apply_f(\sqcup_i d_i) = apply(f, \sqcup_i d_i)$ by def of $apply_f$

$= f(\sqcup_i d_i)$ by def of $apply$

$= \sqcup_i f(d_i)$ by continuity of f

$= \sqcup_i apply(f, d_i)$ by def of $apply$

$= \sqcup_i apply_f(d_i)$ by def of $apply_f$

2. for any $d \in D$ $apply_d \triangleq \lambda f.\ f(d)$ is continuous

take $d \in D$ and a chain $\{f_i\}_{i \in \mathbb{N}}$ in $[D \to E]$

we want to prove $apply_d\left(\bigsqcup_i f_i\right) = \bigsqcup_i apply_d(f_i)$

$apply_d(\sqcup_i f_i) = apply(\sqcup_i f_i, d)$ by def of $apply_d$

$= (\sqcup_i f_i)(d)$ by def of $apply$

$= \sqcup_i f_i(d)$ by def of lub of functions

$= \sqcup_i apply(f_i, d)$ by def of $apply$

$= \sqcup_i apply_d(f_i)$ by def of $apply_d$

Apply: recap

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO

$$apply : [D \to E] \times D \to E$$
$$apply(f, d) \triangleq f(d)$$

$$apply \in [[D \to E] \times D \to E]$$

Fix

$(D, \sqsubseteq_D)$ CPO$_\bot$

$\mathit{fix} : [D \to D] \to D$

$\mathit{fix} \triangleq \lambda f.\ \bigsqcup_{n\in\mathbb{N}} f^n(\bot_D)$

TH. fix is monotone

(try to prove on your own)

TH. fix is continuous

proof. $\mathit{fix} \triangleq \lambda f.\ \bigsqcup_{n\in\mathbb{N}} f^n(\bot_D) = \bigsqcup_{n\in\mathbb{N}} \lambda f.\ f^n(\bot_D)$

by def of lub in functional domains

we prove that $\forall n.\ \lambda f.\ f^n(\bot_D)$ is continuous

(by mathematical induction on n)

then fix is continuous because lub of continuous functions

(see next slides)

(continue) $\forall n.\ \lambda f.\ f^n(\bot_D)$

base case: $\lambda f.\ f^0(\bot_D) = \lambda f.\ \bot_D$

is a constant function (continuous)

inductive case: assume $g \triangleq \lambda f.\ f^n(\bot_D)$ is continuous

we want to prove $h \triangleq \lambda f.\ f^{n+1}(\bot_D)$ is continuous

take a chain $\{f_i\}_{i\in\mathbb{N}}$ in $[D \to D]$

we want to prove $h\left(\bigsqcup_{i\in\mathbb{N}} f_i\right) = \bigsqcup_{i\in\mathbb{N}} h(f_i)$

(see next slide)

(continue) $\forall n.\ \lambda f.\ f^n(\bot_D)$ $\quad g \triangleq \lambda f.\ f^n(\bot_D)$, $h \triangleq \lambda f.\ f^{n+1}(\bot_D)$ $\quad h\left(\bigsqcup_{i\in\mathbb{N}} f_i\right) = \bigsqcup_{i\in\mathbb{N}} h(f_i)$

$h(\sqcup_i f_i) = (\sqcup_i f_i)^{n+1}(\bot_D)$	by def of h
$= (\sqcup_j f_j)((\sqcup_i f_i)^n(\bot_D))$	by def of $(\cdot)^{n+1}$
$= (\sqcup_j f_j)(g(\sqcup_i f_i))$	by def of g
$= (\sqcup_j f_j)(\sqcup_i g(f_i))$	by ind. hyp (g continuous)
$= (\sqcup_j f_j)(\sqcup_i f_i^n(\bot_D))$	by def of g
$= \sqcup_j f_j(\sqcup_i f_i^n(\bot_D))$	by def of lub in functional CPO
$= \sqcup_j \sqcup_i\ f_j(f_i^n(\bot_D))$	by continuity of f_j
$= \sqcup_k f_k(f_k^n(\bot_D))$	by switch lemma
$= \sqcup_k f_k^{n+1}(\bot_D)$	by def of $(\cdot)^{n+1}$
$= \sqcup_k h(f_k)$	by def of h

Fix: recap

$$(D, \sqsubseteq_D) \ \mathrm{CPO}_\perp$$

$$\mathit{fix} : [D \to D] \to D$$

$$\mathit{fix} \triangleq \lambda f.\ \bigsqcup_{n \in \mathbb{N}} f^n(\perp_D)$$

$$\mathit{fix} \in [[D \to D] \to D]$$

TH. $\underline{\text{op}}_\perp$ is monotone and continuous

$$\underline{\text{op}}_\perp : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$v_1 \ \underline{\text{op}}_\perp \ v_2 \triangleq \begin{cases} \lfloor n_1 \ \underline{\text{op}} \ n_2 \rfloor & \text{if } v_1 = \lfloor n_1 \rfloor \text{ and } v_2 = \lfloor n_2 \rfloor \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise } (v_1 = \perp_{\mathbb{Z}_\perp} \text{ or } v_2 = \perp_{\mathbb{Z}_\perp}) \end{cases}$$

We omit monotonicity check

Since the domain has only finite chains, it is also continuous

TH. Cond_τ is monotone and continuous

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

We omit monotonicity check

We prove continuity on each parameter separately

The first parameter is in $\mathbb{Z}_\perp$

only finite chains are possible, hence continuity is guaranteed

We prove continuity over the second parameter (next slides)

For the third parameter the proof is analogous and omitted

(continue)

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

Continuity over the second parameter

take $v \in \mathbb{Z}_\perp, d \in D_\tau, \{d_i\}_{i \in \mathbb{N}} \subseteq D_\tau$

we want to prove $\mathsf{Cond}_\tau\left(v, \bigsqcup_{i \in \mathbb{N}} d_i, d\right) = \bigsqcup_{i \in \mathbb{N}} \mathsf{Cond}_\tau(v, d_i, d)$

we proceed by case analysis on v:
- $\perp_{\mathbb{Z}_\perp}$
- $\lfloor 0 \rfloor$
- $\lfloor n \rfloor$, $n \neq 0$

(continue)

$$\mathsf{Cond}_\tau : \mathbb{Z}_\bot \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \bot_{D_\tau} & \text{if } v = \bot_{\mathbb{Z}_\bot} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

$v = \bot_{\mathbb{Z}_\bot}$

$$\mathsf{Cond}_\tau\left(\bot_{\mathbb{Z}_\bot}, \bigsqcup_{i \in \mathbb{N}} d_i, d\right) = \bot_{D_\tau} = \bigsqcup_{i \in \mathbb{N}} \bot_{D_\tau} = \bigsqcup_{i \in \mathbb{N}} \mathsf{Cond}_\tau(\bot_{\mathbb{Z}_\bot}, d_i, d)$$

(continue)

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

$v = \lfloor 0 \rfloor$

$$\mathsf{Cond}_\tau\left(\lfloor 0 \rfloor, \bigsqcup_{i\in\mathbb{N}} d_i, d\right) = \bigsqcup_{i\in\mathbb{N}} d_i = \bigsqcup_{i\in\mathbb{N}} \mathsf{Cond}_\tau(\lfloor 0 \rfloor, d_i, d)$$

(continue)

$$\mathsf{Cond}_\tau : \mathbb{Z}_\perp \times D_\tau \times D_\tau \to D_\tau$$

$$\mathsf{Cond}_\tau(v, d_1, d_2) \triangleq \begin{cases} \perp_{D_\tau} & \text{if } v = \perp_{\mathbb{Z}_\perp} \\ d_1 & \text{if } v = \lfloor 0 \rfloor \\ d_2 & \text{otherwise } (v = \lfloor n \rfloor \text{ with } n \neq 0) \end{cases}$$

$v = \lfloor n \rfloor \ , \ n \neq 0$

$$\mathsf{Cond}_\tau \left(\lfloor n \rfloor, \bigsqcup_{i \in \mathbb{N}} d_i, d \right) = d = \bigsqcup_{i \in \mathbb{N}} d = \bigsqcup_{i \in \mathbb{N}} \mathsf{Cond}_\tau(\lfloor n \rfloor, d_i, d)$$

TH. lambda abstraction is monotone and continuous

$t : \tau$ $\lambda d.\ [\![t]\!]\rho[{}^{d}/_{x}]$ is continuous

we focus on the stronger property

$\lambda \widetilde{d}.\ [\![t]\!]\rho[{}^{\widetilde{d}}/_{\widetilde{x}}]$ is continuous

the proof is by structural induction on t

(try on your own)

Corollary $t : \tau_0 \to \tau$ $\mathit{fix}\ \lambda d.\ [\![t]\!]\rho[{}^{d}/_{x}]$ is continuous

(the limit of continuous functions is continuous)

More continuity theorems

TH. $(D, \sqsubseteq_D)$ $(E_i, \sqsubseteq_{E_i})$ CPO $\quad f : D \to E_1 \times E_2 \qquad g_i \triangleq \pi_i \circ f$

f is continuous iff g_1, g_2 are continuous

proof. $\Rightarrow)$ f is continuous, π_i is continuous $\Rightarrow$ g_i is continuous

$\Leftarrow)$ note that $\forall d \in D.\ f(d) = (g_1(d), g_2(d))$

assume g_1, g_2 are continuous

we want to prove f is continuous

take a chain $\{d_i\}_{i \in \mathbb{N}}$ in D

we must prove $f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)$

(see next slide)

(continue) $f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \bigsqcup_{i\in\mathbb{N}} f(d_i)$

$$f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \left(g_1\left(\bigsqcup_{i\in\mathbb{N}} d_i\right), g_2\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)\right)$$

by def g_1, g_2

$$= \left(\bigsqcup_{i\in\mathbb{N}} g_1(d_i), \bigsqcup_{i\in\mathbb{N}} g_2(d_i)\right)$$

g_1, g_2 are continuous

$$= \bigsqcup_{i\in\mathbb{N}} (g_1(d_i), g_2(d_i))$$

by def of lub of pairs

$$= \bigsqcup_{i\in\mathbb{N}} f(d_i)$$

by def g_1, g_2

Curry

$(D, \sqsubseteq_D)$

$(E, \sqsubseteq_E)$ CPO

$(F, \sqsubseteq_F)$

$$curry : (D \times E \to F) \to D \to E \to F$$

$$curry\ f\ d\ e \triangleq f(d, e)$$

TH. f continuous $\Rightarrow$ $curry(f)$ continuous

(try to prove on your own)

Curry

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO
$(F, \sqsubseteq_F)$

$curry : (D \times E \to F) \to D \to E \to F$

$curry\ f\ d\ e \triangleq f(d, e)$

TH. f continuous $\Rightarrow$ $curry(f)$ continuous

$h \triangleq curry(f) : D \to E \to F$

given $\{d_i\}_{i \in \mathbb{N}}$ $\quad h\ (\sqcup_i d_i) =^? \sqcup_i (h\ d_i) : E \to F$

take $e \in E$ $\quad h\ (\sqcup_i d_i)\ e =^? (\sqcup_i (h\ d_i))\ e$

$$h\ (\sqcup_i d_i)\ e = curry\ f\ (\sqcup_i d_i)\ e = f((\sqcup_i d_i), e) = f((\sqcup_i d_i), (\sqcup_i e))$$

$$= f(\sqcup_i (d_i, e)) = \sqcup_i f(d_i, e) = \sqcup_i (curry\ f\ d_i\ e)$$

$$= \sqcup_i (h\ d_i\ e) = (\sqcup_i (h\ d_i))\ e$$

Uncurry

$(D, \sqsubseteq_D)$

$(E, \sqsubseteq_E)$ CPO

$(F, \sqsubseteq_F)$

$$uncurry : (D \to E \to F) \to (D \times E) \to F$$

$$uncurry\ f\ (d, e) \triangleq f\ d\ e$$

TH. f continuous $\Rightarrow$ $uncurry(f)$ continuous

(try to prove on your own)

TH. $uncurry$ is the inverse of $curry$

given $g : (D \times E) \to F$ $\qquad uncurry(curry(g)) \stackrel{?}{=} g$

take $(d, e) \in (D \times E)$ $\quad (uncurry(curry(g)))(d, e) \stackrel{?}{=} g(d, e)$

$$(uncurry(curry(g)))(d, e) = (curry(g))\ d\ e = g(d, e)$$

Uncurry

$(D, \sqsubseteq_D)$
$(E, \sqsubseteq_E)$ CPO
$(F, \sqsubseteq_F)$

$$uncurry : (D \to E \to F) \to (D \times E) \to F$$
$$uncurry\ f\ (d, e) \triangleq f\ d\ e$$

TH. f continuous $\Rightarrow$ $uncurry(f)$ continuous

(try to prove on your own)

TH. $uncurry$ is the inverse of $curry$

given $f : D \to E \to F$ $\qquad curry(uncurry(f)) =^? f$

take $d \in D, e \in E$ $\qquad (curry(uncurry(f)))\ d\ e =^? f\ d\ e$

$$(curry(uncurry(f)))\ d\ e = (uncurry(f))(d, e) = f\ d\ e$$

Main properties

Substitution lemma

$x, t_0 : \tau_0$
$t : \tau$

$$[\![t[^{t_0}/_x]]\!]\rho = [\![t]\!]\rho[^{[\![t_0]\!]\rho}/_x]$$

environment update

syntactic substitution

the proof is by structural induction on t

(try on your own)

Compositionality

The substitution lemma $[\![t[^{t_0}/_x]]\!]\rho = [\![t]\!]\rho[^{[\![t_0]\!]\rho}/_x]$ is important: as it guarantees the compositionality of the denotational semantics

TH. $[\![t_1]\!]\rho = [\![t_2]\!]\rho \quad \Rightarrow \quad [\![t[^{t_1}/_x]]\!]\rho = [\![t[^{t_2}/_x]]\!]\rho$

proof. assume $[\![t_1]\!]\rho = [\![t_2]\!]\rho$

$$[\![t[^{t_1}/_x]]\!]\rho = [\![t]\!]\rho[^{[\![t_1]\!]\rho}/_x] = [\![t]\!]\rho[^{[\![t_2]\!]\rho}/_x] = [\![t[^{t_2}/_x]]\!]\rho$$

subs lemma — $[\![t_1]\!]\rho = [\![t_2]\!]\rho$ — subs lemma

Only free variables matter

TH. $t : \tau$

$\forall x \in \mathsf{fv}(t).\ \rho(x) = \rho'(x) \quad \Rightarrow \quad [\![t]\!]\rho = [\![t]\!]\rho'$

the proof is by structural induction on t

(try on your own)

Corollary t closed $\quad \Rightarrow \quad \forall \rho, \rho'.\ [\![t]\!]\rho = [\![t]\!]\rho'$

TH. Canonical terms are not bottom

$$c \in C_\tau \;\Rightarrow\; \forall \rho.\ [\![c]\!]\rho \neq \bot_{D_\tau}$$

proof. by rule induction on the rules for canonical terms

$$P(c \in C_\tau) \triangleq \forall \rho.\ [\![c]\!]\rho \neq \bot_{D_\tau}$$

$$\frac{}{n \in C_{int}} \qquad [\![n]\!]\rho = \lfloor n \rfloor \neq \bot_{D_{int}}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1 \quad t_0, t_1 \text{ closed}}{(t_0, t_1) \in C_{\tau_0 * \tau_1}} \qquad [\![(t_0, t_1)]\!]\rho = \lfloor ([\![t_0]\!]\rho, [\![t_1]\!]\rho) \rfloor \neq \bot_{D_{\tau_0 * \tau_1}}$$

$$\frac{\lambda x.\, t : \tau_0 \to \tau_1 \quad \lambda x.\, t \text{ closed}}{\lambda x.\, t \in C_{\tau_0 \to \tau_1}} \qquad [\![\lambda x.\ t]\!]\rho = \lfloor \lambda d.\ [\![t]\!]\rho[^d/_x] \rfloor \neq \bot_{D_{\tau_0 \to \tau_1}}$$

Additional exercises

Exercise a, part 1

Prove the implication

$$[\![t_1]\!]\rho = [\![t_2]\!]\rho \Rightarrow [\![t_1\ x]\!]\rho = [\![t_2\ x]\!]\rho$$

Assume $x : \tau$ and thus $t_1, t_2 : \tau \to \sigma$

$$[\![t_1\ x]\!]\rho = \text{let } \varphi \Leftarrow [\![t_1]\!]\rho\ .\ \varphi([\![x]\!]\rho)$$

$$= \text{let } \varphi \Leftarrow [\![t_2]\!]\rho\ .\ \varphi([\![x]\!]\rho)$$

$$= [\![t_2\ x]\!]\rho$$

Exercise a, part 2

Give a counterexample to

$$[\![t_1]\!]\rho = [\![t_2]\!]\rho \Leftarrow [\![t_1\ x]\!]\rho = [\![t_2\ x]\!]\rho$$

Take $x : int$

$t_1 = (\lambda y : int\,.\,\text{rec } z : int\,.\,z) : int \to int$

$t_2 = (\text{rec } f : int \to int\,.\,f) : int \to int$

Then $[\![t_i\ x]\!]\rho = \text{let } \varphi \Leftarrow [\![t_i]\!]\rho\,.\,\varphi([\![x]\!]\rho) = \bot$

But $[\![t_1]\!]\rho = \lfloor \lambda d\,.\,\bot \rfloor \neq \bot = [\![t_2]\!]\rho$

Exercise b

Compute the denotational semantics of the HOFL term
$t \triangleq \text{rec } f\,.\ \lambda x\,.\ \text{if } x \text{ then } 1 \text{ else } (f\ (x - x))$

$$[\![t]\!]\rho = fix\ \lambda\varphi\,.\ [\![\lambda x\,.\ \text{if } x \text{ then } 1 \text{ else } (f\ (x - x))]\!]\rho[^{\varphi}/_{f}]$$

$$= fix\ \lambda\varphi\,.\ \lfloor \lambda d\,.\ [\![\text{if } x \text{ then } 1 \text{ else } (f\ (x - x))]\!]\rho[^{\varphi}/_{f}, ^{d}/_{x}]\rfloor$$

$$= fix\ \lambda\varphi\,.\ \lfloor \lambda d\,.\ \text{Cond}(d, \lfloor 1 \rfloor, [\![(f\ (x - x))]\!]\rho[^{\varphi}/_{f}, ^{d}/_{x}])\rfloor$$

$$= fix\ \lambda\varphi\,.\ \lfloor \lambda d\,.\ \text{Cond}(d, \lfloor 1 \rfloor, \text{let } \psi \Leftarrow \varphi\,.\ \psi(d -_{\perp} d))\rfloor$$

Exercise b (ctd.)

Compute the denotational semantics of the HOFL term

$t \triangleq \mathsf{rec}\ f.\ \lambda x.\ \mathsf{if}\ x\ \mathsf{then}\ 1\ \mathsf{else}\ (f\ (x - x))$

$$[\![t]\!]\rho = fix\ \lambda\varphi.\ \lfloor \lambda d.\ \mathrm{Cond}(d, \lfloor 1 \rfloor, \mathrm{let}\ \psi \Leftarrow \varphi.\ \psi(d -_{\perp} d)) \rfloor$$

$$\varphi^0 = \perp$$

$$\varphi^1 = \lfloor \lambda d.\ \mathrm{Cond}(d, \lfloor 1 \rfloor, \mathrm{let}\ \psi \Leftarrow \varphi^0.\ \psi(d -_{\perp} d)) \rfloor$$

$$= \lfloor \lambda d.\ \mathrm{Cond}(d, \lfloor 1 \rfloor, \perp) \rfloor$$

$$\varphi^2 = \lfloor \lambda d.\ \mathrm{Cond}(d, \lfloor 1 \rfloor, \mathrm{let}\ \psi \Leftarrow \varphi^1.\ \varphi(d -_{\perp} d)) \rfloor$$

$$= \lfloor \lambda d.\ \mathrm{Cond}(d, \lfloor 1 \rfloor, \mathrm{Cond}(d -_{\perp} d, \lfloor 1 \rfloor, \perp)) \rfloor$$

$$= \lfloor \lambda d.\ \mathrm{Cond}(d, \lfloor 1 \rfloor, \mathrm{Cond}(\lfloor 0 \rfloor, \lfloor 1 \rfloor, \perp)) \rfloor$$

$$= \lfloor \lambda d.\ \mathrm{Cond}(d, \lfloor 1 \rfloor, \lfloor 1 \rfloor) \rfloor$$

$$\varphi^3 = \lfloor \lambda d.\ \mathrm{Cond}(d, \lfloor 1 \rfloor, \mathrm{let}\ \psi \Leftarrow \varphi^2.\ \varphi(d -_{\perp} d)) \rfloor = \varphi^2$$