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MPP 2025/26 (0077A, 9CFU)

Models for Programming Paradigms

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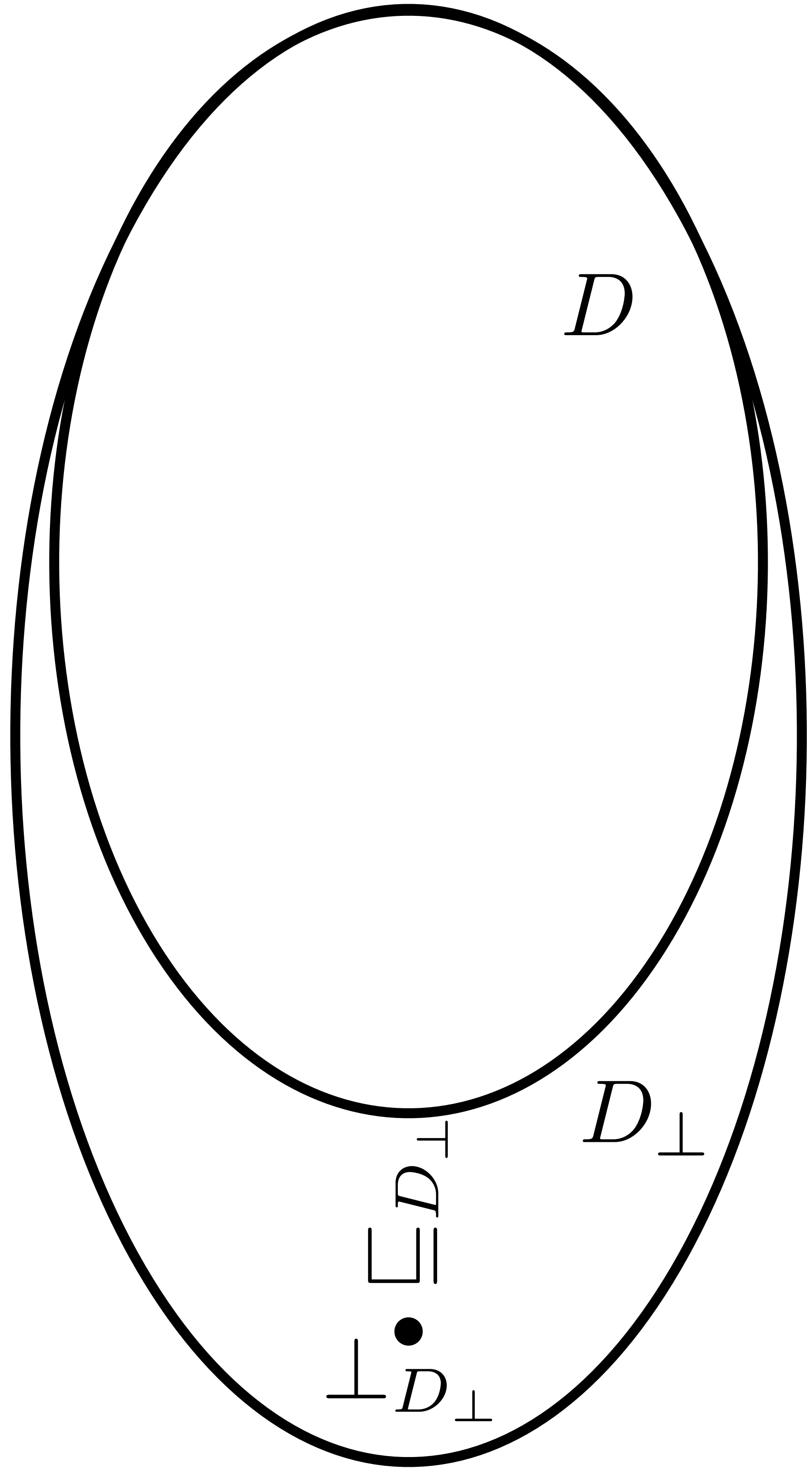
13b - Continuity theorems

Lifted Domains

Lifted Domains



Lifted Domains

$\mathcal{D} = (D, \sqsubseteq_D)$ CPO $\quad \Rightarrow \quad \mathcal{D}_\perp = (D_\perp , \sqsubseteq_{D_\perp})$

$$D_\perp \triangleq \{\perp\} \uplus D$$
$$= \{(0,\perp)\} \cup (\{1\} \times D) = \{(0,\perp)\} \cup \{(1,d) \mid d \in D\}$$

$$\perp_{D_\perp} \triangleq (0,\perp)$$

$$\lfloor\cdot\rfloor : D \to D_\perp$$

$$\lfloor d \rfloor \triangleq (1, d)$$

lifting function

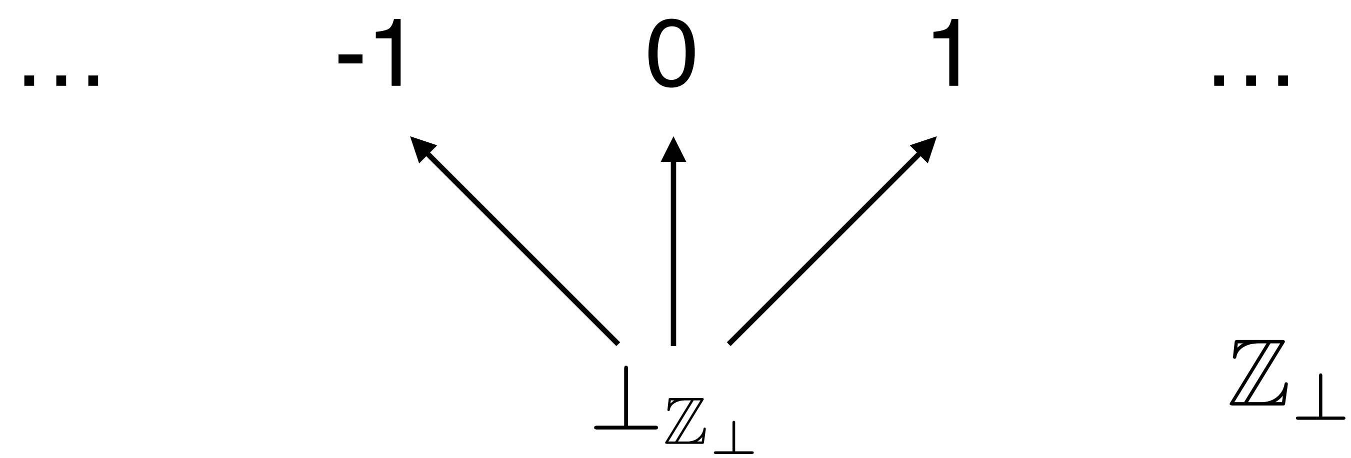
how to order lifted elements?

$$\forall x \in D_\perp.\ \perp_{D_\perp} \sqsubseteq_{D_\perp} x$$

$$\forall d_1, d_2 \in D.\ \lfloor d_1 \rfloor \sqsubseteq_{D_\perp} \lfloor d_2 \rfloor \Leftrightarrow d_1 \sqsubseteq_D d_2$$

Example

$(\mathbb{Z}, =)$



Lifted Domains

TH. $\mathcal{D}_\perp = (\ D_\perp\ ,\ \sqsubseteq_{D_\perp}\ )\ \ \mathrm{CPO}_\perp$

try on your own to prove:
PO,
bottom element,

complete observe that: $\bigsqcup_{i\in\mathbb{N}} \lfloor d_i \rfloor = \left\lfloor \bigsqcup_{i\in\mathbb{N}} d_i \right\rfloor$

it is an upper bound
it is the least upper bound

Lifting operator

$(D, \sqsubseteq_D)$ CPO

$(E, \sqsubseteq_E)$ $\mathrm{CPO}_\perp$

$$(\cdot)^* : [D \to E] \to [D_\perp \to E]$$

$$\forall f \in [D \to E].\ f^*(x) \triangleq \begin{cases} \perp_E & \text{if } x = \perp_{D_\perp} \\ f(d) & \text{if } x = \lfloor d \rfloor \end{cases}$$

for the definition to be well-given
we need to prove:

$$f \in [D \to E] \quad \Rightarrow \quad f^* \in [D_\perp \to E]$$

f continuous implies f^* continuous

TH. the lifting operator is well-defined

proof. assume f continuous, take a chain $\{x_n\}_{n\in\mathbb{N}}$ in $D_\bot$

we need to prove $f^*\left(\bigsqcup_{n\in\mathbb{N}} x_n\right) = \bigsqcup_{n\in\mathbb{N}} f^*(x_n)$

if $\forall n \in \mathbb{N}.\ x_n = \bot_{D_\bot}$ then it is obvious

otherwise, let $k = \min\{i \mid x_i \neq \bot_{D_\bot}\}$

then $\forall m \geq k.\ \exists d_m \in D.\ x_m = \lfloor d_m \rfloor$

and by prefix independence of lub

$$\bigsqcup_{n\in\mathbb{N}} x_n = \bigsqcup_{n\in\mathbb{N}} x_{n+k} \qquad \bigsqcup_{n\in\mathbb{N}} f^*(x_n) = \bigsqcup_{n\in\mathbb{N}} f^*(x_{n+k})$$

we can just prove $f^*\left(\bigsqcup_{n\in\mathbb{N}} x_{n+k}\right) = \bigsqcup_{n\in\mathbb{N}} f^*(x_{n+k})$

(see next slide)

(continue) $f^*\left(\bigsqcup_{n\in\mathbb{N}} x_{n+k}\right) = \bigsqcup_{n\in\mathbb{N}} f^*(x_{n+k})$

$$f^*\left(\bigsqcup_{n\in\mathbb{N}} x_{n+k}\right) = f^*\left(\bigsqcup_{n\in\mathbb{N}} \lfloor d_{n+k} \rfloor\right) \quad \text{by def of } k$$

$$= f^*\left(\left\lfloor \bigsqcup_{n\in\mathbb{N}} d_{n+k} \right\rfloor\right) \quad \text{by lub in a lifted domain}$$

$$= f\left(\bigsqcup_{n\in\mathbb{N}} d_{n+k}\right) \quad \text{by def of lifting}$$

$$= \bigsqcup_{n\in\mathbb{N}} f(d_{n+k}) \quad \text{by continuity of } f$$

$$= \bigsqcup_{n\in\mathbb{N}} f^*(\lfloor d_{n+k} \rfloor) \quad \text{by def of lifting}$$

$$= \bigsqcup_{n\in\mathbb{N}} f^*(x_{n+k}) \quad \text{by def of } k$$

TH. $(\cdot)^*$ is monotone

(try to prove on your own)

TH. $(\cdot)^*$ is continuous

proof. take a chain of continuous functions $\{f_i : D \to E\}_{i\in\mathbb{N}}$

we need to prove $$\left(\bigsqcup_{i\in\mathbb{N}} f_i\right)^* = \bigsqcup_{i\in\mathbb{N}} f_i^*$$

take a generic $x \in D_\perp$

we need to prove $$\left(\bigsqcup_{i\in\mathbb{N}} f_i\right)^*(x) = \left(\bigsqcup_{i\in\mathbb{N}} f_i^*\right)(x)$$

if $x = \perp_{D_\perp}$ it is obvious

if $x = \lfloor d \rfloor$ we have...

(see next slide)

(continue) $\left(\bigsqcup_{i\in\mathbb{N}} f_i\right)^* (\lfloor d \rfloor) = \left(\bigsqcup_{i\in\mathbb{N}} f_i^*\right) (\lfloor d \rfloor)$

$$\left(\bigsqcup_{i\in\mathbb{N}} f_i\right)^* (\lfloor d \rfloor) = \left(\bigsqcup_{i\in\mathbb{N}} f_i\right)(d)$$ by def of lifting

$$= \bigsqcup_{i\in\mathbb{N}} f_i(d)$$ by lub in a functional domain

$$= \bigsqcup_{i\in\mathbb{N}} f_i^*(\lfloor d \rfloor)$$ by def of lifting

$$= \left(\bigsqcup_{i\in\mathbb{N}} f_i^*\right)(\lfloor d \rfloor)$$ by lub in a functional domain

Let notation (de-lifting)

$(E, \sqsubseteq_E)$ $\text{CPO}_\bot$ $\qquad \lambda x.\, e \in [D \to E] \qquad t \in D_\bot$

$$\mathbf{let}\ x \Leftarrow t.\ e \quad \triangleq \quad \underbrace{\underbrace{\underbrace{(\lambda x.\, e)}_{[D \to E]}{}^*}_{[D_\bot \to E]} \underbrace{t}_{D_\bot}}_{E} \quad = \quad \begin{cases} \bot_E & \text{if } t = \bot_{D_\bot} \\ e[^d/_x] & \text{if } t = \lfloor d \rfloor \end{cases}$$

intuitively:

if t is a lifted value $\lfloor d \rfloor$ then we de-lift the value and assign it to x in e

otherwise returns $\bot_E$

Disjoint Union

$$\mathcal{D} = (D, \sqsubseteq_D) \qquad \mathcal{E} = (E, \sqsubseteq_E) \qquad \text{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} + \mathcal{E} = (\ D \uplus E\ ,\ \sqsubseteq_{D \uplus E}\ )$$

$$D \uplus E \triangleq \{(0, d) \mid d \in D\} \cup \{(1, e) \mid e \in E\}$$

how to order elements?

is there a bottom element?

is it a complete order?

how to define (continuous) injections?

$$\iota_D : D \to D \uplus E$$
$$\iota_E : E \to D \uplus E$$