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MPP 2025/26 (0077A, 9CFU)

Models for Programming Paradigms

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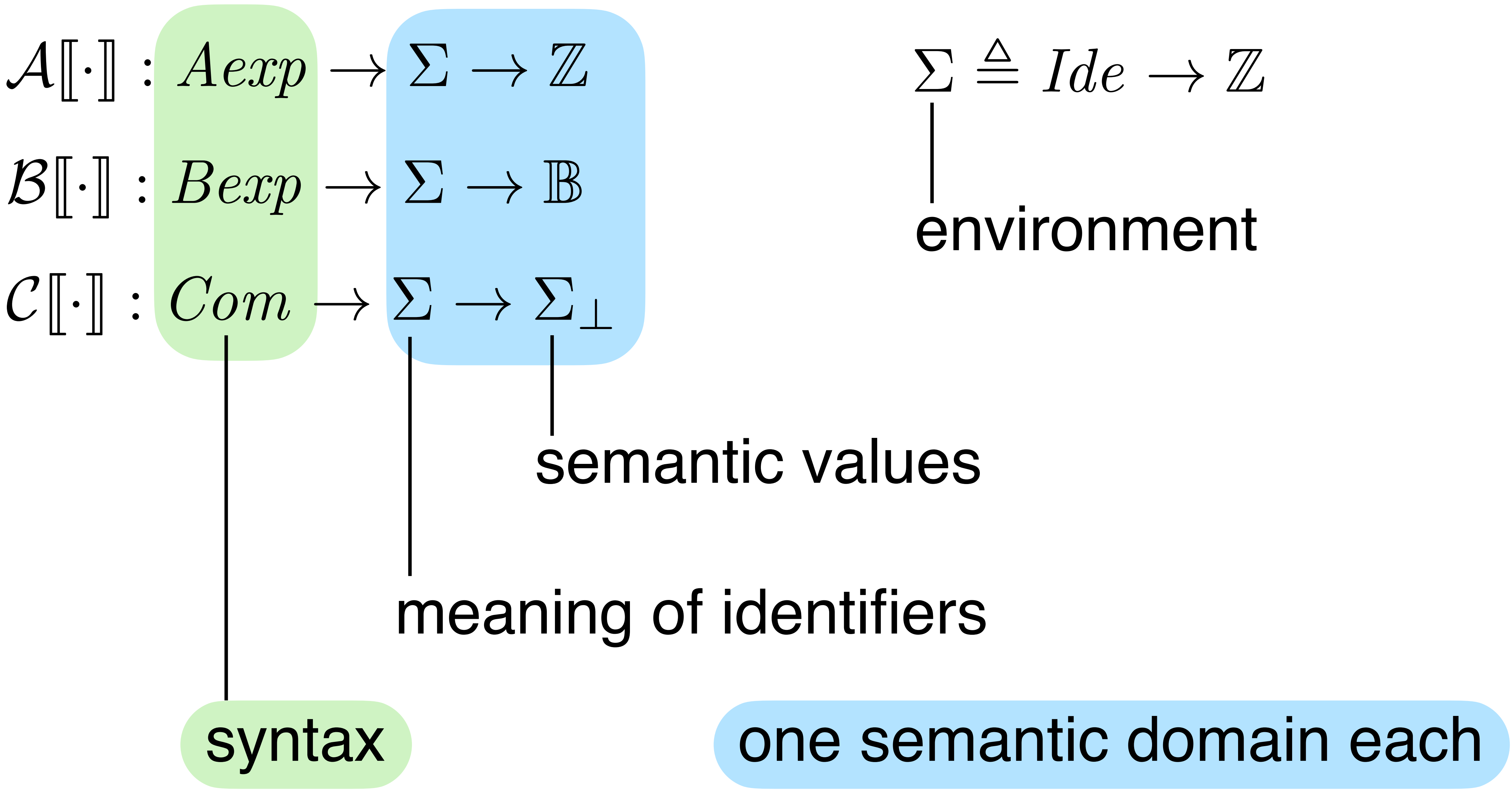
13a - Cartesian and functional domains

HOFL
Towards a denotational semantics

Imp

three syntactic categories (types) $Aexp, Bexp, Com$

one interpretation function each



$$\mathcal{A}[\![\cdot]\!] : Aexp \to \Sigma \to \mathbb{Z}$$

$$\mathcal{B}[\![\cdot]\!] : Bexp \to \Sigma \to \mathbb{B}$$

$$\mathcal{C}[\![\cdot]\!] : Com \to \Sigma \to \Sigma_\perp$$

$$\Sigma \triangleq Ide \to \mathbb{Z}$$

environment

semantic values

meaning of identifiers

syntax

one semantic domain each

HOFL

one syntactic category for pre-terms T

infinitely many types $\tau \ ::= \ int \ \mid \ \tau_0 * \tau_1 \ \mid \ \tau_0 \to \tau_1$

infinitely many categories for typeable terms T_τ

one semantic domain each D_τ

one parametric interpretation function $[\![\cdot]\!]$

variables also have different types $x : \tau$

the environment must be type-sensitive ρ

Requirements

a domain for each type!

$t : \tau$ $\quad$ $[\![t]\!]\rho \in D_\tau$

environment $\quad \rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $\quad x : \tau \Rightarrow \rho(x) \in D_\tau$

t may diverge (e.g. $\mathbf{rec}\ x.\ x$) $\Rightarrow D_\tau$ must include a bottom element $\perp_{D_\tau}$

Requirements

a domain for each type!

$t : \tau \qquad [\![t]\!]\rho \in D_\tau$

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type consistent assignment of values to variables $x : \tau \Rightarrow \rho(x) \in D_\tau$

$$[\![\mathbf{rec}\ x.\ t]\!]\rho = [\![t]\!]\rho[^{[\![\mathbf{rec}\ x.\ t]\!]\rho}/_x] \qquad \Gamma_{x,t} \triangleq \lambda d.\ [\![t]\!]\rho[^d/_x]$$

$$[\![\mathbf{rec}\ x.\ t]\!]\rho = \Gamma_{x,t}\ ([\![\mathbf{rec}\ x.\ t]\!]\rho)$$

to solve recursive equations:

$$[\![\mathbf{rec}\ x.\ t]\!] = \mathit{fix}\ \Gamma_{x,t}$$

D_τ must be a $\mathrm{CPO}_\perp$

$\Gamma_{x,t}$ must be continuous

Requirements

a domain for each type!

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type consistent assignment of values to variables $x : \tau \Rightarrow \rho(x) \in D_\tau$

$$\tau \;::=\; int \;\mid\; \tau_0 * \tau_1 \;\mid\; \tau_0 \to \tau_1$$

we must be able to combine $\mathrm{CPO}_\perp$

using cartesian product and function spaces

Requirements

a domain for each type!

$t : \tau$ $\quad$ $[\![t]\!]\rho \in D_\tau$

environment $\rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $x : \tau \Rightarrow \rho(x) \in D_\tau$

$$\tau \ ::= \ int \ | \ \tau_0 * \tau_1 \ | \ \tau_0 \to \tau_1$$

choose D_{int}

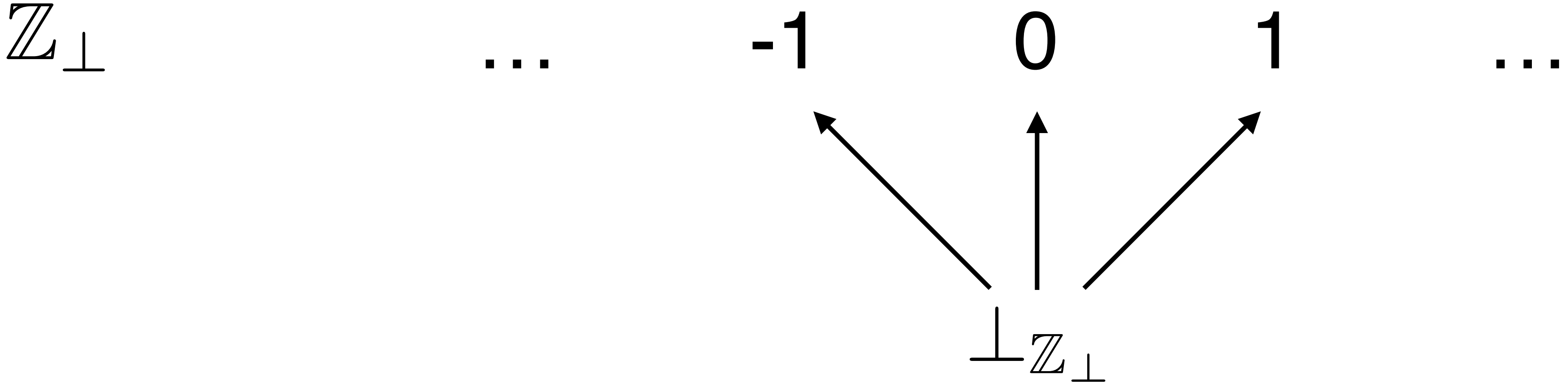
given D_{τ_0}, D_{τ_1} build $D_{\tau_0 * \tau_1}$ $\quad$ $D_{\tau_0 \to \tau_1}$

Flat domain of Integers

Flat domain of Integers

$\mathbb{Z}$ … -1 0 1 …

Flat domain of Integers



$\mathbb{Z}_\perp$

… -1 0 1 …

$\perp_{\mathbb{Z}_\perp}$

PO: flat order

bottom: any flat order has bottom

completeness: any flat order is complete
(only finite chains are possible)

Strict extensions

$$\text{op} \in \{+, -\times\}$$

$$\text{op} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$$

$$\underline{\text{op}}_\perp : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$1 \mathbin{\underline{+}_\perp} \perp = \perp$$

$$\perp \mathbin{\underline{\times}_\perp} 5 = \perp$$

$$\perp \mathbin{\underline{-}_\perp} \perp = \perp$$

$$v_1 \mathbin{\underline{\text{op}}_\perp} v_2 \triangleq \begin{cases} v_1 \mathbin{\underline{\text{op}}} v_2 & \text{if } v_1, v_2 \in \mathbb{Z} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise } (v_1 = \perp_{\mathbb{Z}_\perp} \text{ or } v_2 = \perp_{\mathbb{Z}_\perp}) \end{cases}$$

called *strict* extension

to prove: $\underline{\text{op}}_\perp$ is monotone and continuous

is $\mathbb{Z}_\perp \times \mathbb{Z}_\perp$ a $\text{CPO}_\perp$?

Cartesian product of domains

Cartesian product

$$\mathcal{D} = (D, \sqsubseteq_D)$$

$$\mathcal{E} = (E, \sqsubseteq_E)$$

$$\text{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

how to order pairs?

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1) \quad \text{iff} \quad d_0 \sqsubseteq_D d_1 \wedge e_0 \sqsubseteq_E e_1$$

example $\mathbb{Z}_\perp \times \mathbb{Z}_\perp$

$$(0, 1) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (1, 2)$$ ✗

$$(\perp_{\mathbb{Z}_\perp}, 1) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (1, 1)$$ ✓

$$(2, \perp_{\mathbb{Z}_\perp}) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (2, 0)$$ ✓

$$(0, \perp_{\mathbb{Z}_\perp}) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (\perp_{\mathbb{Z}_\perp}, 0)$$ ✗

Cartesian CPO

$$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

is it a partial order?

reflexivity, antisymmetry, transitivity of $\sqsubseteq_{D \times E}$
follow immediately from those of $\sqsubseteq_D \quad \sqsubseteq_E$

is there a bottom element?

let $\bot_{D \times E} = (\bot_D, \bot_E)$

take any pair $(d, e) \in D \times E$

since $\bot_D \sqsubseteq_D d$ and $\bot_E \sqsubseteq_E e$ then $\bot_{D \times E} = (\bot_D, \bot_E) \sqsubseteq_{D \times E} (d, e)$

Cartesian CPO (ctd)

$$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

is it complete?

take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$ we need to find its lub

we prove its lub is $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$

1. it is an upper bound of the chain
2. it is smaller than or equal to any other upper bound

Cartesian CPO (ctd)

$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$ take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$

1. $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$ is an upper bound of the chain

take a generic element of the chain (d_j, e_j)

we have $d_j \sqsubseteq_D \bigsqcup_{i \in \mathbb{N}} d_i$ $e_j \sqsubseteq_E \bigsqcup_{i \in \mathbb{N}} e_i$ thus $(d_j, e_j) \sqsubseteq_{D \times E} \left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$

Cartesian CPO (ctd)

$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D\times E}\ )$ take a chain $\{(d_i, e_i)\}_{i\in\mathbb{N}}$

2. $\left(\bigsqcup_{i\in\mathbb{N}} d_i, \bigsqcup_{i\in\mathbb{N}} e_i\right)$ is the least among upper bounds

take a generic upper bound (d, e): $\forall i \in \mathbb{N}.\ (d_i, e_i) \sqsubseteq_{D\times E} (d, e)$

by def $\forall i \in \mathbb{N}.\ d_i \sqsubseteq_D d\ \wedge\ \forall i \in \mathbb{N}.\ e_i \sqsubseteq_E e$

i.e., d is an upper bound of $\{d_i\}_{i\in\mathbb{N}}$

e is an upper bound of $\{e_i\}_{i\in\mathbb{N}}$

$$\Rightarrow \quad \bigsqcup_{i\in\mathbb{N}} d_i \sqsubseteq_D d \qquad \bigsqcup_{i\in\mathbb{N}} e_i \sqsubseteq_E e$$

hence $\left(\bigsqcup_{i\in\mathbb{N}} d_i, \bigsqcup_{i\in\mathbb{N}} e_i\right) \sqsubseteq_{D\times E} (d, e)$

Cartesian CPO: recap

$$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1) \quad \text{iff} \quad d_0 \sqsubseteq_D d_1 \wedge e_0 \sqsubseteq_E e_1$$

$$\bot_{D \times E} \triangleq (\bot_D, \bot_E)$$

$$\bigsqcup_{i \in \mathbb{N}} (d_i, e_i) \triangleq \left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$$

is $\mathbb{Z}_\bot \times \mathbb{Z}_\bot$ a $\mathrm{CPO}_\bot$? ✓

Projections

$\pi_1 : D \times E \to D$ $\pi_2 : D \times E \to E$

$\pi_1(d, e) = d$ $\pi_2(d, e) = e$

TH. projections are monotone

proof. take $(d_0, e_0) \sqsubseteq_{D\times E} (d_1, e_1)$

we want to prove $\pi_1(d_0, e_0) \sqsubseteq_D \pi_1(d_1, e_1)$

$\pi_2(d_0, e_0) \sqsubseteq_E \pi_2(d_1, e_1)$

$$\pi_1(d_0, e_0) = d_0 \sqsubseteq_D d_1 = \pi_1(d_1, e_1)$$

$$\Uparrow$$

$$(d_0, e_0) \sqsubseteq_{D\times E} (d_1, e_1)$$

the case of π_2 is analogous

Projections (ctd)

$\pi_1 : D \times E \to D$ $\pi_2 : D \times E \to E$

$\pi_1(d,e) = d$ $\pi_2(d,e) = e$

TH. projections are continuous

proof. take $\{(d_i,e_i)\}_{i\in\mathbb{N}}$

we want to prove $\pi_1\left(\bigsqcup_{i\in\mathbb{N}}(d_i,e_i)\right) = \bigsqcup_{i\in\mathbb{N}}\pi_1(d_i,e_i)$

$$\pi_1\left(\bigsqcup_{i\in\mathbb{N}}(d_i,e_i)\right) = \pi_1\left(\bigsqcup_{i\in\mathbb{N}} d_i, \bigsqcup_{i\in\mathbb{N}} e_i\right) = \bigsqcup_{i\in\mathbb{N}} d_i = \bigsqcup_{i\in\mathbb{N}}\pi_1(d_i,e_i)$$

by def of lub

by def of π_1

by def of π_1

the case of π_2 is analogous

Switch lemma

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$e_{0,0}$

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$e_{0,0} \sqsubseteq e_{0,1}$

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$$e_{0,0} \sqsubseteq e_{0,1} \sqsubseteq e_{0,2} \sqsubseteq \cdots$$

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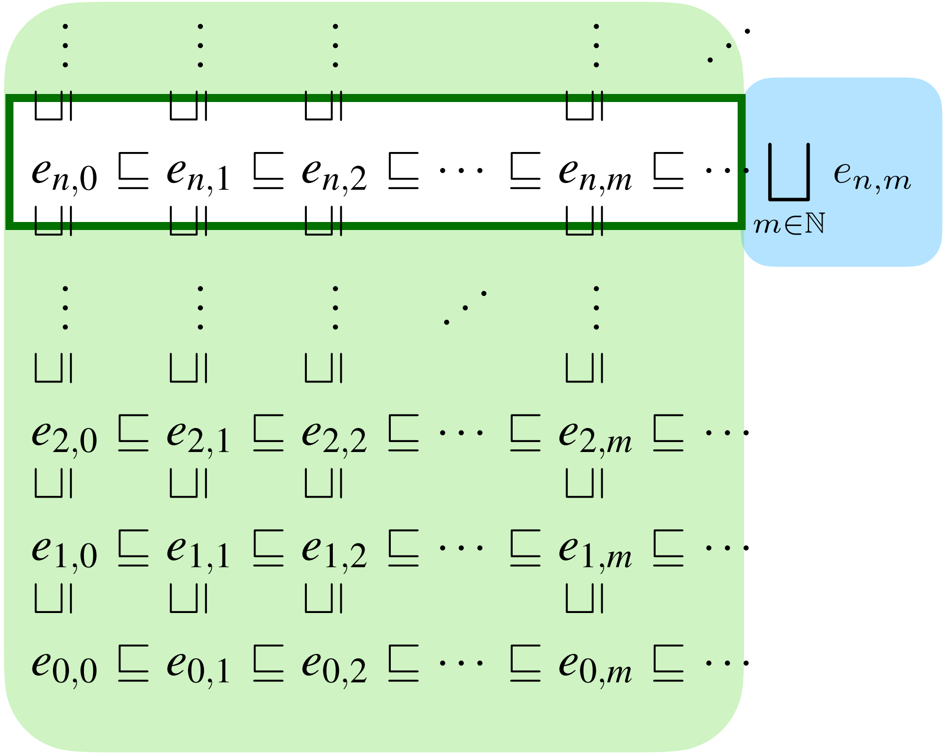
fixed n the set $\{e_{n,m}\}_{m \in \mathbb{N}}$

forms a chain (a row in the picture)

and thus has a lub (E is a CPO)

$$\bigsqcup_{m \in \mathbb{N}} e_{n,m}$$



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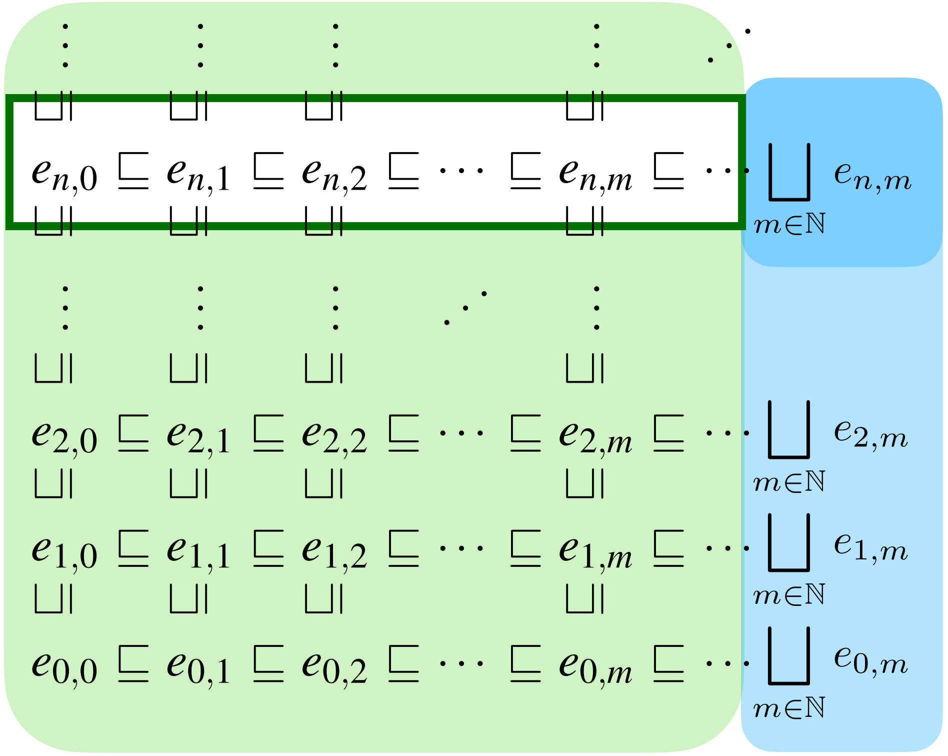

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\end{array}$$

fixed n the set $\{e_{n,m}\}_{m\in\mathbb{N}}$

forms a chain (a row in the picture)

and thus has a lub (E is a CPO)

$$\bigsqcup_{m\in\mathbb{N}} e_{n,m}$$

we form the chain of all row-lubs

$$\left\{ \bigsqcup_{m\in\mathbb{N}} e_{n,m} \right\}_{n\in\mathbb{N}}$$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

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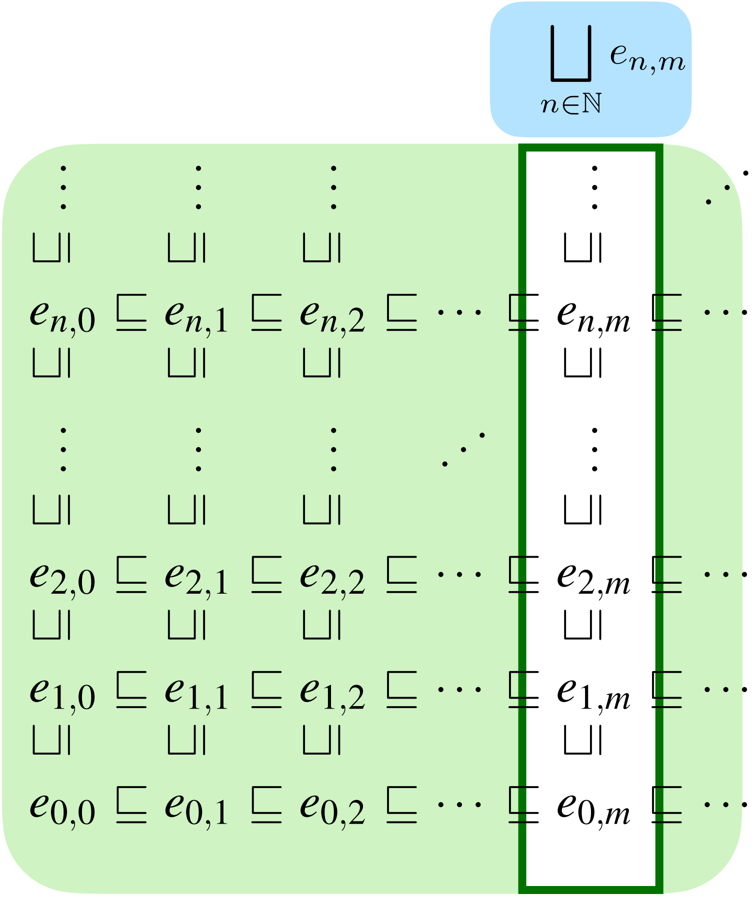
such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$



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\end{array}$$

fixed m the set $\{e_{n,m}\}_{n\in\mathbb{N}}$

forms a chain (a column in the picture)

and thus has a lub (E is a CPO)

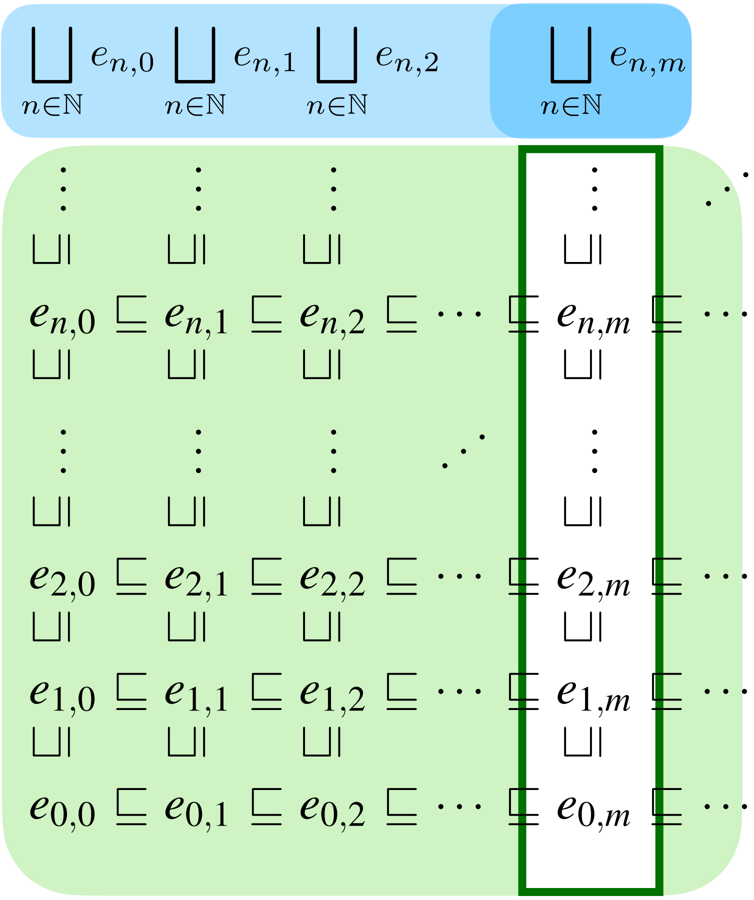
$$\bigsqcup_{n\in\mathbb{N}} e_{n,m}$$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m\in\mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$



$$\begin{array}{ccccccccccc}
\bigsqcup_{n\in\mathbb{N}} e_{n,0} & & \bigsqcup_{n\in\mathbb{N}} e_{n,1} & & \bigsqcup_{n\in\mathbb{N}} e_{n,2} & & & & \bigsqcup_{n\in\mathbb{N}} e_{n,m} & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \iddots \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
\vdots & & \vdots & & \vdots & & \iddots & & \vdots & & \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{1,m} & \sqsubseteq & \cdots \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & & \sqsupseteq & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{0,m} & \sqsubseteq & \cdots
\end{array}$$

fixed m the set $\{e_{n,m}\}_{n\in\mathbb{N}}$

forms a chain (a column in the picture)

and thus has a lub (E is a CPO)

$$\bigsqcup_{n\in\mathbb{N}} e_{n,m}$$

we form the chain of all column-lubs

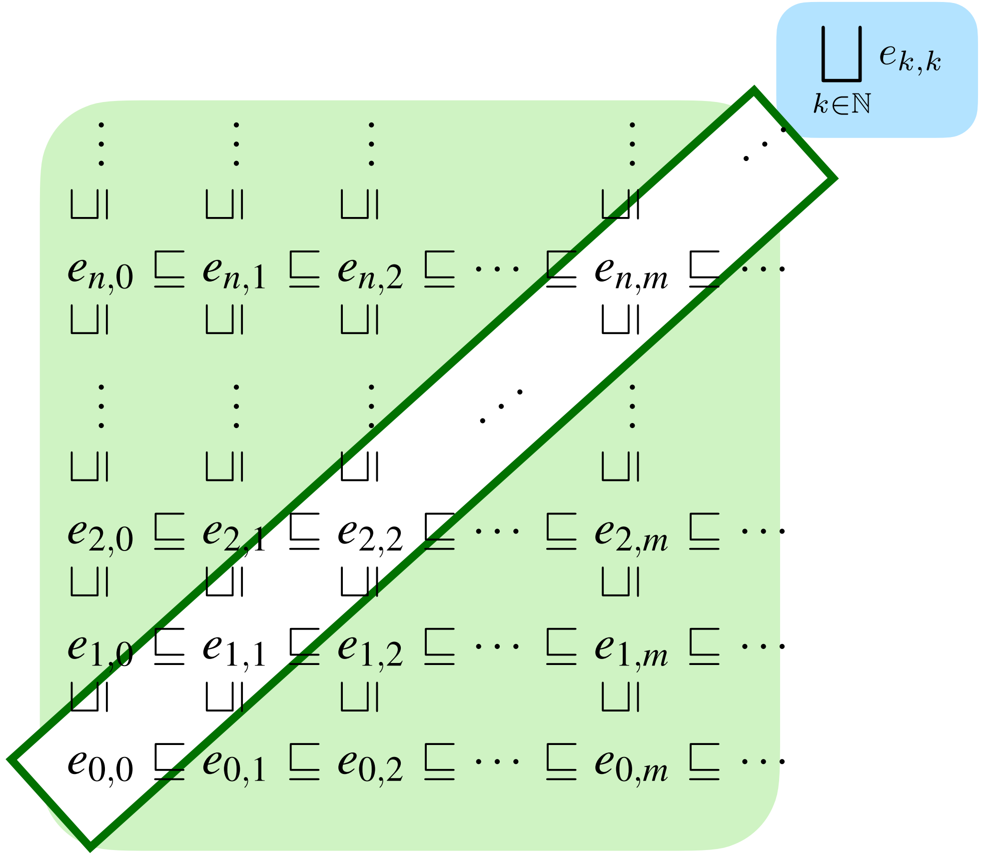
$$\left\{ \bigsqcup_{n\in\mathbb{N}} e_{n,m} \right\}_{m\in\mathbb{N}}$$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m\in\mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$



$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & \iddots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
\vdots & & \vdots & & \vdots & & \iddots & & \vdots & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{1,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{0,m} & \sqsubseteq & \cdots
\end{array}$$

$\bigsqcup_{k\in\mathbb{N}} e_{k,k}$

the diagonal elements $\{e_{k,k}\}_{k\in\mathbb{N}}$

also form a chain

and thus has a lub (E is a CPO)

$$\bigsqcup_{k\in\mathbb{N}} e_{k,k}$$

Switch Lemma

$\mathcal{E} = (E, \sqsubseteq)$ CPO

a set of elements (not a chain) $\{e_{n,m}\}_{n,m \in \mathbb{N}}$

such that $e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & \iddots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
\vdots & & \vdots & & \vdots & & \iddots & & \vdots & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{1,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{0,m} & \sqsubseteq & \cdots
\end{array}$$

we prove that
all sets we have seen
have the same
set of upper bounds
and thus the same
least upper bound

$$\bigsqcup_{n \in \mathbb{N}} \bigsqcup_{m \in \mathbb{N}} e_{n,m} = \bigsqcup_{k \in \mathbb{N}} e_{k,k} = \bigsqcup_{m \in \mathbb{N}} \bigsqcup_{n \in \mathbb{N}} e_{n,m}$$

$$\{e_{n,m}\}_{n,m \in \mathbb{N}}$$

$$\left\{ \bigsqcup_{m \in \mathbb{N}} e_{n,m} \right\}_{n \in \mathbb{N}}$$

$$\left\{ \bigsqcup_{n \in \mathbb{N}} e_{n,m} \right\}_{m \in \mathbb{N}}$$

$$\{e_{k,k}\}_{k \in \mathbb{N}}$$

(i) $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ same u.b. as $\{e_n\}_{n\in\mathbb{N}}$

where $e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m}$

$$
\begin{array}{ccccccccccccl}
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots & \sqsubseteq & e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_2 = \bigsqcup_{m\in\mathbb{N}} e_{2,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_1 = \bigsqcup_{m\in\mathbb{N}} e_{1,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_0 = \bigsqcup_{m\in\mathbb{N}} e_{0,m}
\end{array}
$$

(i) $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ same u.b. as $\{e_n\}_{n\in\mathbb{N}}$

where $e_n = \displaystyle\bigsqcup_{m\in\mathbb{N}} e_{n,m}$

1. take an upper bound e of $\{e_{n,m}\}_{n,m\in\mathbb{N}}$

   we want to prove it is an upper bound for $\{e_n\}_{n\in\mathbb{N}}$

e

$$
\begin{array}{ccccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots & \sqsubseteq & e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_2 = \bigsqcup_{m\in\mathbb{N}} e_{2,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_1 = \bigsqcup_{m\in\mathbb{N}} e_{1,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_0 = \bigsqcup_{m\in\mathbb{N}} e_{0,m}
\end{array}
$$

take any (row) index n

we prove $e_n \sqsubseteq e$

$\underset{\text{a row}}{\{e_{n,m}\}_{m\in\mathbb{N}}} \subseteq \underset{\text{the matrix}}{\{e_{n,m}\}_{n,m\in\mathbb{N}}}$

e is an u.b. of $\{e_{n,m}\}_{m\in\mathbb{N}}$

$e_n = \displaystyle\bigsqcup_{m\in\mathbb{N}} e_{n,m}$ is the lub

therefore $e_n \sqsubseteq e$

(i) $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ same u.b. as $\{e_n\}_{n\in\mathbb{N}}$

where $e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m}$

2. take an upper bound e of $\{e_n\}_{n\in\mathbb{N}}$

we want to prove it is an upper bound for $\{e_{n,m}\}_{n,m\in\mathbb{N}}$

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & & & & & e \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots & \sqsubseteq & e_n = \bigsqcup_{m\in\mathbb{N}} e_{n,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
\vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_2 = \bigsqcup_{m\in\mathbb{N}} e_{2,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_1 = \bigsqcup_{m\in\mathbb{N}} e_{1,m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots & \sqsubseteq & e_0 = \bigsqcup_{m\in\mathbb{N}} e_{0,m}
\end{array}
$$

take any indices n, m

we prove $e_{n,m} \sqsubseteq e$

$$e_{n,m} \sqsubseteq \bigsqcup_{m\in\mathbb{N}} e_{n,m} = e_n \sqsubseteq e$$

one element of a row

the lub of that row

(ii) $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ same u.b. as $\left\{\bigsqcup_{n\in\mathbb{N}} e_{n,m}\right\}_{m\in\mathbb{N}}$

the proof is analogous to the previous case
(reason by columns, not by rows)

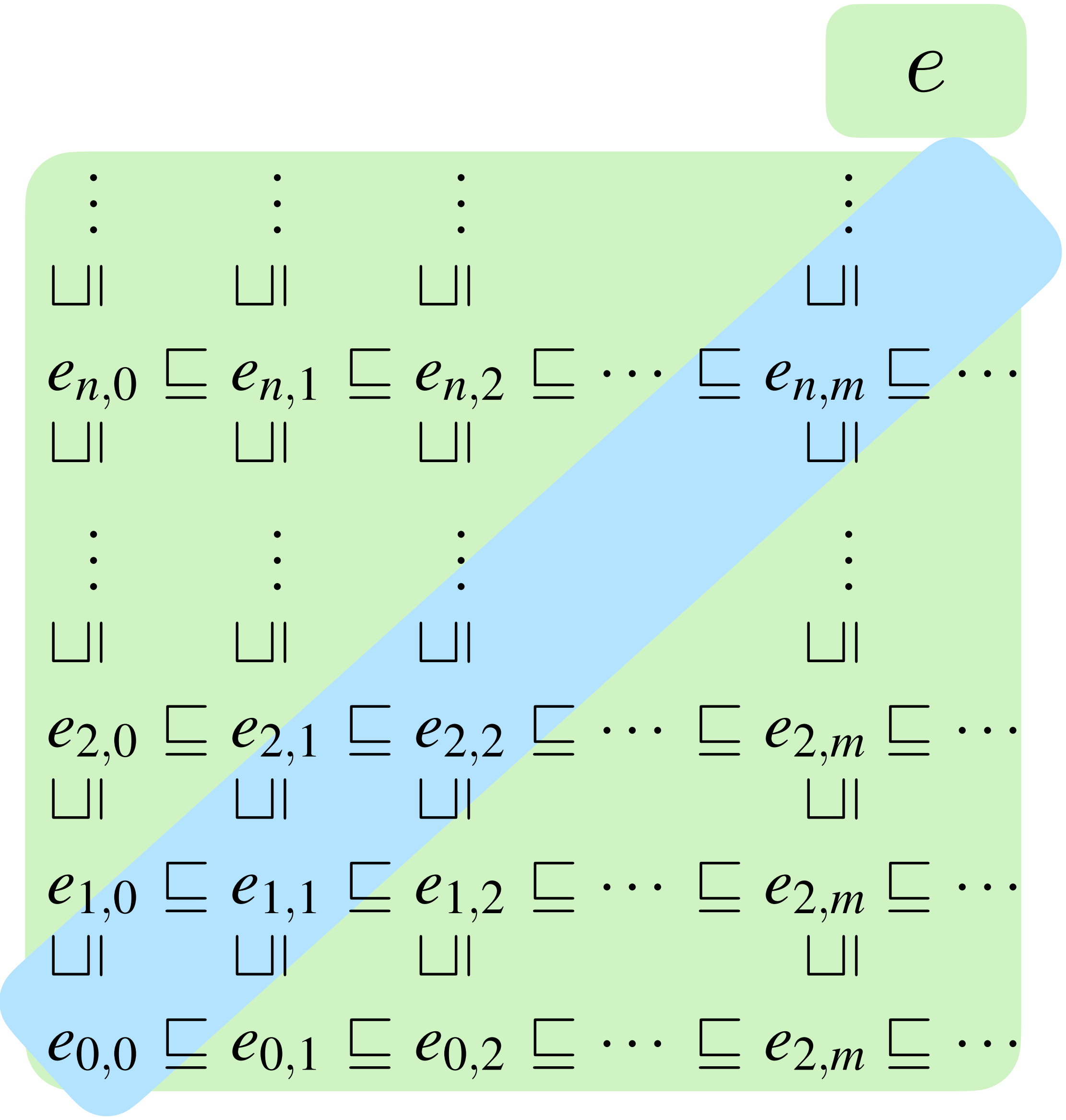
$$\bigsqcup_{n\in\mathbb{N}} e_{n,0} \quad \bigsqcup_{n\in\mathbb{N}} e_{n,1} \quad \bigsqcup_{n\in\mathbb{N}} e_{n,2} \qquad\qquad \bigsqcup_{n\in\mathbb{N}} e_{n,m}$$

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots
\end{array}$$

(iii) $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ same u.b. as $\{e_{k,k}\}_{k\in\mathbb{N}}$

1. take an upper bound e of $\{e_{n,m}\}_{n,m\in\mathbb{N}}$

   we want to prove it is an upper bound for $\{e_{k,k}\}_{k\in\mathbb{N}}$



$$\begin{array}{ccccccccccc}
 & & & & & & & & e & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots
\end{array}$$

but this is immediate, because

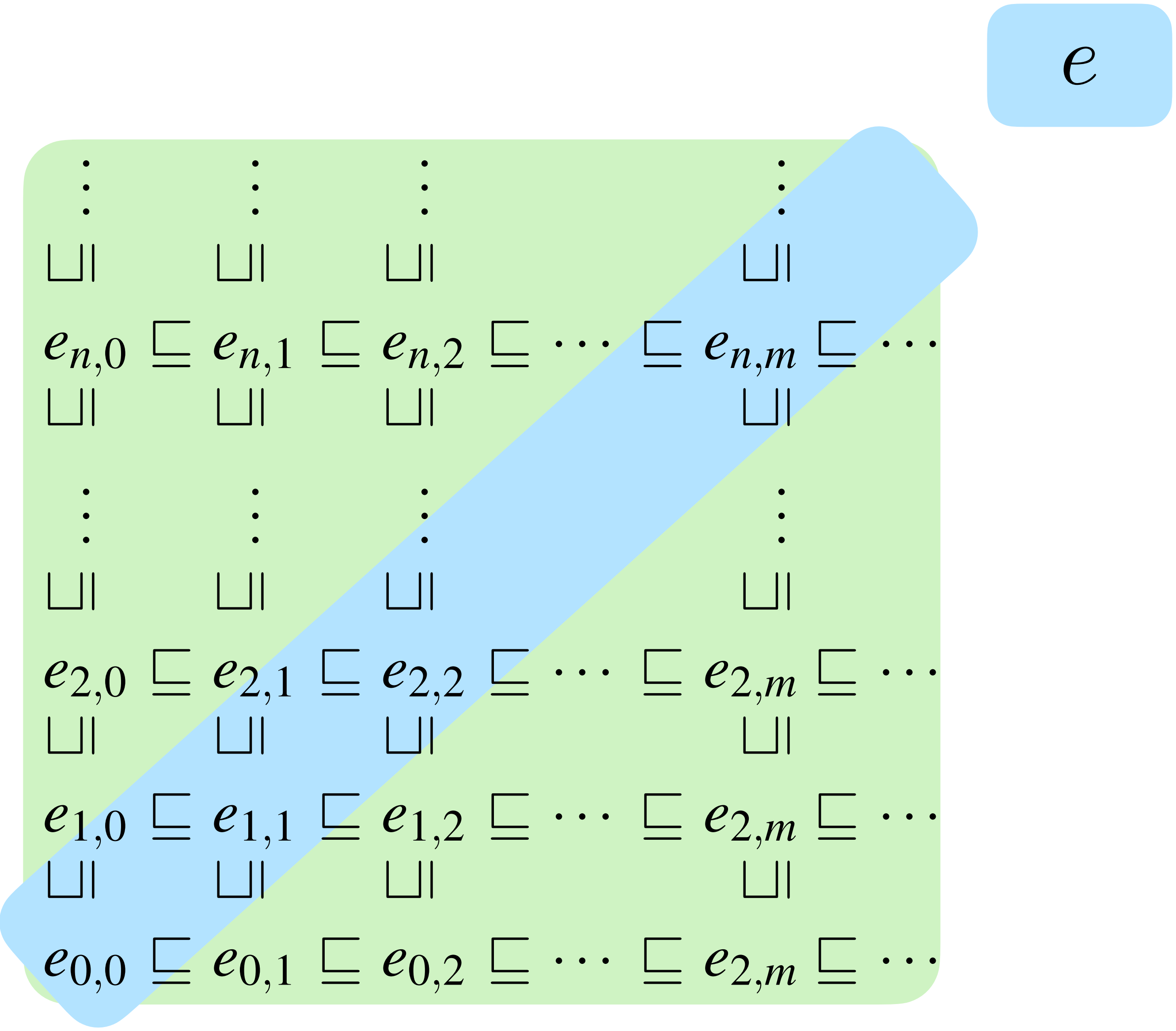
$$\{e_{k,k}\}_{k\in\mathbb{N}} \subseteq \{e_{n,m}\}_{n,m\in\mathbb{N}}$$

the diagonal　　the whole matrix

(iii) $\{e_{n,m}\}_{n,m\in\mathbb{N}}$ same u.b. as $\{e_{k,k}\}_{k\in\mathbb{N}}$

2. take an upper bound e of $\{e_{k,k}\}_{k\in\mathbb{N}}$

we want to prove it is an upper bound for $\{e_{n,m}\}_{n,m\in\mathbb{N}}$



e

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{n,0} & \sqsubseteq & e_{n,1} & \sqsubseteq & e_{n,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{n,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{2,0} & \sqsubseteq & e_{2,1} & \sqsubseteq & e_{2,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{1,0} & \sqsubseteq & e_{1,1} & \sqsubseteq & e_{1,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
e_{0,0} & \sqsubseteq & e_{0,1} & \sqsubseteq & e_{0,2} & \sqsubseteq & \cdots & \sqsubseteq & e_{2,m} & \sqsubseteq & \cdots
\end{array}$$

take any indices n, m

we prove $e_{n,m} \sqsubseteq e$

let $k = \max\{n, m\}$

$$e_{n,m} \sqsubseteq e_{k,k} \sqsubseteq e$$

$$n \leq k \wedge m \leq k$$

e is an u.b. of $\{e_{k,k}\}_{k\in\mathbb{N}}$

Switch Lemma: recap

$$\{e_{n,m}\}_{n,m\in\mathbb{N}}$$

$e_{n,m} \sqsubseteq e_{n',m'}$ if $n \leq n' \wedge m \leq m'$

same set of upper bounds as

$$\left\{\bigsqcup_{m\in\mathbb{N}} e_{n,m}\right\}_{n\in\mathbb{N}} \qquad \{e_{k,k}\}_{k\in\mathbb{N}} \qquad \left\{\bigsqcup_{n\in\mathbb{N}} e_{n,m}\right\}_{m\in\mathbb{N}}$$

$$\bigsqcup_{n\in\mathbb{N}}\bigsqcup_{m\in\mathbb{N}} e_{n,m} = \bigsqcup_{k\in\mathbb{N}} e_{k,k} = \bigsqcup_{m\in\mathbb{N}}\bigsqcup_{n\in\mathbb{N}} e_{n,m}$$

Functional domains

Function space

$$\mathcal{D} = (D, \sqsubseteq_D)$$

$$\mathcal{E} = (E, \sqsubseteq_E)$$

$$\text{CPO}_\perp \;\Rightarrow\; [\mathcal{D} \to \mathcal{E}] = (\ [D \to E]\ ,\ \sqsubseteq_{[D \to E]}\ )$$

$$D \to E \triangleq \{\ f \ \mid\ f : D \to E\ \}$$

$$[D \to E] \triangleq \{\ f \ \mid\ f : D \to E\ ,\ f \text{ continuous}\ \}$$

how to order functions?

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$f(x) \triangleq 0 \qquad f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]} g \qquad g(x) \triangleq \begin{cases} 1 & x \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

❌

not continuous!

$$f(1) = 0 \not\sqsubseteq_{\mathbb{Z}_\perp} 1 = g(1)$$

total functions on $\mathbb{Z}_\perp$ are not comparable

(unless they are equal)

any total function is maximal in $\mathbb{Z}_\perp \to \mathbb{Z}_\perp$

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$f(x) \triangleq \begin{cases} 1 & x \text{ odd} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases}$$

$$g(x) \triangleq \begin{cases} 1 & x \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]} g$$

not continuous!

✓

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$f(x) \triangleq \begin{cases} 1 & x \text{ odd} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases} \qquad g(x) \triangleq \begin{cases} 0 & x \text{ even} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]} g$$

✗

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$f(x) \triangleq \begin{cases} x! & 1 \leq x \leq 10 \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases} \qquad g(x) \triangleq \begin{cases} x! & 1 \leq x \leq 15 \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]} g$$

✓

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$f(x,y) \triangleq \begin{cases} x * y & x, y \neq \perp_{\mathbb{Z}_\perp} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases} \quad g(x,y) \triangleq \begin{cases} (x * y)^2 & x, y \neq \perp_{\mathbb{Z}_\perp} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp]} g$$

❌

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$f(x,y) \triangleq \begin{cases} x * y & x, y \neq \perp_{\mathbb{Z}_\perp} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases} \qquad g(x,y) \triangleq \begin{cases} x * y & x, y \neq \perp_{\mathbb{Z}_\perp} \\ 0 & x = \perp_{\mathbb{Z}_\perp} \\ \perp_{\mathbb{Z}_\perp} & \text{otherwise} \end{cases}$$

$$f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp]} g$$

yes (as functions)

but is g continuous?

$$g(\perp, \perp) = 0 \qquad\qquad g(1,1) = 1$$

not even monotone!

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \to \mathbb{Z}_\perp \times \mathbb{Z}_\perp$$

$$f(x) \triangleq (\perp_{\mathbb{Z}_\perp}, x) \qquad f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp \times \mathbb{Z}_\perp]} g \qquad g(x) \triangleq (x, x)$$

✓

Example

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$f, g : \mathbb{Z}_\perp \to \mathbb{Z}_\perp \times \mathbb{Z}_\perp$$

$$f(x) \triangleq (\perp_{\mathbb{Z}_\perp}, x) \qquad f \overset{?}{\sqsubseteq}_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp \times \mathbb{Z}_\perp]} g \qquad g(x) \triangleq (x, \perp_{\mathbb{Z}_\perp})$$

❌

$$f(0) = (\perp_{\mathbb{Z}_\perp}, 0) \ \not\sqsubseteq_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (0, \perp_{\mathbb{Z}_\perp}) = g(0)$$

Functional CPO

$$[\mathcal{D} \to \mathcal{E}] = (\ [D \to E]\ ,\ \sqsubseteq_{[D\to E]}\ )$$

is it a partial order?

reflexivity, antisymmetry, transitivity of $\sqsubseteq_{[D\to E]}$ follow immediately from those of $\sqsubseteq_E$

is there a bottom element?

let $\bot_{[D\to E]} = \lambda d.\ \bot_E$

take any function $f \in [D \to E]$

for any $d \in D$ we have $\bot_{[D\to E]}\ d = \bot_E \sqsubseteq_E f(d)$

Functional CPO (ctd)

$$[\mathcal{D} \to \mathcal{E}] = (\ [D \to E]\ ,\ \sqsubseteq_{[D \to E]}\ )$$

is it complete?

first we show that any chain of functions
(not necessarily continuous)
has a limit in $D \to E$

then we show that the limit in $D \to E$
of any chain of continuous functions
is also continuous

Functional CPO (ctd)

$\{f_n : D \to E\}_{n\in\mathbb{N}}$ a chain of functions
(not necessarily continuous)

we prove its lub is $h \triangleq \lambda d.\ \bigsqcup_{n\in\mathbb{N}} f_n(d)$

i.e. $h(d) \triangleq \bigsqcup_{n\in\mathbb{N}} f_n(d)$

1. it is an upper bound of the chain
2. it is smaller than or equal to any other upper bound

Functional CPO (ctd)

take a chain $\{f_n : D \to E\}_{n\in\mathbb{N}}$ (not necessarily continuous)

1. $h \triangleq \lambda d.\ \bigsqcup_{n\in\mathbb{N}} f_n(d)$ is an upper bound of the chain

take any $n \in \mathbb{N}$

for any $d \in D$ $\quad f_n(d) \sqsubseteq_E \bigsqcup_{n\in\mathbb{N}} f_n(d) = h(d)$

therefore $\quad f_n \sqsubseteq_{D\to E} h$

Functional CPO (ctd)

take a chain $\{f_n : D \to E\}_{n\in\mathbb{N}}$ (not necessarily continuous)

2. $h \triangleq \lambda d.\ \bigsqcup_{n\in\mathbb{N}} f_n(d)$ is the least among upper bounds

take any g such that $\forall n.\ f_n \sqsubseteq_{D\to E} g$

we want to prove $h \sqsubseteq_{D\to E} g$

take any $d \in D$ $\forall n.\ f_n(d) \sqsubseteq_E g(d)$

thus $g(d)$ is an u.b. of $\{f_n(d)\}_{n\in\mathbb{N}}$

and therefore $h(d) = \bigsqcup_{n\in\mathbb{N}} f_n(d) \sqsubseteq_E g(d)$

Functional CPO (ctd)

TH. take a chain $\{f_n : D \to E\}_{n\in\mathbb{N}}$ of continuous functions

then $h \triangleq \lambda d. \bigsqcup_{n\in\mathbb{N}} f_n(d)$ is continuous

proof. let $\{d_i\}_{i\in\mathbb{N}}$ a chain in D

we prove $h\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \bigsqcup_{i\in\mathbb{N}} h(d_i)$

$$h\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \bigsqcup_{n\in\mathbb{N}} f_n\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$$ by def of h

$$= \bigsqcup_{n\in\mathbb{N}} \bigsqcup_{i\in\mathbb{N}} f_n(d_i)$$ by continuity of f_n

$$= \bigsqcup_{i\in\mathbb{N}} \bigsqcup_{n\in\mathbb{N}} f_n(d_i)$$ by switch lemma (applicable?)

$$= \bigsqcup_{i\in\mathbb{N}} h(d_i)$$ by def of h

Functional CPO (ctd)

if $n \leq m \wedge i \leq j$ then $f_n(d_i) \sqsubseteq_E f_m(d_j)$? ✓

$$\Downarrow$$

$$f_n \sqsubseteq_{[D\to E]} f_m \wedge d_i \sqsubseteq d_j$$

$$f_n(d_i) \underset{\substack{f_n \\ \text{monotone}}}{\sqsubseteq_E} f_n(d_j) \underset{\substack{| \\ f_n \sqsubseteq_{[D\to E]} f_m}}{\sqsubseteq_E} f_m(d_j)$$

$$= \bigsqcup_{n\in\mathbb{N}} \bigsqcup_{i\in\mathbb{N}} f_n(d_i)$$

✓

$$= \bigsqcup_{i\in\mathbb{N}} \bigsqcup_{n\in\mathbb{N}} f_n(d_i)$$

by switch lemma (applicable?)

Functional CPO (ctd)

TH. $[\mathcal{D} \to \mathcal{E}] = (\ [D \to E]\ ,\ \sqsubseteq_{[D \to E]}\ )$ is complete

proof. take a chain $\{f_n : [D \to E]\}_{n \in \mathbb{N}}$

$h \triangleq \lambda d.\ \bigsqcup_{n \in \mathbb{N}} f_n(d)$ is the lub in $D \to E$

is continuous $h \in [D \to E]$

since $[D \to E] \subseteq D \to E$

h is the lub in $[D \to E]$

Functional CPO: recap

$$[\mathcal{D} \to \mathcal{E}] = (\ [D \to E]\ ,\ \sqsubseteq_{[D \to E]}\ )$$

$$f \sqsubseteq_{[D \to E]} g \quad \text{iff} \quad \forall d \in D.\ f(d) \sqsubseteq_E g(d)$$

$$\bot_{[D \to E]} \triangleq \lambda d.\ \bot_E$$

$$\bigsqcup_{n \in \mathbb{N}} f_n \triangleq \lambda d.\ \bigsqcup_{n \in \mathbb{N}} f_n(d)$$

$$f \in [D \to E], g \in [E \to F] \quad \Rightarrow \quad g \circ f \in [D \to F]$$

the composition of continuous functions is continuous

Exercise: smashed prod

$$\begin{matrix} \mathcal{D} = (D, \sqsubseteq_D) \\ \mathcal{E} = (E, \sqsubseteq_E) \end{matrix} \quad \mathrm{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} \otimes \mathcal{E} = (\ D \otimes E\ ,\ \sqsubseteq_{D \otimes E}\ )$$

$$D \otimes E \triangleq \{(d, e) \mid (d, e) \in D \times E, d = \perp_D \Leftrightarrow e = \perp_E\}$$

how to order pairs?

bottom element?

complete order?

Ex. CPO of fixpoints

Let $(D, \sqsubseteq_D)$ be a CPO and $f : D \to D$ be a continuous function. Prove that the set of fixpoints of f is itself a CPO (ordered by $\sqsubseteq_D$).

Ex. CPO of fixpoints

$(D, \sqsubseteq_D)$ CPO $\qquad f : D \to D$ continuous

$\mathsf{FP}_f \triangleq \{\ d \mid d = f(d)\ \}$ set of all fixpoints of f

$(\mathsf{FP}_f, \sqsubseteq) \qquad \sqsubseteq \triangleq \sqsubseteq_D \cap (\mathsf{FP}_f \times \mathsf{FP}_f)$ CPO?

it is a PO (because $\mathsf{FP}_f \subseteq D$)

we prove it is complete take a chain $\{d_i\}_{i \in \mathbb{N}} \subseteq \mathsf{FP}_f$

we show that $\bigsqcup_{i \in \mathbb{N}} d_i$ as computed in D is a fixpoint of f

$$f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = \bigsqcup_{i \in \mathbb{N}} f(d_i) \qquad \text{by continuity}$$

$$= \bigsqcup_{i \in \mathbb{N}} d_i \qquad \text{each } d_i \text{ is a fixpoint}$$