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Models for Programming Paradigms

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12b - HOFL Operational Semantics

Disclaim

$$t ::= x \mid n \mid t_0 \text{ op } t_1 \mid \textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1$$
$$\mid (t_0, t_1) \mid \textbf{fst}(t) \mid \textbf{snd}(t)$$
$$\mid \lambda x.\ t \mid t_0\ t_1$$
$$\mid \textbf{rec } x.\ t$$

$$\tau ::= int \mid \tau_0 * \tau_1 \mid \tau_0 \to \tau_1$$

$$\frac{}{x : \widehat{x}} \quad \frac{}{n : int} \quad \frac{t_0 : int \quad t_1 : int}{t_0 \text{ op } t_1 : int} \quad \frac{t : int \quad t_0 : \tau \quad t_1 : \tau}{\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 : \tau}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1}{(t_0, t_1) : \tau_0 * \tau_1} \quad \frac{t : \tau_0 * \tau_1}{\textbf{fst}(t) : \tau_0} \quad \frac{t : \tau_0 * \tau_1}{\textbf{snd}(t) : \tau_1}$$

$$\frac{x : \tau_0 \quad t : \tau_1}{\lambda x.\ t : \tau_0 \to \tau_1} \quad \frac{t_1 : \tau_0 \to \tau_1 \quad t_0 : \tau_0}{t_1\ t_0 : \tau_1}$$

$$\frac{x : \tau \quad t : \tau}{\textbf{rec } x.\ t : \tau}$$

we assign semantics only to terms that are: well-formed and closed

$$t : \tau \qquad \text{fv}(t) = \varnothing$$

Canonical forms

Statements

reads "reduces to"

$$t \to c$$

a closed term

a canonical form
(a special kind of terms)

Big step operational semantics

computation of canonical form
(by term manipulation)

Canonical forms

set of canonical forms $C_\tau \subseteq T_\tau$
with type τ

(laziness)
not required to be
in canonical forms

t not necessarily
a closed term

$$\frac{}{n \in C_{int}}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1 \quad t_0, t_1 \text{ closed}}{(t_0, t_1) \in C_{\tau_0 * \tau_1}}$$

$$\frac{\lambda x.\, t : \tau_0 \to \tau_1 \quad \lambda x.\, t \text{ closed}}{\lambda x.\, t \in C_{\tau_0 \to \tau_1}}$$

Canonical forms?

$$\frac{}{n \in C_{int}} \qquad \frac{t_0 : \tau_0 \quad t_1 : \tau_1 \quad t_0, t_1 \text{ closed}}{(t_0, t_1) \in C_{\tau_0 * \tau_1}} \qquad \frac{\lambda x.\, t : \tau_0 \to \tau_1 \quad \lambda x.\, t \text{ closed}}{\lambda x.\, t \in C_{\tau_0 \to \tau_1}}$$

$1 + 2 \times 3$ ✗

$\textbf{if } 0 \textbf{ then } 0 \textbf{ else } 0$ ✗

$(1, 2)$ ✓

$\lambda x.\ 1$ ✓

$(1 + 2, 2 - 1)$ ✓

$\lambda x.\ 1 + 2 \times 3$ ✓

$\textbf{fst}(1, 2)$ ✗

$\lambda x.\ \textbf{fst}(1, 2)$ ✓

HOFL
Lazy operational semantics

Operational semantics: axioms

$$\frac{c \in C_\tau}{c \to c}$$

i.e., expanding the various cases

$$\frac{}{n \to n} \qquad \frac{t_0 : \tau_0 \quad t_1 : \tau_1 \quad t_0, t_1 \text{ closed}}{(t_0, t_1) \to (t_0, t_1)} \qquad \frac{\lambda x.\, t : \tau_0 \to \tau_1 \quad \lambda x. t \text{ closed}}{\lambda x.\, t \to \lambda x.\, t}$$

integers, pairs and abstractions
are already in canonical form

Lazy op semantics

$$\frac{}{n \rightarrow n} \qquad \frac{t_0 : \tau_0 \quad t_1 : \tau_1 \quad t_0, t_1 \text{ closed}}{(t_0, t_1) \rightarrow (t_0, t_1)} \qquad \frac{\lambda x.\, t : \tau_0 \rightarrow \tau_1 \quad \lambda x. t \text{ closed}}{\lambda x.\, t \rightarrow \lambda x.\, t}$$

$$\frac{t_0 \rightarrow n_0 \quad t_1 \rightarrow n_1}{t_0 \text{ op } t_1 \rightarrow n_0 \underline{\text{op}}\, n_1} \qquad \frac{t \rightarrow 0 \quad t_0 \rightarrow c_0}{\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 \rightarrow c_0} \qquad \frac{t \rightarrow (t_0, t_1) \quad t_0 \rightarrow c_0}{\textbf{fst}(t) \rightarrow c_0}$$

$$\frac{t[^{\textbf{rec } x.\, t}/_x] \rightarrow c}{\textbf{rec } x.\, t \rightarrow c} \qquad \frac{t \rightarrow n \quad n \neq 0 \quad t_1 \rightarrow c_1}{\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 \rightarrow c_1} \qquad \frac{t \rightarrow (t_0, t_1) \quad t_1 \rightarrow c_1}{\textbf{snd}(t) \rightarrow c_1}$$

$$\frac{t_1 \rightarrow \lambda x.\, t_1' \quad t_1'[^{t_0}/_x] \rightarrow c}{(t_1\ t_0) \rightarrow c} \quad \text{(lazy)}$$

Type system (remind)

$$\frac{}{x : \widehat{x}} \qquad \frac{}{n : int} \qquad \frac{t_0 : int \quad t_1 : int}{t_0 \;\mathsf{op}\; t_1 : int} \qquad \frac{t : int \quad t_0 : \tau \quad t_1 : \tau}{\mathbf{if}\; t \;\mathbf{then}\; t_0 \;\mathbf{else}\; t_1 : \tau}$$

$$\frac{t_0 : \tau_0 \quad t_1 : \tau_1}{(t_0, t_1) : \tau_0 * \tau_1} \qquad \frac{t : \tau_0 * \tau_1}{\mathbf{fst}(t) : \tau_0} \qquad \frac{t : \tau_0 * \tau_1}{\mathbf{snd}(t) : \tau_1}$$

$$\frac{x : \tau_0 \quad t : \tau_1}{\lambda x.\; t : \tau_0 \to \tau_1} \qquad \frac{t_1 : \tau_0 \to \tau_1 \quad t_0 : \tau_0}{t_1 \; t_0 : \tau_1}$$

$$\frac{x : \tau \quad t : \tau}{\mathbf{rec}\; x.\; t : \tau}$$

Example

$$t \triangleq \underbrace{\lambda \underbrace{x}_{int}.\ \underbrace{\underbrace{x}_{int} + \underbrace{1}_{int}}_{int}}_{int \to int} \quad : int \to int$$

$$\lambda x.\ x + 1 \to c \quad \nwarrow_{c = \lambda x.\ x+1} \ \square$$

Example

$$t \triangleq \underbrace{\underbrace{(\lambda x.\ x+1}_{int \to int},\ \underbrace{\underbrace{1}_{int} + \underbrace{2}_{int}}_{int})}_{(int \to int) * int} : (int \to int) * int$$

$$(\lambda x.\ x+1, 1+2) \to c \ \nwarrow_{c=(\lambda x.\ x+1,1+2)} \ \square$$

laziness:
no need to evaluate 1+2

Example

$$t \triangleq \underbrace{\lambda \underbrace{x}_{int * int}.\ \underbrace{\textbf{if } \underbrace{\textbf{fst}(\underbrace{x}_{int * \tau_1})}_{int} \textbf{ then } \underbrace{1}_{int} \textbf{ else } \underbrace{\textbf{snd}(\underbrace{x}_{int * \tau_1})}_{int = \tau_1}}_{int}}_{(int * int) \rightarrow int} \quad : (int * int) \rightarrow int$$

Example (ctd)

$$t \triangleq \lambda x.\ \mathbf{if}\ \mathbf{fst}(x)\ \mathbf{then}\ 1\ \mathbf{else}\ \mathbf{snd}(x)$$

$$t\ (1,2) \to c\ \nwarrow\ t \to \lambda x'.\ t'\ ,\ t'[^{(1,2)}/_{x'}] \to c$$

$$\nwarrow_{x'=x,\ t'=\mathbf{if}\ \ldots}\ (\mathbf{if}\ \mathbf{fst}(x)\ \mathbf{then}\ 1\ \mathbf{else}\ \mathbf{snd}(x))[^{(1,2)}/_{x}] \to c$$

$$= \mathbf{if}\ \mathbf{fst}(1,2)\ \mathbf{then}\ 1\ \mathbf{else}\ \mathbf{snd}(1,2) \to c$$

$$\nwarrow\ \mathbf{fst}(1,2) \to n\ ,\ n \neq 0\ ,\ \mathbf{snd}(1,2) \to c$$

$$\nwarrow\ (1,2) \to (n_1, n_2)\ ,\ n_1 \to n\ ,\ n \neq 0\ ,\ \mathbf{snd}(1,2) \to c$$

$$\nwarrow^{*}_{n_1=1, n_2=2, n=1}\ \mathbf{snd}(1,2) \to c$$

$$\nwarrow\ (1,2) \to (n_3, n_4)\ ,\ n_4 \to c$$

$$\nwarrow^{*}_{n_3=1, n_4=2, c=2}\ \Box \qquad\qquad t\ (1,2) \to 2$$

Example

$$t \triangleq \underbrace{\mathbf{rec}\ \underbrace{x}_{\tau}.\ \underbrace{x}_{\tau}}_{\tau}\ :\tau$$

$$\mathbf{rec}\ x.\ x \to c \ \ \nwarrow \ \ x[^{\mathbf{rec}\ x.\ x}/_{x}] \to c$$

$$= \mathbf{rec}\ x.\ x \to c$$

same goal from which we started
no other option to explore:
divergence!

Example

$$fact \triangleq \mathbf{rec}\ f.\ \lambda x.\ \mathbf{if}\ x\ \mathbf{then}\ 1\ \mathbf{else}\ x \times (f\ (x-1))$$

$$fact \to c \ \nwarrow\ (\lambda x.\ \mathbf{if}\ x\ \mathbf{then}\ 1\ \mathbf{else}\ x \times (f(x-1)))[{}^{fact}/{}_{f}] \to c$$

$$= \lambda x.\ \mathbf{if}\ x\ \mathbf{then}\ 1\ \mathbf{else}\ x \times (\overbrace{(\mathbf{rec} f.\ ...)}^{fact}(x-1)) \to c$$

$$\nwarrow_{c=\lambda x.\ \mathbf{if}\ x\ \mathbf{then}\ 1\ \mathbf{else}\ x\times(fact(x-1))} \quad \square$$

Example

$$fact \triangleq \textbf{rec } f.\ \lambda x.\ \textbf{if } x \textbf{ then } 1 \textbf{ else } x \times (f\ (x-1))$$

$$(fact\ 1) \to c \ \nwarrow\ fact \to \lambda x'.\ t'\ ,\ t'[{}^1/_{x'}] \to c$$

$$\nwarrow^*_{x'=x, t'=\textbf{if}\ldots}\ (\textbf{if } x \textbf{ then } 1 \textbf{ else } x \times (fact\ (x-1)))[{}^1/_x] \to c$$

$$= \textbf{if } 1 \textbf{ then } 1 \textbf{ else } 1 \times (fact\ (1-1)) \to c$$

$$\nwarrow\ 1 \to n\ ,\ n \neq 0\ ,\ 1 \times (fact\ (1-1)) \to c$$

$$\nwarrow^*_{n=1, c=n_1 \underline{\times} n_2}\ 1 \to n_1\ ,\ (fact\ (1-1)) \to n_2$$

laziness evident here

$$\nwarrow_{n_1=1}\ fact \to \lambda x''.\ t''\ ,\ t''[{}^{1-1}/_{x''}] \to n_2$$

$$\nwarrow^*_{x''=x, t''=\textbf{if}\ldots}\ (\textbf{if } x \textbf{ then } 1 \textbf{ else } x \times (fact\ (x-1)))[{}^{1-1}/_x] \to n_2$$

$$= \textbf{if } 1-1 \textbf{ then } 1 \textbf{ else } (1-1) \times (fact\ ((1-1)-1)) \to n_2$$

$$\nwarrow\ 1-1 \to 0\ ,\ 1 \to n_2$$

$$\nwarrow^*_{n_2=1}\ \square$$

$$c = n_1 \underline{\times} n_2 = 1 \underline{\times} 1 = 1$$

HOFL
Eager operational semantics

Lazy vs Eager

$$\frac{t_1 \to \lambda x.\, t_1' \quad t_1'[{}^{t_0}/_{x}] \to c}{(t_1\ t_0) \to c} \quad \text{(lazy)}$$

$$\frac{t_1 \to \lambda x.\, t_1' \quad t_0 \to c_0 \quad t_1'[{}^{c_0}/_{x}] \to c}{(t_1\ t_0) \to c} \quad \text{(eager)}$$

Lazy vs Eager

$$t \triangleq (\lambda x.\ 1)\ (\mathbf{rec}\ y.\ y)\ :\ int$$

$x : \tau$

$y : \tau$

$$t \to c \ \nwarrow\ \lambda x.\ 1 \to \lambda x'.\ t'\ ,\ t'[{}^{\mathbf{rec}\ y.\ y}/_{x'}] \to c$$

lazy

$$\nwarrow_{x'=x,\ t'=1}\ 1[{}^{\mathbf{rec}\ y.\ y}/_{x}] \to c$$

$$= 1 \to c$$

$$\nwarrow_{c=1}\ \square$$

$$t \to c \ \nwarrow\ \lambda x.\ 1 \to \lambda x'.\ t'\ ,\ \mathbf{rec}\ y.\ y \to c'\ ,\ t'[{}^{c'}/_{x'}] \to c$$

eager

$$\nwarrow_{x'=x,\ t'=1}\ \mathbf{rec}\ y.\ y \to c'\ ,\ 1[{}^{c'}/_{x}] \to c$$

$$\nwarrow\ \mathbf{rec}\ y.\ y \to c'\ ,\ 1[{}^{c'}/_{x}] \to c$$

divergence!

Lazy vs Eager

$$t \triangleq (\lambda x.\ x + x)\ (1 \times 2)\ : int \qquad x : int$$

$$t \to c \ \nwarrow\ \ \lambda x.\ x + x \to \lambda x'.\ t'\ ,\ t'[^{1\times 2}/_{x'}] \to c$$

lazy $\nwarrow_{x'=x,\ t'=x+x}\ \ (x+x)[^{1\times 2}/_{x}] \to c$

$$= (1\times 2) + (1 \times 2) \to c$$

evaluated twice

$$\nwarrow_{c=c_1 \underline{+} c_2}\ \boxed{(1\times 2) \to c_1\ ,\ (1 \times 2) \to c_2}$$

$$\nwarrow^{*}_{c_1=2, c_2=2}\ \square \qquad c = c_1 \underline{+} c_2 = 2 \underline{+} 2 = 4$$

$$t \to c \ \nwarrow\ \ \lambda x.\ x + x \to \lambda x'.\ t'\ ,\ 1 \times 2 \to c'\ ,\ t'[^{c'}/_{x'}] \to c$$

eager $\nwarrow_{x'=x,\ t'=x+x}\ \ 1 \times 2 \to c'\ ,\ (x+x)[^{c'}/_{x}] \to c$

$$\nwarrow^{*}_{c'=2}\ \ (x+x)[^{2}/_{x}] \to c$$

$$= 2 + 2 \to c$$

$$\nwarrow^{*}_{c=4}\ \ \square$$

HOFL
Properties of operational semantics

Termination

termination? $\forall t.\ \exists c.\ t \to c?$ $\textbf{rec}\ x.\ x$

Determinacy?

determinacy? $\forall t.\ \forall c_1, c_2.\ t \to c_1 \wedge t \to c_2 \Rightarrow c_1 = c_2\,?$ ✓

$$P(t \to c) \triangleq \forall c_1.\ t \to c_1 \Rightarrow c_1 = c$$

by rule induction (try by yourself)

Subject reduction

(statically assigned types do not change at runtime)

subject reduction? $\forall t.\ \forall c.\ \forall \tau.\ t \to c \wedge t : \tau \Rightarrow c : \tau\ ?$ ✔

$$P(t \to c) \triangleq \forall \tau.\ t : \tau \Rightarrow c : \tau$$

by rule induction (try by yourself)

Congruence?

$$t_1 \equiv_{\text{op}} t_2 \quad \text{iff} \quad \forall c.\ (t_1 \rightarrow c \Leftrightarrow t_2 \rightarrow c)$$

is it a congruence?

$$2 \equiv_{\text{op}} 1 + 1$$

$$\lambda x.\ 2 \not\equiv_{\text{op}} \lambda x.\ 1 + 1$$

$$\lambda x.\ 2\ ,\ \lambda x.\ 1 + 1 \in C_{\tau \rightarrow int}$$

$$\lambda x.\ 2 \rightarrow \lambda x.\ 2$$

$$\lambda x.\ 1 + 1 \rightarrow \lambda x.\ 1 + 1$$

HOFL, type inference and operational semantics

[**Ex. 1**] Determine the type of the HOFL term

$$t \stackrel{\text{def}}{=} \mathbf{rec}\ x.\ ((\lambda y.\ \mathbf{if}\ y\ \mathbf{then}\ 0\ \mathbf{else}\ 0)\ x).$$

Then compute its (lazy) canonical form.

Ex. 1, typing

$$t \triangleq \underbrace{\mathbf{rec}\ \underbrace{x}_{int}.\ \underbrace{(\ \underbrace{(\ \lambda \underbrace{y}_{int}.\ \underbrace{\mathbf{if}\ \underbrace{y}_{int}\ \mathbf{then}\ \underbrace{0}_{int}\ \mathbf{else}\ \underbrace{0}_{int}}_{int}\ )}_{int \to int}\ \underbrace{x}_{int}\ )}_{int}}_{int} \quad : int$$

Ex. 1, canonical form?

$$t \triangleq \textbf{rec}\ x.\ (\ (\ \lambda y.\ \textbf{if}\ y\ \textbf{then}\ 0\ \textbf{else}\ 0\ )\ x\ ) \quad : int$$

$$t \to c \nwarrow \ (\ (\ \lambda y.\ \textbf{if}\ y\ \textbf{then}\ 0\ \textbf{else}\ 0\ )\ x\ )[^t/_x] \to c$$

$$= (\ (\ \lambda y.\ \textbf{if}\ y\ \textbf{then}\ 0\ \textbf{else}\ 0\ )\ t\ ) \to c$$

$$\nwarrow \ \lambda y.\ \textbf{if}\ y\ \textbf{then}\ 0\ \textbf{else}\ 0 \to \lambda x'.\ t'\ ,\ t'[^t/_{x'}] \to c$$

$$\nwarrow_{x'=y, t'=\textbf{if}\ldots}\ (\ \textbf{if}\ y\ \textbf{then}\ 0\ \textbf{else}\ 0\ )[^t/_y] \to c$$

$$= (\ \textbf{if}\ t\ \textbf{then}\ 0\ \textbf{else}\ 0\ ) \to c$$

$$\nwarrow \ t \to n,\ 0 \to c$$

(it doesn't matter if $n = 0$)

same goal from which we started:
no canonical form

[Ex. 2] Determine the type of the HOFL term

$$map \stackrel{\text{def}}{=} \lambda f.\ \lambda x.\ ((f\ \mathbf{fst}(x)), (f\ \mathbf{snd}(x)))$$

Then, compute the (lazy) canonical forms of the terms

$$t_1 \stackrel{\text{def}}{=} map\ (\lambda z.\ 2 \times z)\ (1, 2) \qquad\qquad t_2 \stackrel{\text{def}}{=} \mathbf{fst}\ (\ map\ (\lambda z.\ 2 \times z)\ (1, 2)\ )$$

Ex. 2, typing

$$map \triangleq \lambda f \;.\; \lambda x \;.\; (\;(\;f\;\mathbf{fst}(x)\;)\;,\;(\;f\;\mathbf{snd}(x)\;)\;)$$

$$\tau_1 \to \tau \qquad \tau_1 * \tau_1 \qquad \tau_1 \to \tau \qquad \tau_1 * \tau_2 \qquad \tau_1 \to \tau \qquad \tau_1 * \tau_2$$

$$\tau_1 \qquad \tau_2 = \tau_1$$

$$\tau \qquad \tau$$

$$\tau * \tau$$

$$\tau_1 * \tau_1 \to \tau * \tau$$

$$(\tau_1 \to \tau) \to \tau_1 * \tau_1 \to \tau * \tau$$

$$map \;:\; (\tau_1 \to \tau) \to \tau_1 * \tau_1 \to \tau * \tau$$

Ex. 2a, canonical form

$$map \triangleq \lambda f\ .\ \lambda x\ .\ (\ (\ f\ \mathbf{fst}(x)\ )\ ,\ (\ f\ \mathbf{snd}(x)\ )\ )$$

$$t_1 \triangleq map\ (\lambda z.\ 2 \times z)\ (1,2)$$

$$t_1 \to c\ \nwarrow\ \ (map\ (\lambda z.\ 2 \times z)) \to \lambda x'.\ t'\ ,\ t'[^{(1,2)}/_{x'}] \to c$$

$$\nwarrow\ map \to \lambda f'.\ t''\ ,\ t''[^{\lambda z.\ 2\times z}/_{f'}] \to \lambda x'.\ t'\ ,\ t'[^{(1,2)}/_{x'}] \to c$$

$$\nwarrow_{f'=f,t''=\lambda x\ldots}\ (\lambda x.\ ((f\ \mathbf{fst}(x)),(f\ \mathbf{snd}(x))))[^{\lambda z.\ 2\times z}/_{f}] \to \lambda x'.\ t'\ ,\ t'[^{(1,2)}/_{x'}] \to c$$

$$= (\lambda x.\ (((\lambda z.\ 2 \times z)\ \mathbf{fst}(x)),((\lambda z.\ 2 \times z)\ \mathbf{snd}(x)))) \to \lambda x'.\ t'\ ,\ t'[^{(1,2)}/_{x'}] \to c$$

$$\nwarrow_{x'=x,t'=(\ldots,\ldots)}\ (((\lambda z.\ 2 \times z)\ \mathbf{fst}(x)),((\lambda z.\ 2 \times z)\ \mathbf{snd}(x)))[^{(1,2)}/_{x}] \to c$$

$$= (((\lambda z.\ 2 \times z)\ \mathbf{fst}(1,2)),((\lambda z.\ 2 \times z)\ \mathbf{snd}(1,2))) \to c$$

$$\nwarrow_{c=(((\lambda z.\ 2\times z)\ \mathbf{fst}(1,2)),((\lambda z.\ 2\times z)\ \mathbf{snd}(1,2)))}\ \square$$

Ex. 2b, canonical form

$$t_1 \to (\ ((\lambda z.\ 2 \times z)\ \mathbf{fst}(1,2))\ ,\ ((\lambda z.\ 2 \times z)\ \mathbf{snd}(1,2))\ )$$

$$\mathbf{fst}(t_1) \to c \ \nwarrow\ \ t_1 \to (t_1', t_2')\ ,\ t_1' \to c$$

$$\nwarrow^{*}_{t_1'=(\lambda z.\ 2\times z)\ \mathbf{fst}(1,2)\ ,\ t_2'=(\lambda z.\ 2\times z)\ \mathbf{snd}(1,2)}\ (\lambda z.\ 2 \times z)\ \mathbf{fst}(1,2) \to c$$

$$\nwarrow\ \lambda z.\ 2 \times z \to \lambda z'.\ t'\ ,\ t'[^{\mathbf{fst}(1,2)}/_{z'}] \to c$$

$$\nwarrow_{z'=z, t'=2\times z}\ (2 \times z)[^{\mathbf{fst}(1,2)}/_{z}] \to c$$

$$= (2 \times \mathbf{fst}(1,2)) \to c$$

$$\nwarrow_{c=n_1 \underline{\times} n_2}\ 2 \to n_1\ ,\ \mathbf{fst}(1,2) \to n_2$$

$$\nwarrow^{*}_{n_1=2}\ (1,2) \to (t_1'', t_2'')\ ,\ t_1'' \to n_2$$

$$\nwarrow_{t_1''=1, t_2''=2}\ 1 \to n_2$$

$$\nwarrow_{n_2=1}\ \square \qquad c = n_1 \underline{\times} n_2 = 2 \underline{\times} 1 = 2$$

Ex. 3, part 1

Determine the type of the HOFL term

$$\lambda f.\ \lambda x.\ (f\ \mathsf{fst}(x))\ \mathsf{snd}(x)$$

$f : \tau_1 \to \tau_2 \to \tau_3$

$x : \tau_1 * \tau_2$

$f : \tau_1 \to \tau$

$x : \tau_1 * \tau_2$, $\mathsf{fst}(x) : \tau_1$

$x : \tau_1 * \tau_2$, $\mathsf{snd}(x) : \tau_2$

$(f\ \mathsf{fst}(x)) : \tau = \tau_2 \to \tau_3$

$(f\ \mathsf{fst}(x))\ \mathsf{snd}(x) : \tau_3$

$\lambda x.\ (f\ \mathsf{fst}(x))\ \mathsf{snd}(x) : \tau_1 * \tau_2 \to \tau_3$

$\lambda f.\ \lambda x.\ (f\ \mathsf{fst}(x))\ \mathsf{snd}(x) : (\tau_1 \to \tau_2 \to \tau_3) \to \tau_1 * \tau_2 \to \tau_3$

Which function is it? $uncurry$

Ex. 3, part 2

$$uncurry \triangleq \lambda f\,.\,\lambda x\,.\,(f\ \mathsf{fst}(x))\ \mathsf{snd}(x)$$

Prove that for every $t : \tau_1 \to \tau_2 \to \tau_3$ and $t_0 : \tau_1 * \tau_2$ then

$$(t\ \mathsf{fst}(t_0))\ \mathsf{snd}(t_0) \equiv_{\mathrm{op}} (uncurry\ t)\ t_0$$

$$(uncurry\ t)\ t_0 \to c \nwarrow (uncurry\ t) \to \lambda x'\,.\,t', \quad t'[{}^{t_0}/_{x'}] \to c$$

$$\nwarrow uncurry \to \lambda f'\,.\,t'', \quad t''[{}^{t}/_{f'}] \to \lambda x'\,.\,t', \quad t'[{}^{t_0}/_{x'}] \to c$$

$$\nwarrow (\lambda x\,.\,(f\ \mathsf{fst}(x))\ \mathsf{snd}(x))[{}^{t}/_{f}] \to \lambda x'\,.\,t', \quad t'[{}^{t_0}/_{x'}] \to c$$

$$= (\lambda x\,.\,(t\ \mathsf{fst}(x))\ \mathsf{snd}(x)) \to \lambda x'\,.\,t', \quad t'[{}^{t_0}/_{x'}] \to c$$

$$\nwarrow ((t\ \mathsf{fst}(x))\ \mathsf{snd}(x))[{}^{t_0}/_{x}] \to c$$

$$= (t\ \mathsf{fst}(t_0))\ \mathsf{snd}(t_0) \to c$$