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Models for Programming Paradigms

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08c - Immediate Consequences Operator

Recap

Kleene's theorem

$$p \triangleq f(p)$$

a recursive definition

is there such a p?

is there a least p?

if $\quad f : D \to D$ continuous $\qquad D \;\; \mathrm{CPO}_\perp$

then $$p \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp_D)$$

also written $\mathit{fix}(f)$

McCarthy 91 function

$f = M\ f$ $(\mathbf{Pf}(\mathbb{N},\mathbb{N}), \sqsubseteq)$ $\bot = \varnothing$ $\mathrm{CPO}_\bot$

$$M\ f\ n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n+11)) & \text{if } n \leq 100 \text{ and } f(n+11)\downarrow \\ \bot & \text{otherwise} \end{cases}$$

M monotone? $f \sqsubseteq g \overset{?}{\Rightarrow} M(f) \sqsubseteq M(g)$

M continuous? $M\left(\bigsqcup_i f_i\right) \overset{?}{\sqsubseteq} \bigsqcup_i M(f_i)$

McCarthy 91 function

$$M\ f\ n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n+11)) & \text{if } n \leq 100 \text{ and } f(n+11) \downarrow \\ \bot & \text{otherwise} \end{cases}$$

M monotone? ✓

we assume $f \sqsubseteq g$ and prove $M(f) \sqsubseteq M(g)$

take $(n, m) \in M(f)$ we prove $(n, m) \in M(g)$

if $n > 100$ $m = n - 10$ and $(n, n - 10) \in M(g)$

if $n \leq 100$ $m = f(\overbrace{f(n+11)}^{k})$

thus $\exists k.\ \{(n+11, k), (k, m)\} \subseteq f \subseteq g$

and $(n, m) \in M(g)$

McCarthy 91 function

$$M\ f\ n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n+11)) & \text{if } n \leq 100 \text{ and } f(n+11)\downarrow \\ \bot & \text{otherwise} \end{cases}$$

M continuous? ✔

$$M\left(\bigsqcup_i f_i\right) \overset{?}{\sqsubseteq} \bigsqcup_i M(f_i)$$

let $f \triangleq \bigsqcup_i f_i$

take $(n, m) \in M(f)$

we prove $(n, m) \in \bigsqcup_i M(f_i)$

if $n > 100$ $\quad m = n - 10$ and $(n, n-10) \in M(f_1) \subseteq \bigsqcup_i M(f_i)$

if $n \leq 100$ $\quad m = f(f(n+11))$

thus $\exists k.\ \{(n+11, k), (k, m)\} \subseteq f$

thus $\exists j_1, j_2.\ (n+11, k) \in f_{j_1}, (k, m) \in f_{j_2}$

let $j \triangleq \max\{j_1, j_2\}$ thus $\{(n+11, k), (k, m)\} \subseteq f_j$

and $(n, m) \in M(f_j) \subseteq \bigsqcup_i M(f_i)$

McCarthy 91 function

$f = M\ f$ $\qquad (\mathbf{Pf}(\mathbb{N},\mathbb{N}), \sqsubseteq)$ $\qquad \bot = \varnothing$ $\qquad \mathrm{CPO}_\bot$

$$M\ f\ n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n+11)) & \text{if } n \leq 100 \text{ and } f(n+11)\downarrow \\ \bot & \text{otherwise} \end{cases}$$

$$M^0(\varnothing) = \varnothing$$

$$M^1(\varnothing) = \{(n, n-10) \mid n > 100\} = \{(101,91),(102,92),...\}$$

$$M^2(\varnothing) = \{(100,91),(101,91),(102,92),...\}$$

$$M^3(\varnothing) = \{(99,91),(100,91),(101,91),(102,92),...\}$$

$$\cdots$$

$$M^{11}(\varnothing) = \{(91,91),(92,91),...,(101,91),(102,92),...\}$$

$$M^{12}(\varnothing) = \{(80,91),(81,91),...,(101,91),(102,92),...\}$$

$$\cdots$$

$$M^{20}(\varnothing) = \{(0,91),(1,91),...,(101,91),(102,92),...\}$$

fixpoint reached! (total function)

Alternative perspective

$f = M\ f$ $(\mathbf{Pf}(\mathbb{N},\mathbb{N}), \sqsubseteq)$ $\bot = \varnothing$ $\mathrm{CPO}_\bot$

$$M\ f\ n = \begin{cases} n - 10 & \text{if } n > 100 \\ f(f(n+11)) & \text{if } n \leq 100 \text{ and } f(n+11) \downarrow \\ \bot & \text{otherwise} \end{cases}$$

formulas: $(n, m) \in f$

or just (n, m) for brevity

$$R_M \triangleq \left\{ \frac{}{(n, n-10)}\ n > 100 \quad , \quad \frac{(n+11, k) \quad (k, m)}{(n, m)}\ n \leq 100 \right\}$$

$$f(n) = m \qquad \Leftrightarrow \qquad (n, m) \in I_{R_M}$$

(set of theorems of R_M)

Immediate Consequences Operator (ICO)

Immediate Consequences Operator

F a set of formulas

R a logical system
(with some restrictions)

$(\wp(F), \subseteq)$ $\mathrm{CPO}_\perp$

$\widehat{R} : \wp(F) \to \wp(F)$ ICO

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \dots \ x_n}{y} \in R.\ \{x_1, \dots, x_n\} \subseteq S \right\}$$

$\widehat{R}$ applies the rules to the facts in S in all possible ways

$\widehat{R}(S)$: all conclusions we can draw in one step from hypotheses on S

We prove the least fixpoint of $\widehat{R}$ is I_R

(set of theorems of R)

Example

Strings of balanced parentheses $F = \{s \in \mathcal{L} \mid s \in \{(,)\}^*\}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} , \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} , \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \emptyset$$

$$\widehat{R}(S) = \{ \ \epsilon \in \mathcal{L} \ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \ldots \ x_n}{y} \in R. \ \{x_1, \ldots, x_n\} \subseteq S \right\}$$

Exercise

Strings of balanced parentheses $F = \{s \in \mathcal{L} \mid s \in \{(,)\}^*\}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} \,,\; \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} \,,\; \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{\ \epsilon \in \mathcal{L}\ \}$$

$$\widehat{R}(S) = \text{?}\, \{\ \epsilon \in \mathcal{L}\ ,\ () \in \mathcal{L}\ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \ldots\ x_n}{y} \in R.\ \{x_1, \ldots, x_n\} \subseteq S \right\}$$

Exercise

Strings of balanced parentheses $\quad F = \{ s \in \mathcal{L} \mid s \in \{(,)\}^* \}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} \, , \; \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} \, , \; \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{ \; () \in \mathcal{L} \; \}$$

$$\widehat{R}(S) = \text{?} \, \{ \, \epsilon \in \mathcal{L} \; , \; (()) \in \mathcal{L} \; , \; ()() \in \mathcal{L} \; \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \; \ldots \; x_n}{y} \in R. \; \{x_1, \ldots, x_n\} \subseteq S \right\}$$

Exercise

Strings of balanced parentheses $F = \{s \in \mathcal{L} \mid s \in \{(,)\}^*\}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}}, \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}}, \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{ \ )(\in \mathcal{L} \ \}$$

$$\widehat{R}(S) = ?\ \{ \ \epsilon \in \mathcal{L} \ , \ \ ()() \in \mathcal{L} \ , \ \ )()(\in \mathcal{L} \ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \ldots \ x_n}{y} \in R.\ \{x_1, \ldots, x_n\} \subseteq S \right\}$$

Example

Strings of balanced parentheses $F = \{s \in \mathcal{L} \mid s \in \{(,)\}^*\}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} \,,\; \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} \,,\; \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{\ )(\in \mathcal{L}\ ,\ \ () \in \mathcal{L}\ \}$$

$$\widehat{R}(S) = \{\ \epsilon \in \mathcal{L}\ ,\ ()() \in \mathcal{L}\ ,\ (()) \in \mathcal{L}\ ,\ )()(\in \mathcal{L}\ ,\ )(() \in \mathcal{L}\ ,\ ())(\in \mathcal{L}\ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \ldots\ x_n}{y} \in R.\ \{x_1, \ldots, x_n\} \subseteq S \right\}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$ $\quad$ $F = \{(n,m) \mid n,m \in \mathbb{N}\}$

$$R = \left\{ \ \frac{}{(0,0)} \ , \ \frac{}{(1,1)} \ , \ \frac{(n,h) \quad (n+1,k)}{(n+2, h+k)} \ \right\}$$

$$S = \emptyset$$

$$\widehat{R}(S) = \{ \ (0,0) \ , \ (1,1) \ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \dots \ x_n}{y} \in R. \ \{x_1, \dots, x_n\} \subseteq S \right\}$$

Exercise

$\mathbf{Pf}(\mathbb{N},\mathbb{N})$ $\qquad F = \{(n,m) \mid n,m \in \mathbb{N}\}$

$$R = \left\{ \frac{}{(0,0)} \;,\; \frac{}{(1,1)} \;,\; \frac{(n,h) \quad (n+1,k)}{(n+2,h+k)} \right\}$$

$$S = \{\ (2,1)\ \}$$

$$\widehat{R}(S) = \text{?}\ \{\ (0,0)\ ,\ (1,1)\ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \dots\ x_n}{y} \in R.\ \{x_1,\dots,x_n\} \subseteq S \right\}$$

Exercise

$\mathbf{Pf}(\mathbb{N},\mathbb{N})$ $\qquad F = \{(n,m) \mid n,m \in \mathbb{N}\}$

$$R = \left\{ \frac{}{(0,0)}\ ,\ \frac{}{(1,1)}\ ,\ \frac{(n,h) \quad (n+1,k)}{(n+2,h+k)} \right\}$$

$$S = \{\ (5,5)\ ,\ (6,8)\ \}$$

$$\widehat{R}(S) = \ ?\ \{\ (0,0)\ ,\ (1,1)\ ,\ (7,13)\ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \dots \ x_n}{y} \in R.\ \{x_1,\dots,x_n\} \subseteq S \right\}$$

ICO is monotone

TH. $\widehat{R}$ is monotone

proof. Take $S_1 \subseteq S_2$

we want to prove $\widehat{R}(S_1) \subseteq \widehat{R}(S_2)$

take $y \in \widehat{R}(S_1)$ we want to prove $y \in \widehat{R}(S_2)$

$$\Downarrow$$

$$\exists \frac{x_1 \ \dots \ x_n}{y} \in R \text{ with } \{x_1, \dots, x_n\} \subseteq S_1$$

$$\Downarrow \quad S_1 \subseteq S_2$$

$$\{x_1, \dots, x_n\} \subseteq S_2$$

$$\Downarrow$$

$$y \in \widehat{R}(S_2)$$

ICO is continuous

TH. $\widehat{R}$ is continuous (under some hypotheses)

proof. Take $\{S_i\}_{i\in\mathbb{N}}$ a chain in $\wp(F)$

We want to prove $\bigcup_{i\in\mathbb{N}} \widehat{R}(S_i) = \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$

$$\bigcup_{i\in\mathbb{N}} \widehat{R}(S_i) \subseteq \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$$ because $\widehat{R}$ is monotone

$$\bigcup_{i\in\mathbb{N}} \widehat{R}(S_i) \supseteq \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$$?

ICO is continuous (ctd)

$$\bigcup_{i\in\mathbb{N}} \widehat{R}(S_i) \supseteq \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$$

Take $y \in \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$ we want to prove $y \in \bigcup_{i\in\mathbb{N}} \widehat{R}(S_i)$

$\Downarrow$

$\exists \dfrac{x_1 \ \dots \ x_n}{y} \in R$ with $\{x_1, \dots, x_n\} \subseteq \bigcup_{i\in\mathbb{N}} S_i$

thus $\forall j \in [1,n].\ \exists k_j \in \mathbb{N}.\ x_j \in S_{k_j}$ take $k = \max\{k_1, \dots, k_n\}$

clearly $\{x_1, \dots, x_n\} \subseteq S_k$

thus $y \in \widehat{R}(S_k) \subseteq \bigcup_{i\in\mathbb{N}} \widehat{R}(S_i)$

possible iff each rule has finitely many premises

Provable theorems

I_R set of theorems in R

I_R^n theorems provable with a derivation of height at most n

$$I_R^0 = \emptyset$$

$$I_R^{n+1} = I_R^n \cup \widehat{R}(I_R^n)$$

provable using one more inference step

theorems provable with a derivation of height at most n

clearly $I_R = \bigcup_{n \in \mathbb{N}} I_R^n$

n-depth theorems

TH. Let $P(n) \triangleq I_R^n = \widehat{R}^n(\emptyset)$ $\forall n \in \mathbb{N}.\ P(n)$

proof. by mathematical induction

$P(0) \triangleq I_R^0 = \widehat{R}^0(\emptyset)$ $I_R^0 = \emptyset = \widehat{R}^0(\emptyset)$

$\forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)$ take a generic n
we assume $P(n) \triangleq I_R^n = \widehat{R}^n(\emptyset)$

we want to prove $P(n+1) \triangleq I_R^{n+1} = \widehat{R}^{n+1}(\emptyset)$

$I_R^{n+1} = I_R^n \cup \widehat{R}(I_R^n)$ by def

$= \widehat{R}^n(\emptyset) \cup \widehat{R}(\widehat{R}^n(\emptyset))$ by inductive hypothesis

$= \widehat{R}^n(\emptyset) \cup \widehat{R}^{n+1}(\emptyset)$ by def

$= \widehat{R}^{n+1}(\emptyset)$ because $\widehat{R}^n(\emptyset) \subseteq \widehat{R}^{n+1}(\emptyset)$

ICO's fixpoint

TH. $\mathit{fix}(\widehat{R}) = I_R$ (under some hypotheses)

each rule must have finitely many premises

proof.

by Kleene's fixpoint theorem we know that $\mathit{fix}(\widehat{R})$ exists

$$\mathit{fix}(\widehat{R}) = \bigcup_{n \in \mathbb{N}} \widehat{R}^n(\emptyset) \quad \text{by def}$$

$$= \bigcup_{n \in \mathbb{N}} I_R^n \quad \text{by previous result}$$

$$= I_R \quad \text{by def}$$

Example

Strings of balanced parentheses $F = \{s \in \mathcal{L} \mid s \in \{(,)\}^*\}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} \;,\; \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} \;,\; \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$\widehat{R}^0(\emptyset) = \emptyset$$

$$\widehat{R}^1(\emptyset) = \{\ \epsilon \in \mathcal{L}\ \}$$

$$\widehat{R}^2(\emptyset) = \{\ \epsilon \in \mathcal{L}\ ,\ () \in \mathcal{L}\ \}$$

$$\widehat{R}^3(\emptyset) = \{\ \epsilon \in \mathcal{L}\ ,\ () \in \mathcal{L}\ ,\ (()) \in \mathcal{L}\ ,\ ()() \in \mathcal{L}\ \}$$

$$\begin{aligned}\widehat{R}^4(\emptyset) = \{\ & \epsilon \in \mathcal{L}\ ,\ () \in \mathcal{L}\ ,\ (()) \in \mathcal{L}\ ,\ ()() \in \mathcal{L}\ ,\ ((())) \in \mathcal{L}\ , \\ & (()()) \in \mathcal{L}\ ,\ ()(()) \in \mathcal{L}\ ,\ (())() \in \mathcal{L}\ ,\ ()()() \in \mathcal{L}\ , \\ & (())()() \in \mathcal{L}\ ,\ ()()(()) \in \mathcal{L}\ ,\ (())(()) \in \mathcal{L}\ ,\ ()()()() \in \mathcal{L}\ \}\end{aligned}$$

Example

$\mathbf{Pf}(\mathbb{N},\mathbb{N})$ $\qquad F = \{(n,m) \mid n,m \in \mathbb{N}\}$

$$R = \left\{ \ \frac{}{(0,0)}\ ,\ \frac{}{(1,1)}\ ,\ \frac{(n,h) \quad (n+1,k)}{(n+2,h+k)} \ \right\}$$

$$\widehat{R}^0(\emptyset) = \emptyset$$

$$\widehat{R}^1(\emptyset) = \{\ (0,0)\ ,\ (1,1)\ \}$$

$$\widehat{R}^2(\emptyset) = \{\ (0,0)\ ,\ (1,1)\ ,\ (2,1)\ \}$$

$$\widehat{R}^3(\emptyset) = \{\ (0,0)\ ,\ (1,1)\ ,\ (2,1)\ ,\ (3,2)\ \}$$

$$\widehat{R}^4(\emptyset) = \{\ (0,0)\ ,\ (1,1)\ ,\ (2,1)\ ,\ (3,2)\ ,\ (4,3)\ \}$$

$$\widehat{R}^5(\emptyset) = \{\ (0,0)\ ,\ (1,1)\ ,\ (2,1)\ ,\ (3,2)\ ,\ (4,3)\ ,\ (5,5)\ \}$$

$$\widehat{R}^6(\emptyset) = \{\ (0,0)\ ,\ (1,1)\ ,\ (2,1)\ ,\ (3,2)\ ,\ (4,3)\ ,\ (5,5)\ ,\ (6,8)\ \}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$ $\qquad F = \{(n, m) \mid n, m \in \mathbb{N}\}$

(111,101) (110,100) (101,91) (100,?)

$$R_M \triangleq \left\{ \quad \frac{}{(n, n-10)}\; n > 100 \quad , \quad \frac{(n+11, k) \quad (k, m)}{(n, m)}\; n \leq 100 \right\}$$

(100,91) (99,?)

$\hat{R}^0_M(\emptyset) = \emptyset$

$\hat{R}^1_M(\emptyset) = \{\ (n, n-10) \mid n > 100\ \} = \{\ (101,91), (102,92), \ \ldots\ \}$

$\hat{R}^2_M(\emptyset) = \{\ (100,91), (101,91), (102,92), \ \ldots\ \}$

$\hat{R}^3_M(\emptyset) = \{\ (99,91), (100,91), (101,91), (102,92), \ \ldots\ \}$

$\ldots$

$\hat{R}^{11}_M(\emptyset) = \{\ (91,91), (92,91), \ \ldots, (99,91), (100,91), (101,91), (102,92), \ \ldots\ \}$

$\hat{R}^{12}_M(\emptyset) = \{\ (80,91), (81,91), \ \ldots, (99,91), (100,91), (101,91), (102,92), \ \ldots\ \}$

$\ldots$

$\hat{R}^{20}_M(\emptyset) = \{\ (0,91), (1,91), \ \ldots, (99,91), (100,91), (101,91), (102,92), \ \ldots\ \}$

fixpoint reached! (total function)