MPP 2025/26 (0077A, 9CFU)

Models for Programming Paradigms

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08b - Kleene's fixed point theorem

Monotone functions

Monotone function

$(D, \sqsubseteq_D)$ PO $\quad (E, \sqsubseteq_E)$ PO $\quad f : D \to E$

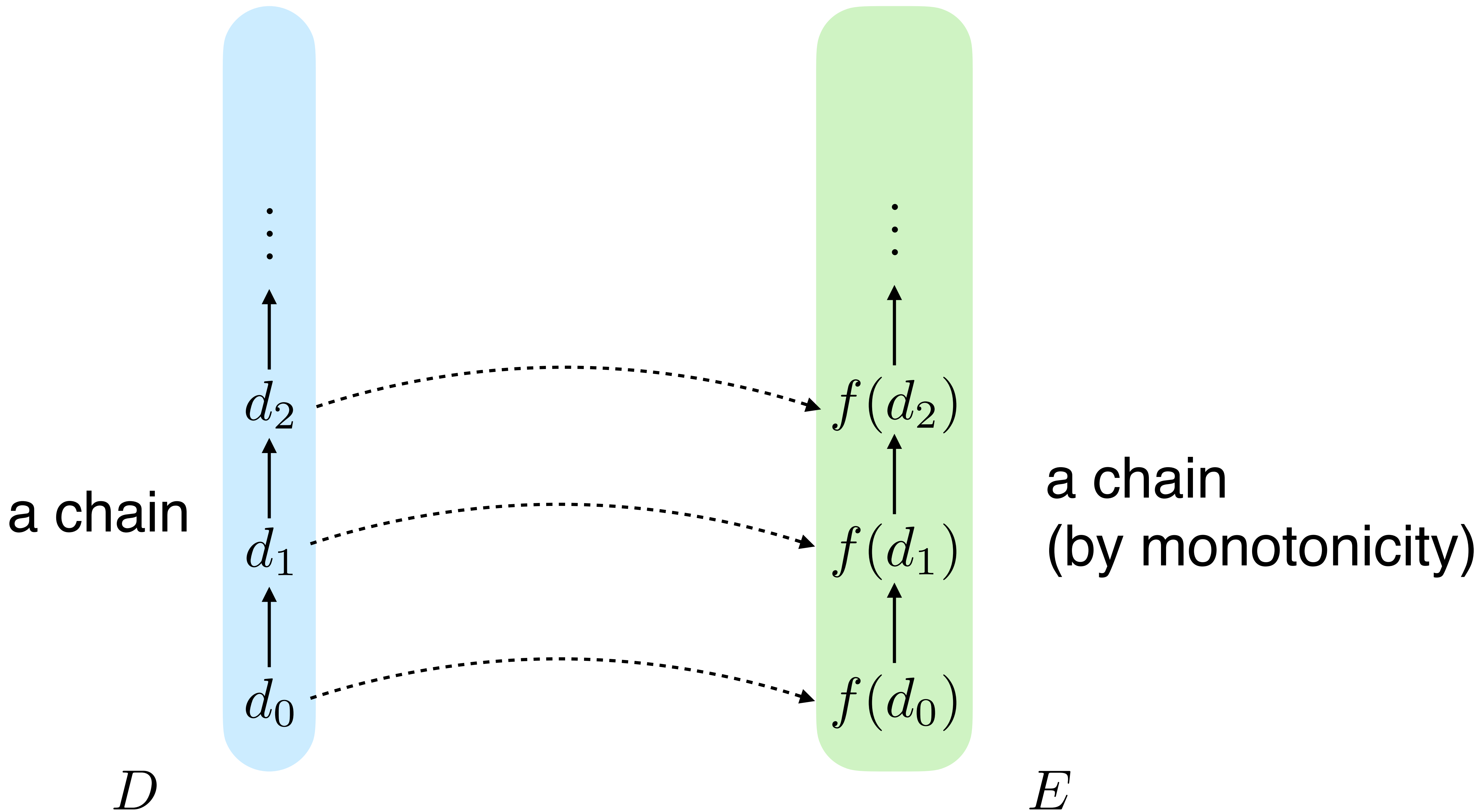
f is monotone if $\forall d_1, d_2 \in D.\ d_1 \sqsubseteq_D d_2 \Rightarrow f(d_1) \sqsubseteq_E f(d_2)$

Monotone = Order preserving

$$\left.\begin{array}{l} \{d_i\}_{i \in \mathbb{N}} \text{ a chain in } D \\ f \text{ monotone} \end{array}\right\} \Rightarrow \{f(d_i)\}_{i \in \mathbb{N}} \text{ a chain in } E$$

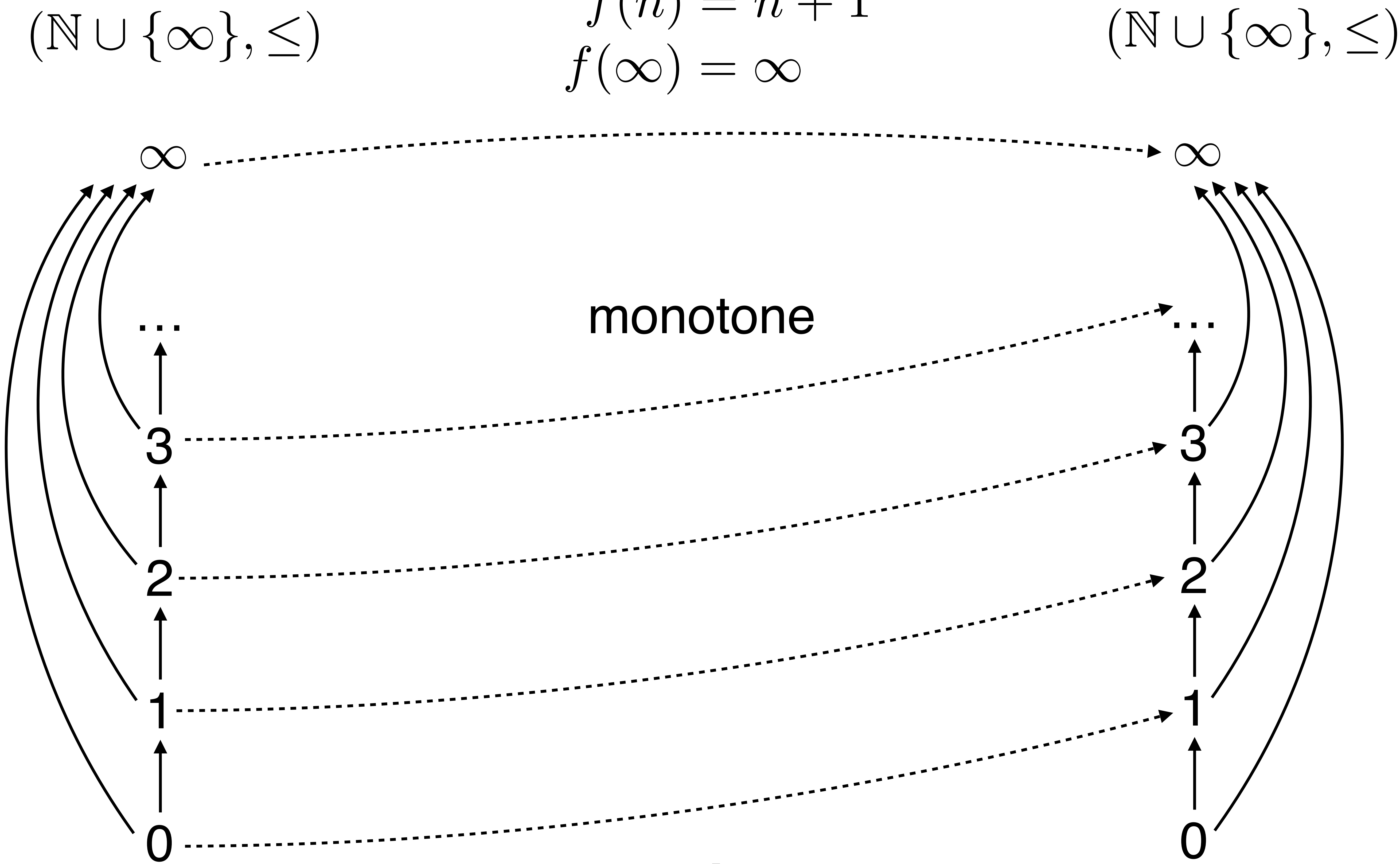
When $D = E$ we say $f : D \to D$ is a function on D

Monotonicity illustrated



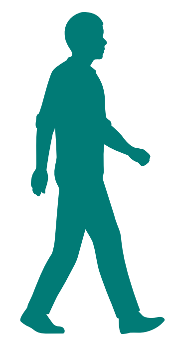
Example

$$f(n) = n + 1$$
$$f(\infty) = \infty$$



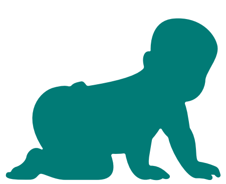
Skill levels

advanced
multitasking



intermediate
does many things

novice
does one thing



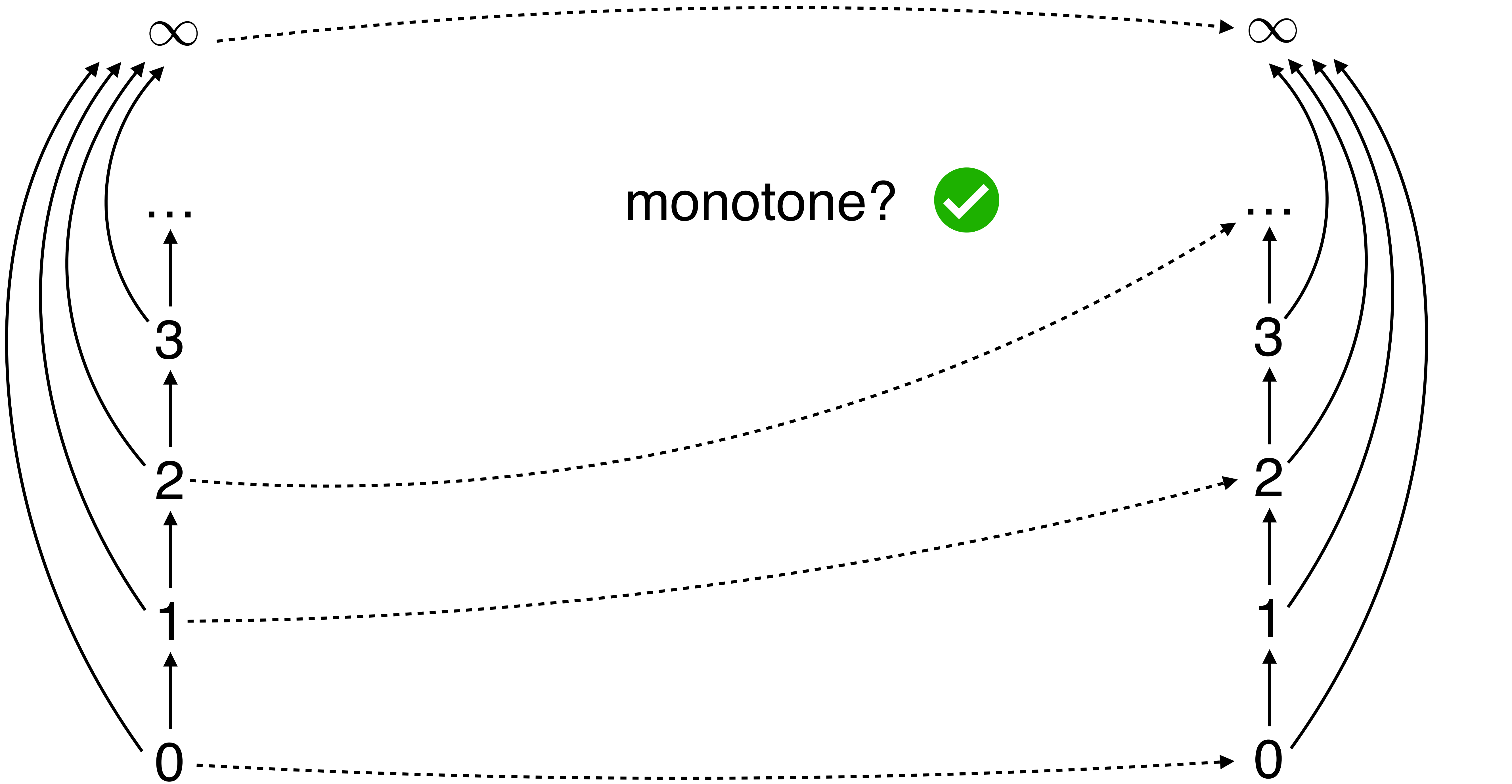
beginner
does nothing

Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$

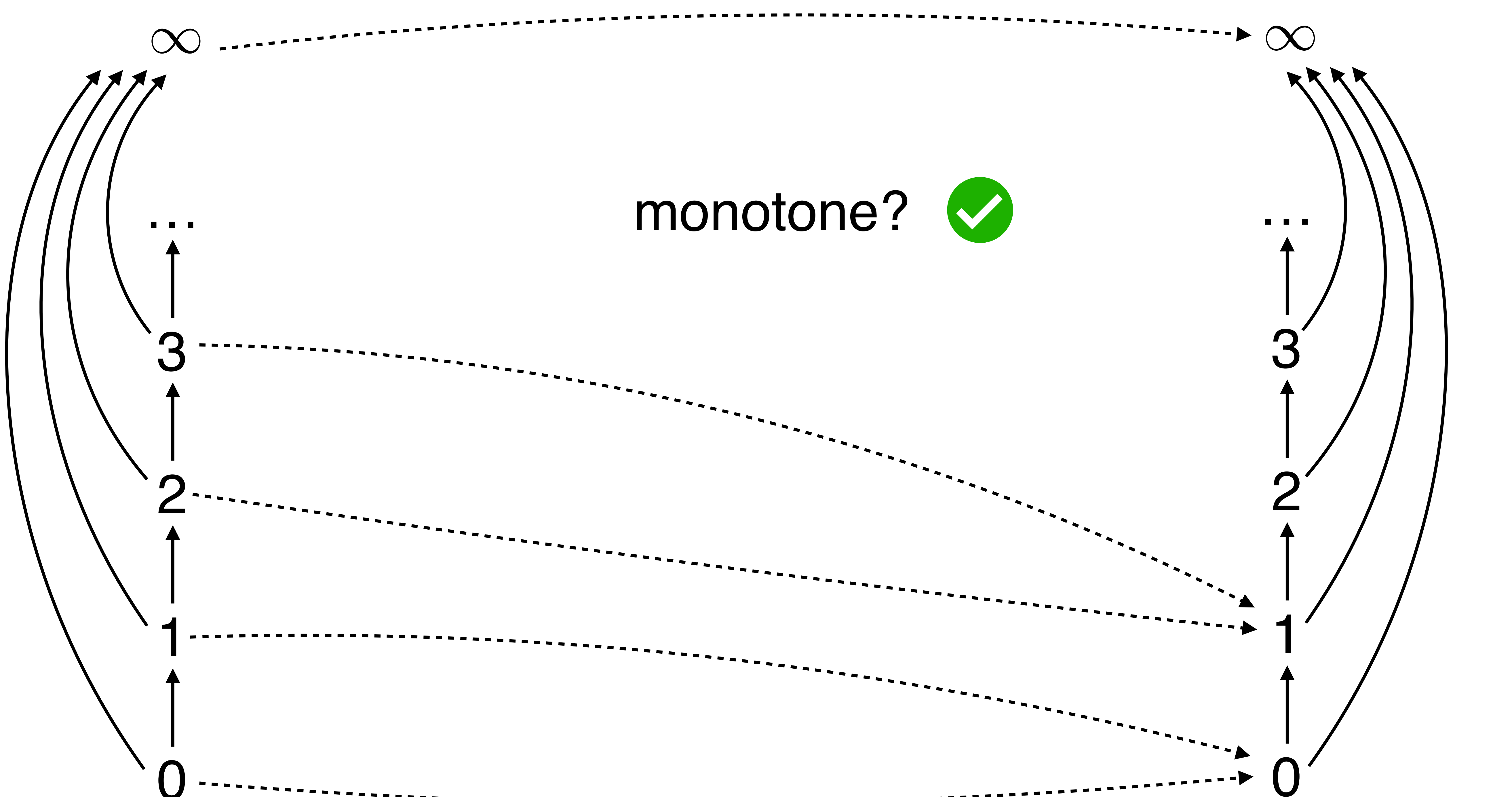


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n/2$$
$$f(\infty) = \infty$$

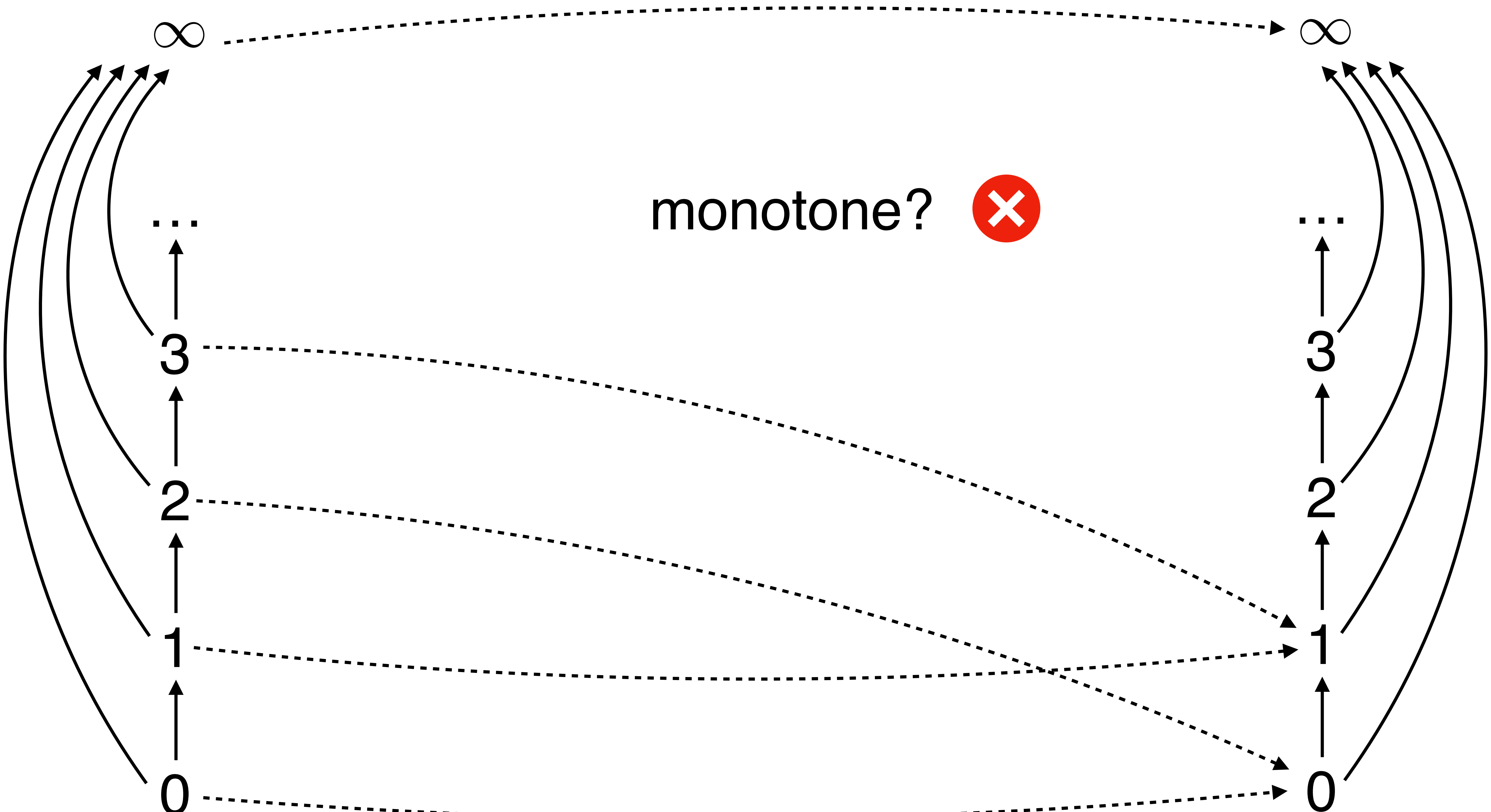
$(\mathbb{N} \cup \{\infty\}, \leq)$



Exercise

$$f(n) = n \;\%\; 2$$
$$f(\infty) = \infty$$

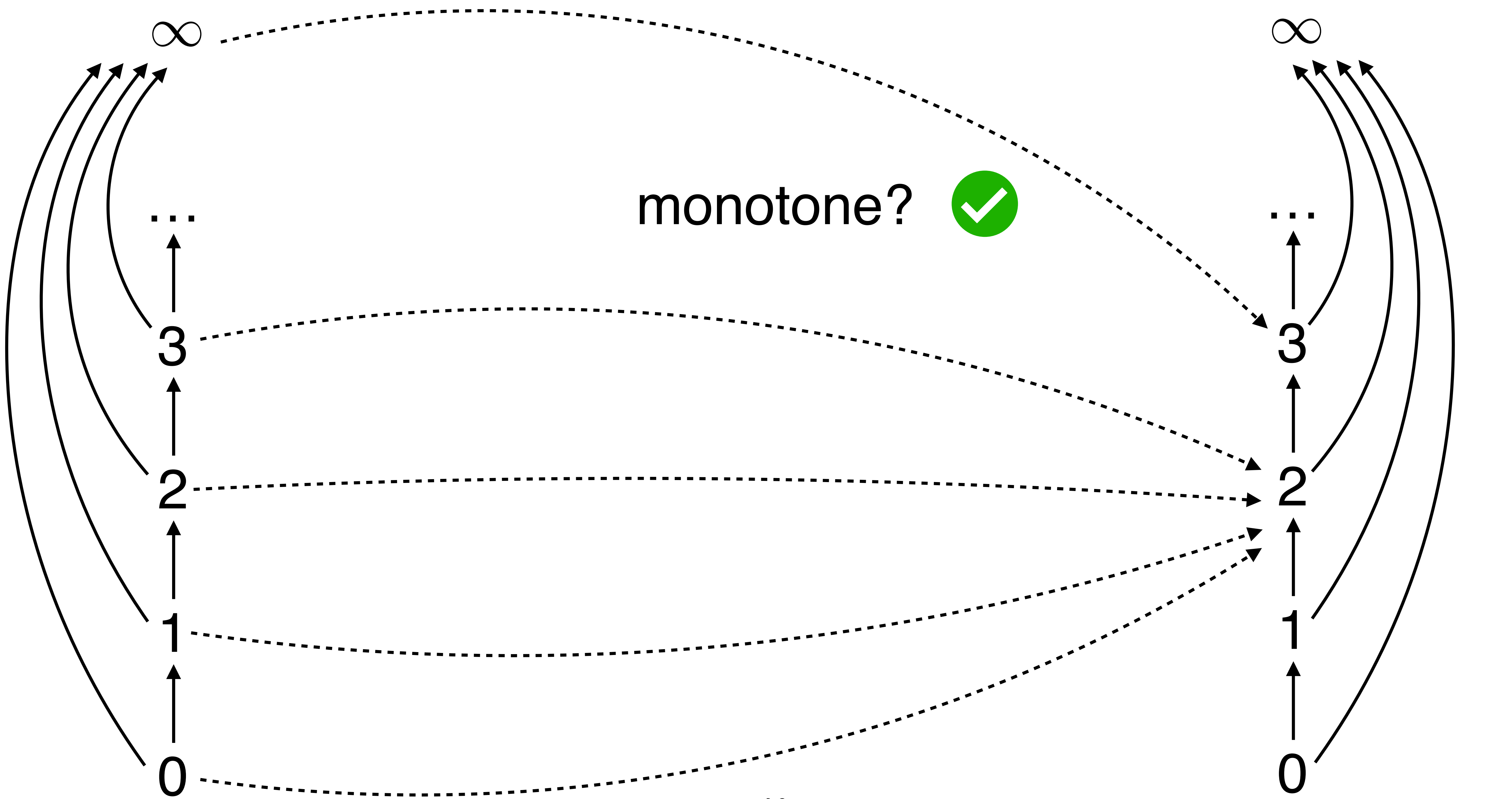
$(\mathbb{N} \cup \{\infty\}, \leq)$ $(\mathbb{N} \cup \{\infty\}, \leq)$



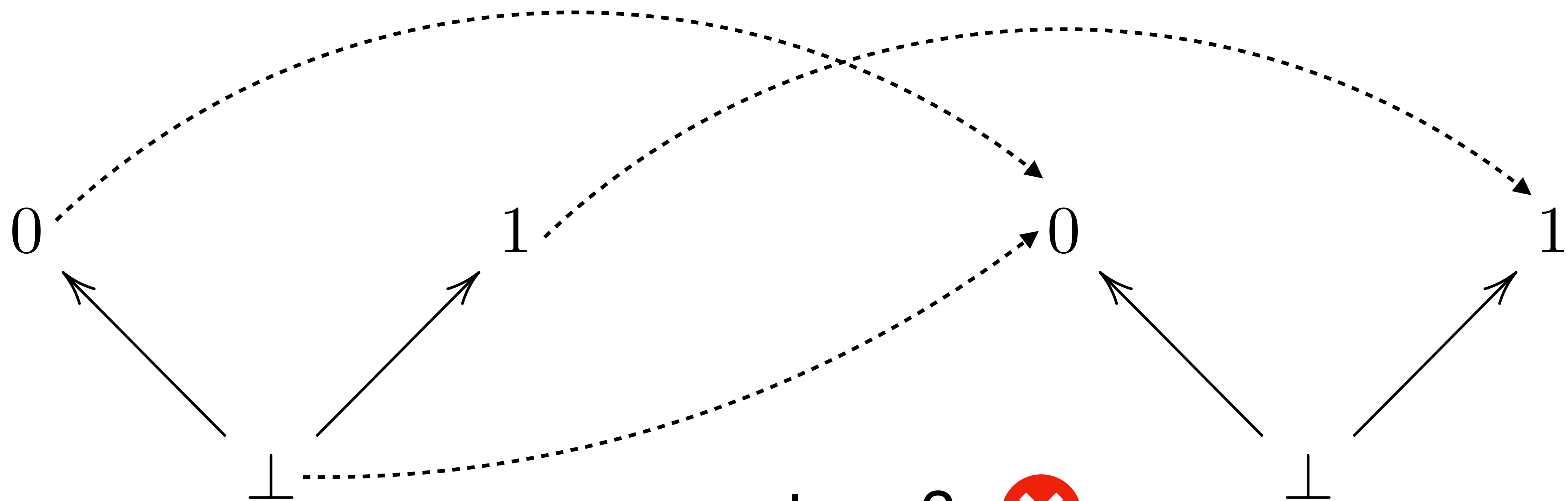
Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$ $\quad f(n) = 2$ $\quad (\mathbb{N} \cup \{\infty\}, \leq)$

$f(\infty) = 3$



Exercise



monotone?

D $\qquad$ $f : D \to D$ $\qquad$ D

$$f(\perp) = f(0) = 0$$
$$f(1) = 1$$

$\perp \sqsubseteq 1$ $\qquad$ $f(\perp) = 0 \not\sqsubseteq 1 = f(1)$

Exercise

$$(\wp(\mathbb{N}), \subseteq) \qquad f(S) = \{\, m \in \mathbb{N} \mid \exists n \in S, m \leq n \,\} \qquad (\wp(\mathbb{N}), \subseteq)$$

monotone? ✓

Exercise

$$(\wp(\mathbb{N}), \subseteq) \quad f(S) = \{\, m \in \mathbb{N} \mid \forall n \in S, n < m \,\} \quad (\wp(\mathbb{N}), \subseteq)$$

monotone?

Composition

TH. Any composition of monotone functions is monotone

$(D, \sqsubseteq_D)$ PO
$(E, \sqsubseteq_E)$ PO
$(F, \sqsubseteq_F)$ PO

$f : D \to E$ monotone
$g : E \to F$ monotone

$\Rightarrow$ $h = g \circ f : D \to F$ monotone

proof. we need to prove $\forall x, y \in D.\ x \sqsubseteq_D y \Rightarrow h(x) \sqsubseteq_F h(y)$

take $x \sqsubseteq_D y$

we want to prove $h(x) \sqsubseteq_F h(y)$

then $f(x) \sqsubseteq_E f(y)$ because f is monotone

then $\underbrace{g(f(x))}_{= h(x)} \sqsubseteq_F \underbrace{g(f(y))}_{= h(y)}$ because g is monotone

Continuous functions

Continuous function

$(D, \sqsubseteq_D)$ CPO $(E, \sqsubseteq_E)$ CPO $f : D \to E$ monotone

f is continuous if $\forall \{d_i\}_{i\in\mathbb{N}}.\ f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \bigsqcup_{i\in\mathbb{N}} f(d_i)$

chain

limit in D

limit in E

Continuous = limit preserving

Continuity illustrated

$$
\begin{tikzcd}
\text{lub} & \bigsqcup_{i\in\mathbb{N}} d_i \arrow[r, dashed] & f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) & \text{u.b. (by monotonicity)} \\
& & \bigsqcup_{i\in\mathbb{N}} f(d_i) \arrow[u, "=?"'] & \text{lub} \\
& \vdots & \vdots & \\
& d_2 \arrow[u] \arrow[r, dashed] & f(d_2) \arrow[u] & \\
\text{a chain} & d_1 \arrow[u] \arrow[r, dashed] & f(d_1) \arrow[u] & \text{a chain (by monotonicity)} \\
& d_0 \arrow[u] \arrow[r, dashed] & f(d_0) \arrow[u] &
\end{tikzcd}
$$

Continuity illustrated

$$
\begin{tikzcd}
\text{lub} & \bigsqcup_{i\in\mathbb{N}} d_i \arrow[rr, dashed] & & f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) \\
& & & \bigsqcup_{i\in\mathbb{N}} f(d_i) \arrow[u, "=?"'] \\
& \vdots & & \vdots \\
& d_2 \arrow[u] \arrow[rr, dashed] & & f(d_2) \arrow[u] \\
\text{a chain} & d_1 \arrow[u] \arrow[rr, dashed] & & f(d_1) \arrow[u] \\
& d_0 \arrow[u] \arrow[rr, dashed] & & f(d_0) \arrow[u]
\end{tikzcd}
$$

$$\bigsqcup_{i\in\mathbb{N}} f(d_i) \sqsubseteq_E f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$$

follows from monotonicity (and CPO)

Continuity illustrated

$$
\begin{tikzcd}
\text{lub} & \bigsqcup_{i\in\mathbb{N}} d_i \arrow[rr, dashed, bend left] & & f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) \\
& & & \bigsqcup_{i\in\mathbb{N}} f(d_i) \arrow[u, "=?"'] \\
& \vdots & & \vdots \\
& d_2 \arrow[u] \arrow[rr, dashed, bend left] & & f(d_2) \arrow[u] \\
\text{a chain} & d_1 \arrow[u] \arrow[rr, dashed, bend left] & & f(d_1) \arrow[u] \\
& d_0 \arrow[u] \arrow[rr, dashed, bend left] & & f(d_0) \arrow[u]
\end{tikzcd}
$$

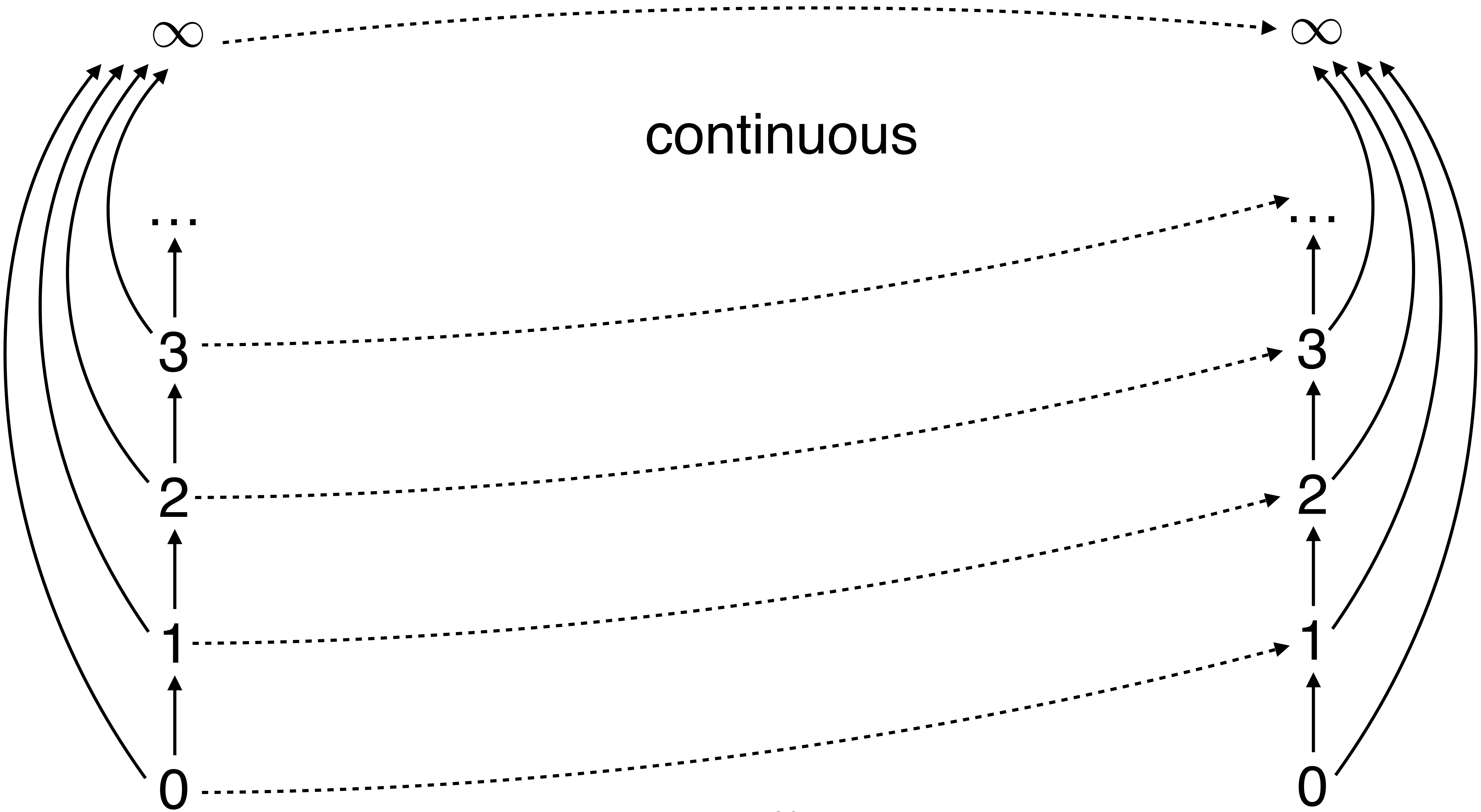
$$\bigsqcup_{i\in\mathbb{N}} f(d_i) \overset{?}{\sqsupseteq}_E f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$$

continuity

Example

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$ $(\mathbb{N} \cup \{\infty\}, \leq)$

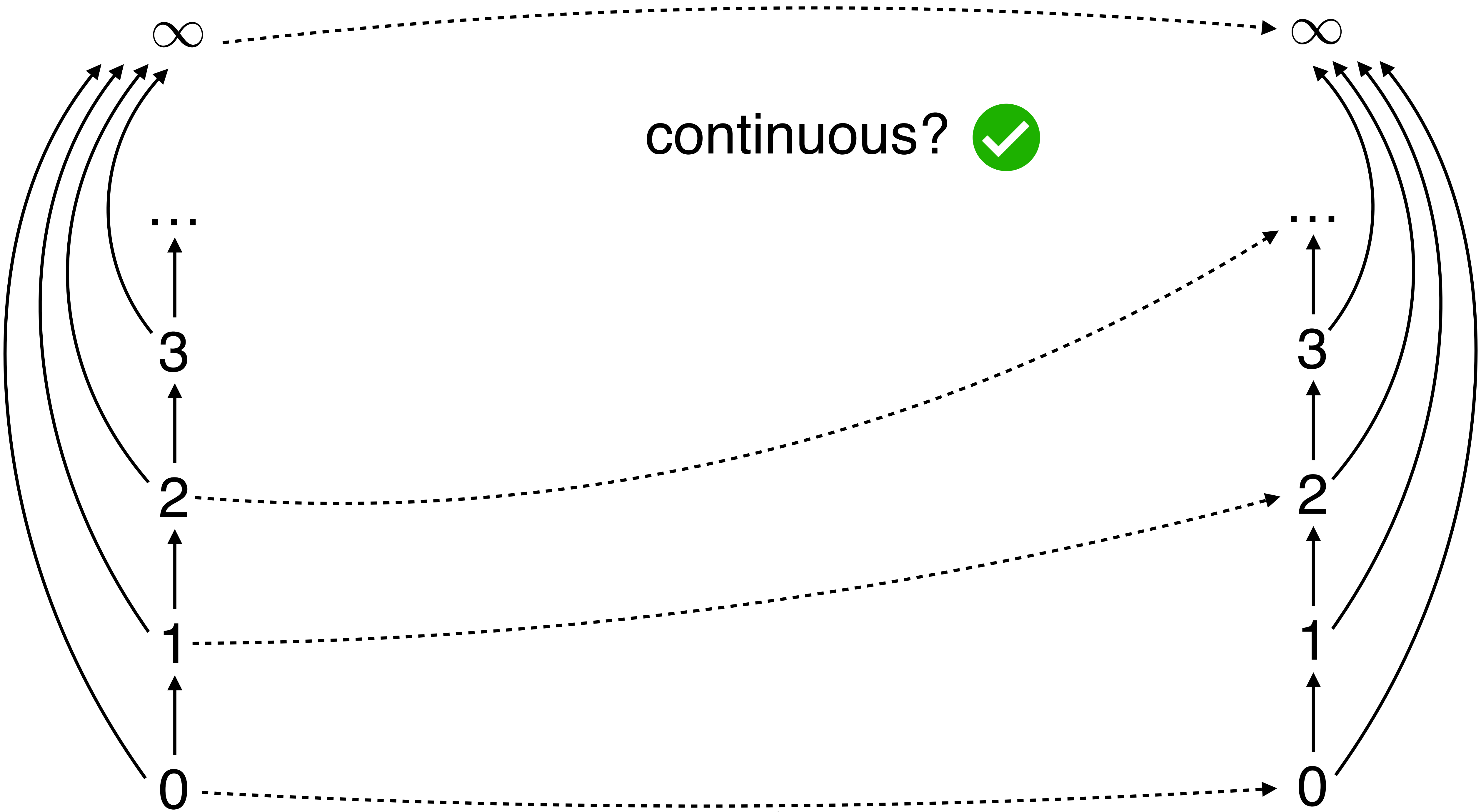


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$

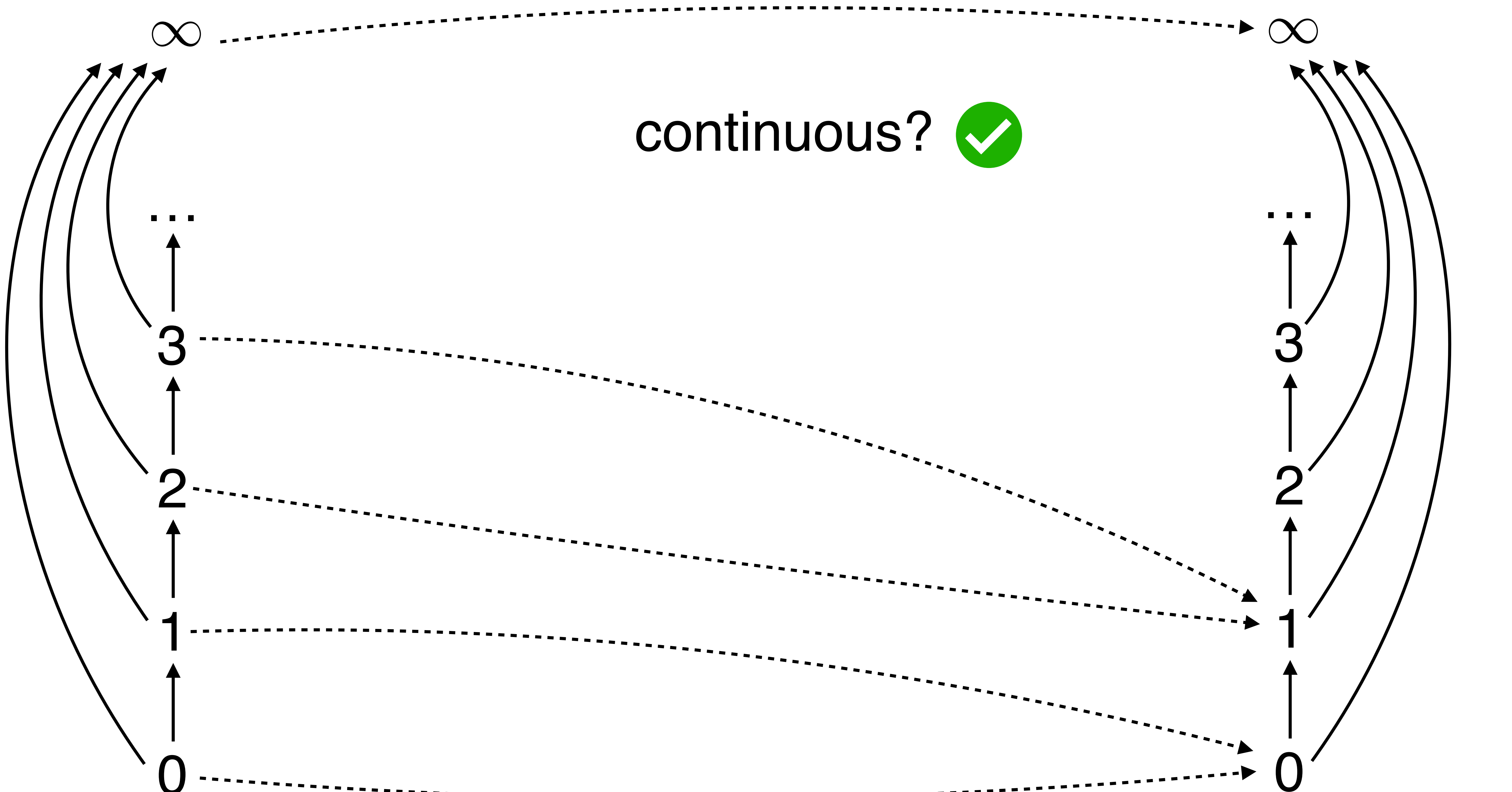


Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n/2$$
$$f(\infty) = \infty$$

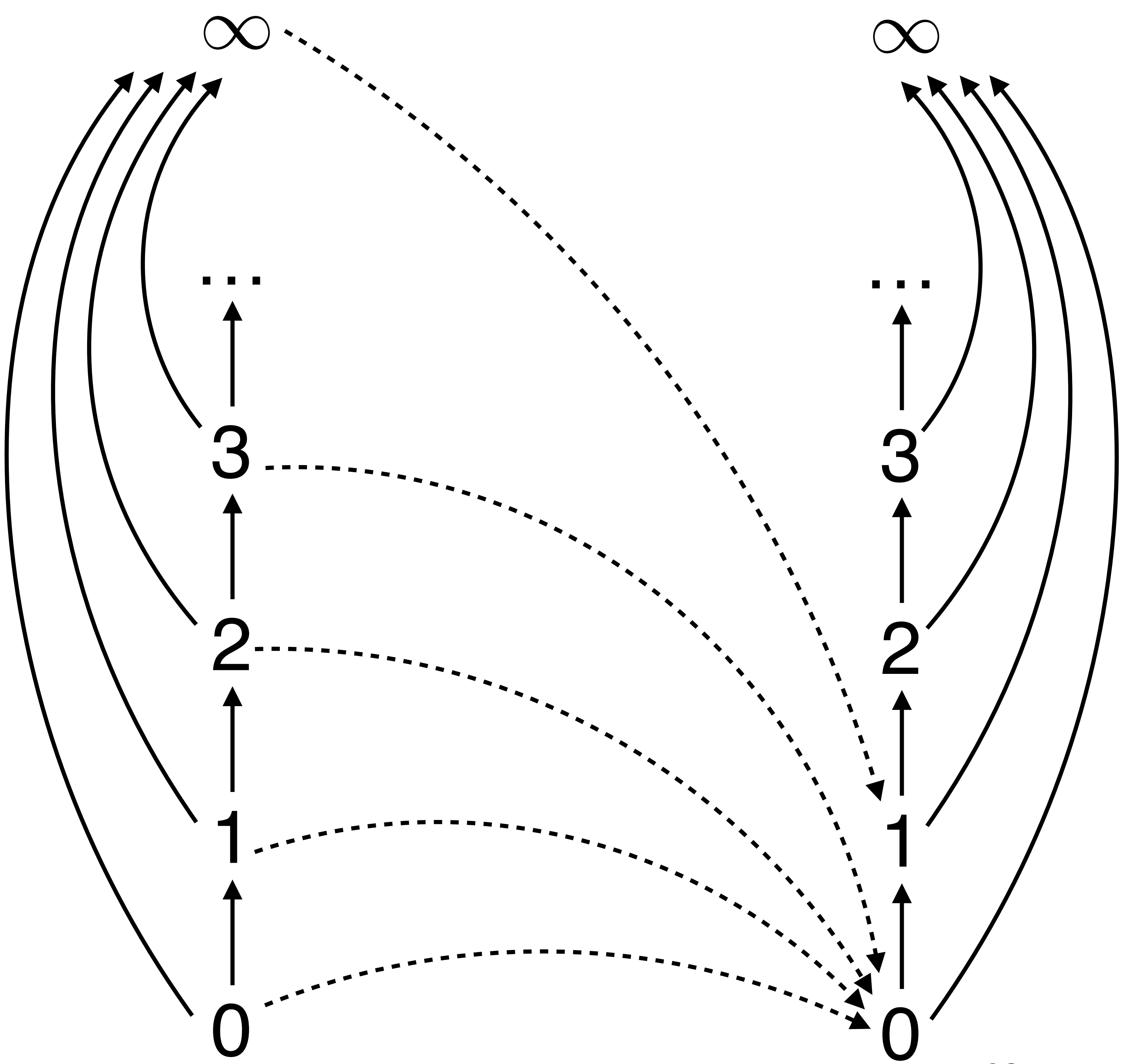
$(\mathbb{N} \cup \{\infty\}, \leq)$



Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

monotone function, not continuous



$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{N} \\ 1 & \text{if } x = \infty \end{cases}$$

$$d_i = 2 \cdot i \qquad \bigsqcup_{i \in \mathbb{N}} d_i = \infty$$

$$f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = f(\infty) = 1$$

$$f(d_i) = 0$$

$$\bigsqcup_{i \in \mathbb{N}} f(d_i) = \bigsqcup_{i \in \mathbb{N}} 0 = 0$$

Exercise

$$(\wp(\mathbb{N}), \subseteq) \quad f(S) = \{\, m \in \mathbb{N} \mid \exists n \in S, m \leq n \,\} \quad (\wp(\mathbb{N}), \subseteq)$$

continuous? ✔

Exercise

$(\wp(\mathbb{N}), \subseteq)$

$$f(S) = \begin{cases} S & \text{if } S \text{ finite} \\ \mathbb{N} & \text{otherwise} \end{cases}$$

$(\wp(\mathbb{N}), \subseteq)$

continuous? ❌

Lemma

$(D, \sqsubseteq_D)$ CPO
no infinite chains

$(E, \sqsubseteq_E)$ PO

$f : D \to E$
monotone

$\Rightarrow$

f
continuous

proof. Take a chain $\{d_i\}_{i \in \mathbb{N}}$

$$\{d_i\}_{i\in\mathbb{N}} \text{ is finite} \quad \Rightarrow \quad \exists k \in \mathbb{N}.\ \bigsqcup_{i\in\mathbb{N}} d_i = d_k$$

$$\Downarrow$$

$$\{f(d_i)\}_{i\in\mathbb{N}} \text{ is finite} \quad \Rightarrow \bigsqcup_{i\in\mathbb{N}} f(d_i) = f(d_k) = f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$$

Composition

TH. Any composition of continuous functions is continuous

$(D, \sqsubseteq_D)$ CPO
$(E, \sqsubseteq_E)$ CPO
$(F, \sqsubseteq_F)$ CPO

$f : D \to E$ continuous
$g : E \to F$ continuous

$\Rightarrow$ $h = g \circ f : D \to F$ continuous

proof. take a chain $\{d_i\}_{i \in \mathbb{N}}$

we need to prove $h\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = \bigsqcup_{i\in\mathbb{N}} h(d_i)$

$$h\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) = g\left(f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)\right) = g\left(\bigsqcup_{i\in\mathbb{N}} f(d_i)\right) = \bigsqcup_{i\in\mathbb{N}} g(f(d_i))$$

$$= \bigsqcup_{i\in\mathbb{N}} h(d_i)$$

Kleene’s fixpoint theorem

Repeated application

$$f : D \to D$$

$$f^0(d) \triangleq d$$

$$f^{n+1}(d) \triangleq f(f^n(d))$$

$$f^n(d) = \overbrace{f(\cdots(f}^{n \text{ times}}(d))\cdots)$$

$$f^n : D \to D$$

Lemma

$(D, \sqsubseteq)$ $\mathrm{PO}_\perp$ $f : D \to D$ monotone $\Rightarrow$ $\{f^n(\perp)\}_{n\in\mathbb{N}}$ is a chain

proof. we need to prove $\forall n \in \mathbb{N}.\ f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

by mathematical induction $P(n) \triangleq f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

$P(0) \triangleq f^0(\perp) \sqsubseteq f^1(\perp)$ $f^0(\perp) = \perp \sqsubseteq f^1(\perp)$

$\forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)$ take a generic n

assume $P(n) \triangleq f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

we want to prove $P(n+1) \triangleq f^{n+1}(\perp) \sqsubseteq f^{n+2}(\perp)$

$$f^n(\perp) \sqsubseteq f^{n+1}(\perp)$$
$$\Downarrow$$
$$f^{n+1}(\perp) = f(f^n(\perp)) \sqsubseteq f(f^{n+1}(\perp)) = f^{n+2}(\perp)$$

Towards Kleene's theo.

when $(D, \sqsubseteq)$ is a $\mathrm{CPO}_\perp$

then $\{f^n(\perp)\}_{n\in\mathbb{N}}$ is a chain

it must have a limit

$\{f^n(d)\}_{n\in\mathbb{N}}$
not necessarily
a chain!

Kleene's fix point theorem states that
if f is continuous, then the limit of the above chain
is the least fixpoint of f

Pre-fixpoints

$(D, \sqsubseteq)$ PO $\qquad f : D \rightarrow D$ monotone

fixpoint $\qquad p \in D \qquad f(p) = p$

pre-fixpoint $\qquad p \in D \qquad f(p) \sqsubseteq p$

Clearly any fixpoint is also a pre-fixpoint

Kleene's theorem

$(D, \sqsubseteq)$ $\mathrm{CPO}_\perp$ $\quad f : D \to D$ continuous

let $\mathit{fix}(f) \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp)$

1. $\mathit{fix}(f)$ is a fix point of f

$$f(\mathit{fix}(f)) = \mathit{fix}(f)$$

2. $\mathit{fix}(f)$ is the least pre-fixpoint of f

$$\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d$$

if d is a pre-fixpoint then $\mathit{fix}(f)$ is smaller than d

Kleene's theorem: 1

1. $f(\mathit{fix}(f)) = \mathit{fix}(f)$

proof.

$$f(\mathit{fix}(f)) = f\left(\bigsqcup_{n\in\mathbb{N}} f^n(\bot)\right)$$ by def of fix

$$= \bigsqcup_{n\in\mathbb{N}} f(f^n(\bot))$$ by continuity

$$= \bigsqcup_{n\in\mathbb{N}} f^{n+1}(\bot)$$ by def of f^n

$$= \bigsqcup_{n\in\mathbb{N}} f^n(\bot)$$ by prefix independence of limits

$$= \mathit{fix}(f)$$ by def of fix

Kleene's theorem: 2

2. $\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d$

proof.

we prove that any pre-fixpoint is an upper bound of the chain

$$\{f^n(\perp)\}_{n \in \mathbb{N}}$$

by definition $\mathit{fix}(f)$ is the lub of the same chain

and thus smaller than any other upper bound

Kleene's theorem: 2

2. $\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d$

take any $d \in D$ such that $f(d) \sqsubseteq d$

we prove $\forall n \in \mathbb{N}.\ f^n(\bot) \sqsubseteq d$ (d is an upper bound)

$P(n) \triangleq f^n(\bot) \sqsubseteq d$ by mathematical induction

$P(0) \triangleq f^0(\bot) \sqsubseteq d$ $\quad f^0(\bot) = \bot \sqsubseteq d$

$\forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)$

take a generic n

assume $P(n) \triangleq f^n(\bot) \sqsubseteq d$

we want to prove $P(n+1) \triangleq f^{n+1}(\bot) \sqsubseteq d$

$$f^n(\bot) \sqsubseteq d$$

$$\Downarrow \text{(monot.)}$$

$$f^{n+1}(\bot) \overset{\text{(by def)}}{=} f(f^n(\bot)) \sqsubseteq f(d) \overset{\text{(pre-fixpoint)}}{\sqsubseteq} d$$

Example

$n = 2 \cdot n$ $(\mathbb{N} \cup \{\infty\}, \leq)$ $\perp = 0$ $\mathrm{CPO}_\perp$

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

monotone? ok
continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 2 \cdot 0 = 0$$

fixpoint reached!

Example

$n = n + 1$ $\qquad (\mathbb{N} \cup \{\infty\}, \leq)$ $\qquad \bot = 0$ $\qquad \mathrm{CPO}_\bot$

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

monotone? ok
continuous? ok

$$f^0(0) = 0$$
$$f^1(0) = f(0) = 0 + 1 = 1$$
$$f^2(0) = f(f^1(0)) = f(1) = 1 + 1 = 2$$
$$f^3(0) = f(f^2(0)) = f(2) = 2 + 1 = 3$$

$$\bigsqcup_{n \in \mathbb{N}} f^n(0) = \bigsqcup_{n \in \mathbb{N}} n = \infty$$ fixpoint

Example

$X = X \cap \{1\}$ $\quad (\wp(\mathbb{N}), \subseteq)$ $\quad \bot = \emptyset$ $\quad \mathrm{CPO}_\bot$

$$f(X) = X \cap \{1\}$$

monotone? ok
continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cap \{1\} = \emptyset$$

fixpoint reached!

Example

$X = \mathbb{N} \setminus X$ $\quad (\wp(\mathbb{N}), \subseteq)$ $\quad \bot = \emptyset$ $\quad \mathrm{CPO}_\bot$

$$f(X) = \mathbb{N} \setminus X$$

monotone? NO

the larger X the smaller $f(X)$

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \mathbb{N} \setminus \emptyset = \mathbb{N}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\mathbb{N}) = \mathbb{N} \setminus \mathbb{N} = \emptyset$$

not a chain!

Example

$X = X \cup \{1\}$ $(\wp(\mathbb{N}), \subseteq)$ $\perp = \emptyset$ $\mathrm{CPO}_\perp$

$f(X) = X \cup \{1\}$

monotone? ok
continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cup \{1\} = \{1\}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\{1\}) = \{1\} \cup \{1\} = \{1\}$$

fixpoint reached!