

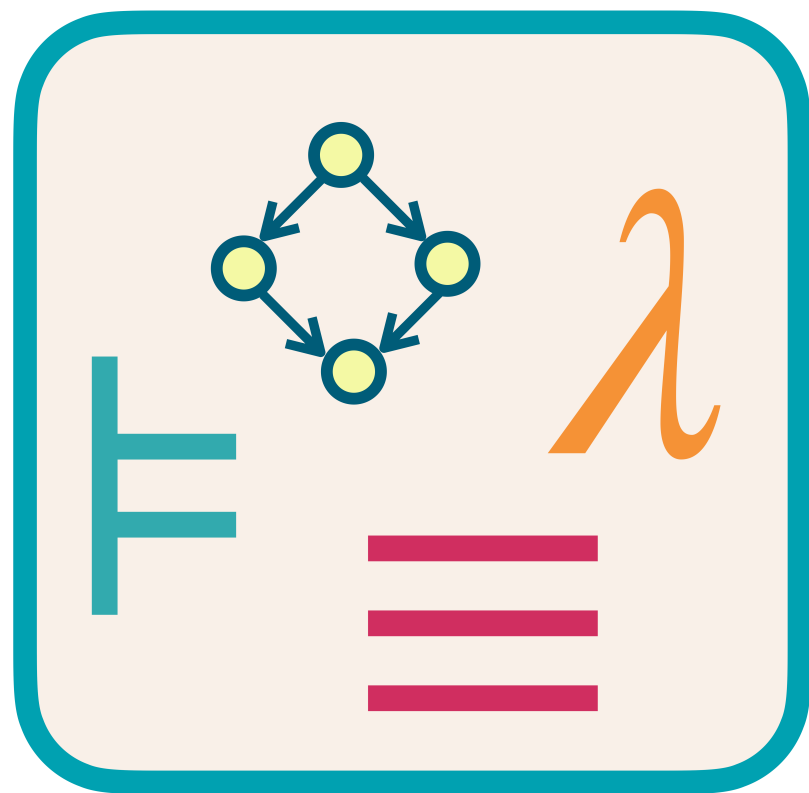
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Models for Programming Paradigms

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05c - Rule Induction

Induction on derivations

Derivations

Given a logical system R , a **derivation in R** , is written

$$d \Vdash_R y$$

where

- either $d = \left(\frac{}{y} \right) \in R$ is an axiom of R ;
- or $d = \left(\frac{d_1, \dots, d_n}{y} \right)$ for some derivations $d_1 \Vdash_R x_1, \dots, d_n \Vdash_R x_n$ such that $\left(\frac{x_1, \dots, x_n}{y} \right) \in R$ is an inference rule of R .

$$D_R \triangleq \{d \mid d \Vdash_R y\}$$

Immediate subderivation

Take $A = D_R$

$$\prec = \left\{ \left(d_i, \frac{d_1, \dots, d_n}{y} \right) \mid d_1 \Vdash_R x_1, \dots, d_n \Vdash_R x_n, \left(\frac{x_1, \dots, x_n}{y} \right) \in R \right\}$$

(immediate subderivation relation)

Example

$$R = \left\{ \frac{}{N \longrightarrow n}, \frac{E_0 \longrightarrow n_0 \quad E_1 \longrightarrow n_1}{E_0 \oplus E_1 \longrightarrow n_0 + n_1}, \frac{E_0 \longrightarrow n_0 \quad E_1 \longrightarrow n_1}{E_0 \otimes E_1 \longrightarrow n_0 \cdot n_1} \right\}$$

$$\frac{}{2 \longrightarrow 2} \prec \frac{\frac{}{1 \longrightarrow 1} \quad \frac{}{2 \longrightarrow 2}}{(1 \oplus 2) \longrightarrow 3} \prec \frac{\frac{\frac{}{1 \longrightarrow 1} \quad \frac{}{2 \longrightarrow 2}}{(1 \oplus 2) \longrightarrow 3} \quad \frac{\frac{}{3 \longrightarrow 3} \quad \frac{}{4 \longrightarrow 4}}{(3 \oplus 4) \longrightarrow 7}}{(1 \oplus 2) \otimes (3 \oplus 4) \longrightarrow 21}$$

Lemma

$D_R, \prec$ is w.f.

Let $height : D_R \rightarrow \mathbb{N}$ defined as:

$$\begin{aligned} height\left(\frac{}{y}\right) &\triangleq 1 && \text{if } (\frac{}{y}) \in R \\ height\left(\frac{d_1, \dots, d_n}{y}\right) &\triangleq 1 + \max_{i \in [1, n]} height(d_i) && \text{if } d_1 \Vdash_R x_1, \dots, d_n \Vdash_R x_n, (\frac{x_1, \dots, x_n}{y}) \in R \end{aligned}$$

By definition, if $d \prec d'$ then $height(d) < height(d')$

Any descending chain in $\prec$ induces a descending chain in $<$

Since $<$ is w.f., so is $\prec$

Induction on derivation principle

$$\frac{\forall \frac{x_1, \dots, x_n}{y} \in R. \forall d_1 \Vdash_R x_1, \dots, d_1 \Vdash_R x_n. (P(d_1) \wedge \dots \wedge P(d_n)) \Rightarrow P(\frac{d_1, \dots, d_n}{y})}{\forall d. P(d)}$$

Corollary

$$D_R, \prec^+ \text{ is w.f.}$$

Because $\prec^+$ is the transitive closure of a w.f. relation

Example

$$R = \left\{ \frac{}{N \longrightarrow n}, \frac{E_0 \longrightarrow n_0 \quad E_1 \longrightarrow n_1}{E_0 \oplus E_1 \longrightarrow n_0 + n_1}, \frac{E_0 \longrightarrow n_0 \quad E_1 \longrightarrow n_1}{E_0 \otimes E_1 \longrightarrow n_0 \cdot n_1} \right\}$$

$$\frac{}{2 \longrightarrow 2} \prec^+ \frac{\frac{\frac{}{1 \longrightarrow 1} \quad \frac{}{2 \longrightarrow 2}}{(1 \oplus 2) \longrightarrow 3} \quad \frac{\frac{}{3 \longrightarrow 3} \quad \frac{}{4 \longrightarrow 4}}{(3 \oplus 4) \longrightarrow 7}}{(1 \oplus 2) \otimes (3 \oplus 4) \longrightarrow 21}$$

Rule induction

Typical properties

It is very often the case that the property of a derivation is only concerned with the conclusion of the derivation

$$d \Vdash_R y \quad \Rightarrow \quad P(d) \Leftrightarrow Q(y)$$

$$P\left(\frac{d_1, \dots, d_n}{y}\right) \triangleq Q(y)$$

in such cases we can avoid to mention derivations at all

Rule induction principle

we assume derivations exist
and that we can build a larger one
but don't need to mention this fact

$$\frac{\forall \frac{x_1, \dots, x_n}{y} \in R. (\{x_1, \dots, x_n\} \subseteq I_R \wedge P(x_1) \wedge \dots \wedge P(x_n)) \Rightarrow P(y)}{\forall x \in I_R. P(x)}$$

$$I_R \triangleq \{y \mid \vdash_R y\}$$

Rule induction simplified

assuming that premises are theorems
may be not even necessary

$$\frac{\forall \frac{x_1, \dots, x_n}{y} \in R. (P(x_1) \wedge \dots \wedge P(x_n)) \Rightarrow P(y)}{\forall x \in I_R. P(x)}$$

Induction schemes

properties of numbers $P(n)$ mathematical induction

two proof obligations: $P(0)$ and $P(n) \Rightarrow P(n + 1)$

properties of terms $P(t)$ structural induction

one proof obligation for each function symbol

properties of formulas $P(F)$ rule induction

one proof obligation for each inference rule

Determinacy: two views

properties of terms

$P(t)$

structural induction

$$P(c) \triangleq \forall \sigma, \sigma_1, \sigma_2. \langle c, \sigma \rangle \longrightarrow \sigma_1 \wedge \langle c, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

properties of formulas

$P(F)$

rule induction

$$P(\langle c, \sigma \rangle \longrightarrow \sigma_1) \triangleq \forall \sigma_2. \langle c, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

Determinacy of commands

$c ::= \text{skip} \mid x := a \mid c; c \mid \text{if } b \text{ then } c \text{ else } c \mid \text{while } b \text{ do } c$

$$\frac{}{\langle \text{skip}, \sigma \rangle \longrightarrow \sigma} \quad \frac{\langle a, \sigma \rangle \longrightarrow n}{\langle x := a, \sigma \rangle \longrightarrow \sigma[n/x]} \quad \frac{\langle c_0, \sigma \rangle \longrightarrow \sigma'' \quad \langle c_1, \sigma'' \rangle \longrightarrow \sigma'}{\langle c_0; c_1, \sigma \rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma \rangle \longrightarrow \text{ff} \quad \langle c_1, \sigma \rangle \longrightarrow \sigma'}{\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma'} \quad \frac{\langle b, \sigma \rangle \longrightarrow \text{tt} \quad \langle c_0, \sigma \rangle \longrightarrow \sigma'}{\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma'}$$

$$\frac{\langle b, \sigma \rangle \longrightarrow \text{ff}}{\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma} \quad \frac{\langle b, \sigma \rangle \longrightarrow \text{tt} \quad \langle c, \sigma \rangle \longrightarrow \sigma'' \quad \langle \text{while } b \text{ do } c, \sigma'' \rangle \longrightarrow \sigma'}{\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma'}$$

$$P(\langle c, \sigma \rangle \longrightarrow \sigma_1) \stackrel{\Delta}{=} \forall \sigma_2. \langle c, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2 \quad \forall c, \sigma, \sigma_1. P(\langle c, \sigma \rangle \longrightarrow \sigma_1) ?$$

Base case

$$\frac{}{\langle \mathbf{skip}, \sigma \rangle \longrightarrow \sigma}$$

We want to prove

$$P(\langle \mathbf{skip}, \sigma \rangle \longrightarrow \sigma) \triangleq \forall \sigma_2. \langle \mathbf{skip}, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma = \sigma_2$$

Take σ_2 s.t. $\langle \mathbf{skip}, \sigma \rangle \longrightarrow \sigma_2$

We want to prove $\sigma = \sigma_2$

Consider the goal $\langle \mathbf{skip}, \sigma \rangle \longrightarrow \sigma_2$

Only the rule $\frac{}{\langle \mathbf{skip}, \sigma \rangle \longrightarrow \sigma}$ is applicable, hence $\sigma_2 = \sigma$

Base case

$$\frac{\langle a, \sigma \rangle \longrightarrow n}{\langle x := a, \sigma \rangle \longrightarrow \sigma[n/x]}$$

We assume $\langle a, \sigma \rangle \longrightarrow n$

We want to prove

$$P(\langle x := a, \sigma \rangle \longrightarrow \sigma[n/x]) \stackrel{\Delta}{=} \forall \sigma_2. \langle x := a, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma[n/x] = \sigma_2$$

Take σ_2 s.t. $\langle x := a, \sigma \rangle \longrightarrow \sigma_2$

We want to prove $\sigma[n/x] = \sigma_2$

Consider the goal $\langle x := a, \sigma \rangle \longrightarrow \sigma_2$

Only the rule $\frac{\langle a, \sigma \rangle \longrightarrow n}{\langle x := a, \sigma \rangle \longrightarrow \sigma[n/x]}$ is applicable, hence $\sigma_2 = \sigma[m/x]$
with $\langle a, \sigma \rangle \longrightarrow m$

since we assumed $\langle a, \sigma \rangle \longrightarrow n$

by determinacy of Aexp we have $n = m$ and thus $\sigma_2 = \sigma[m/x] = \sigma[n/x]$

Inductive case

$$\frac{\langle c_0, \sigma \rangle \longrightarrow \sigma'' \quad \langle c_1, \sigma'' \rangle \longrightarrow \sigma'}{\langle c_0 ; c_1, \sigma \rangle \longrightarrow \sigma'}$$

We assume (inductive hypotheses)

$$P(\langle c_0, \sigma \rangle \longrightarrow \sigma'') \triangleq \forall \sigma_2''. \langle c_0, \sigma \rangle \longrightarrow \sigma_2'' \Rightarrow \sigma'' = \sigma_2''$$

$$P(\langle c_1, \sigma'' \rangle \longrightarrow \sigma') \triangleq \forall \sigma_2'. \langle c_1, \sigma'' \rangle \longrightarrow \sigma_2' \Rightarrow \sigma' = \sigma_2'$$

We want to prove

$$P(\langle c_0 ; c_1, \sigma \rangle \longrightarrow \sigma') \triangleq \forall \sigma_2. \langle c_0 ; c_1, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma' = \sigma_2$$

Take σ_2 such that $\langle c_0 ; c_1, \sigma \rangle \longrightarrow \sigma_2$

We want to prove $\sigma' = \sigma_2$

Inductive case (ctd)

$$P(\langle c_0, \sigma \rangle \longrightarrow \sigma'') \triangleq \forall \sigma_2''. \langle c_0, \sigma \rangle \longrightarrow \sigma_2'' \Rightarrow \sigma'' = \sigma_2''$$

$$P(\langle c_1, \sigma'' \rangle \longrightarrow \sigma') \triangleq \forall \sigma_2'. \langle c_1, \sigma'' \rangle \longrightarrow \sigma_2' \Rightarrow \sigma' = \sigma_2'$$

Consider the goal $\langle c_0 ; c_1, \sigma \rangle \longrightarrow \sigma_2$

Only the rule $\frac{\langle c_0, \sigma \rangle \longrightarrow \sigma'' \quad \langle c_1, \sigma'' \rangle \longrightarrow \sigma'}{\langle c_0 ; c_1, \sigma \rangle \longrightarrow \sigma'}$ is applicable

hence $\sigma_2 = \sigma_2'$ with $\langle c_0, \sigma \rangle \longrightarrow \sigma_2''$ and $\langle c_1, \sigma_2'' \rangle \longrightarrow \sigma_2'$

By inductive hypothesis $P(\langle c_0, \sigma \rangle \longrightarrow \sigma'')$, we have $\sigma'' = \sigma_2''$

and thus $\langle c_1, \sigma'' \rangle \longrightarrow \sigma_2'$

By inductive hypothesis $P(\langle c_1, \sigma'' \rangle \longrightarrow \sigma')$, we then have $\sigma' = \sigma_2'$

Inductive case

$$\frac{\langle b, \sigma \rangle \longrightarrow \text{ff} \quad \langle c_1, \sigma \rangle \longrightarrow \sigma'}{\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma'}$$

We assume

$$\langle b, \sigma \rangle \longrightarrow \text{ff} \quad (\text{inductive hypothesis})$$

$$P(\langle c_1, \sigma \rangle \longrightarrow \sigma') \triangleq \forall \sigma_2. \langle c_1, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma' = \sigma_2$$

We want to prove

$$P(\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma') \triangleq \forall \sigma_2. \langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma' = \sigma_2$$

Take σ_2 such that $\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma_2$

We want to prove $\sigma' = \sigma_2$

Inductive case (ctd)

$$\langle b, \sigma \rangle \longrightarrow \text{ff}$$

$$P(\langle c_1, \sigma \rangle \longrightarrow \sigma') \triangleq \forall \sigma_2. \langle c_1, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma' = \sigma_2$$

Consider the goal $\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma_2$

By determinacy of Bexp

only the rule $\frac{\langle b, \sigma \rangle \longrightarrow \text{ff} \quad \langle c_1, \sigma \rangle \longrightarrow \sigma'}{\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma'}$ is applicable

hence $\sigma_2 = \sigma'_2$ with $\langle c_1, \sigma \rangle \longrightarrow \sigma'_2$

By inductive hypothesis $P(\langle c_1, \sigma \rangle \longrightarrow \sigma')$, we then have $\sigma' = \sigma'_2 = \sigma_2$

Inductive case

$$\frac{\langle b, \sigma \rangle \longrightarrow \text{tt} \quad \langle c_0, \sigma \rangle \longrightarrow \sigma'}{\langle \text{if } b \text{ then } c_0 \text{ else } c_1, \sigma \rangle \longrightarrow \sigma'}$$

Analogous to the previous case and thus omitted

Base case

$$\frac{\langle b, \sigma \rangle \longrightarrow \text{ff}}{\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma}$$

We assume

$$\langle b, \sigma \rangle \longrightarrow \text{ff}$$

We want to prove

$$P(\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma) \triangleq \forall \sigma_2. \langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma = \sigma_2$$

Take σ_2 such that $\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma_2$

We want to prove $\sigma = \sigma_2$

Inductive case (ctd)

$$\langle b, \sigma \rangle \longrightarrow \text{ff}$$

Consider the goal $\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma_2$

By determinacy of Bexp

Only the rule $\frac{\langle b, \sigma \rangle \longrightarrow \text{ff}}{\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma}$ is applicable hence $\sigma_2 = \sigma$

Inductive case

$$\frac{\langle b, \sigma \rangle \longrightarrow \mathbf{tt} \quad \langle c, \sigma \rangle \longrightarrow \sigma'' \quad \langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma'' \rangle \longrightarrow \sigma'}{\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma \rangle \longrightarrow \sigma'}$$

We assume

$$\langle b, \sigma \rangle \longrightarrow \mathbf{tt} \quad (\text{inductive hypotheses})$$

$$P(\langle c, \sigma \rangle \longrightarrow \sigma'') \triangleq \forall \sigma_2''. \langle c, \sigma \rangle \longrightarrow \sigma_2'' \Rightarrow \sigma'' = \sigma_2''$$

$$P(\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma'' \rangle \longrightarrow \sigma') \triangleq \forall \sigma_2'. \langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma'' \rangle \longrightarrow \sigma_2' \Rightarrow \sigma' = \sigma_2'$$

We want to prove

$$P(\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma \rangle \longrightarrow \sigma') \triangleq \forall \sigma_2. \langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma' = \sigma_2$$

Take σ_2 such that $\langle \mathbf{while} \ b \ \mathbf{do} \ c, \sigma \rangle \longrightarrow \sigma_2$

We want to prove $\sigma' = \sigma_2$

Inductive case (ctd)

$$\langle b, \sigma \rangle \longrightarrow \texttt{tt}$$

$$P(\langle c, \sigma \rangle \longrightarrow \sigma'') \triangleq \forall \sigma_2''. \langle c, \sigma \rangle \longrightarrow \sigma_2'' \Rightarrow \sigma'' = \sigma_2''$$

$$P(\langle \textbf{while } b \textbf{ do } c, \sigma'' \rangle \longrightarrow \sigma') \triangleq \forall \sigma_2'. \langle \textbf{while } b \textbf{ do } c, \sigma'' \rangle \longrightarrow \sigma_2' \Rightarrow \sigma' = \sigma_2'$$

Consider the goal $\langle \textbf{while } b \textbf{ do } c, \sigma \rangle \longrightarrow \sigma_2$

By determinacy of Bexp

only the rule
$$\frac{\langle b, \sigma \rangle \longrightarrow \texttt{tt} \quad \langle c, \sigma \rangle \longrightarrow \sigma'' \quad \langle \textbf{while } b \textbf{ do } c, \sigma'' \rangle \longrightarrow \sigma'}{\langle \textbf{while } b \textbf{ do } c, \sigma \rangle \longrightarrow \sigma'}$$
 is applicable

hence $\sigma_2 = \sigma_2'$ with $\langle c, \sigma \rangle \longrightarrow \sigma_2''$ and $\langle \textbf{while } b \textbf{ do } c, \sigma_2'' \rangle \longrightarrow \sigma_2'$

By inductive hypothesis $P(\langle c, \sigma \rangle \longrightarrow \sigma'')$, we have $\sigma'' = \sigma_2''$

thus $\langle \textbf{while } b \textbf{ do } c, \sigma'' \rangle \longrightarrow \sigma_2'$

By inductive hypothesis $P(\langle \textbf{while } b \textbf{ do } c, \sigma'' \rangle \longrightarrow \sigma')$

we conclude $\sigma' = \sigma_2' = \sigma_2$

Determinacy of commands

$$\forall c, \sigma, \sigma_1. P(\langle c, \sigma \rangle \longrightarrow \sigma_1)$$

$$P(\langle c, \sigma \rangle \longrightarrow \sigma_1) \triangleq \forall \sigma_2. \langle c, \sigma \rangle \longrightarrow \sigma_2 \Rightarrow \sigma_1 = \sigma_2$$

Structural induction, rule induction and divergence

Exercise: $a_0 \sqcap a_1$

Extend the syntax of arithmetic expressions with the operator $a_0 \sqcap a_1$ whose big-step operational semantics is defined by the rule:

$$\frac{\langle a_0, \sigma \rangle \longrightarrow n \quad \langle a_1, \sigma \rangle \longrightarrow n}{\langle a_0 \sqcap a_1, \sigma \rangle \longrightarrow n}$$

1. Prove termination or exhibit a counterexample.
2. Prove determinacy or exhibit a counterexample.

Exercise: $a_0 \sqcup a_1$

Extend the syntax of arithmetic expressions with the operator $a_0 \sqcup a_1$ whose big-step operational semantics is defined by the rule:

$$\frac{\langle a_0, \sigma \rangle \longrightarrow n_0}{\langle a_0 \sqcup a_1, \sigma \rangle \longrightarrow n_0} \quad \frac{\langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \sqcup a_1, \sigma \rangle \longrightarrow n_1}$$

1. Prove termination or exhibit a counterexample.
2. Prove determinacy or exhibit a counterexample.

Exercise: Divergence

Consider the command:

$$w \stackrel{\text{def}}{=} \text{while } b \text{ do } c$$

Prove by rule induction that:

$$\forall \sigma, \sigma'. \langle w, \sigma \rangle \longrightarrow \sigma' \Rightarrow \langle b, \sigma' \rangle \longrightarrow \text{false}$$

Exercise: Divergence

Consider the command:

$$w \stackrel{\text{def}}{=} \textbf{while } x > y \textbf{ do } (x := x + 1 ; y := y - 1)$$

Find out the set S of memories σ such that $\langle w, \sigma \rangle \not\rightarrow$ and prove that this is the case by using the inference rule for divergence.