

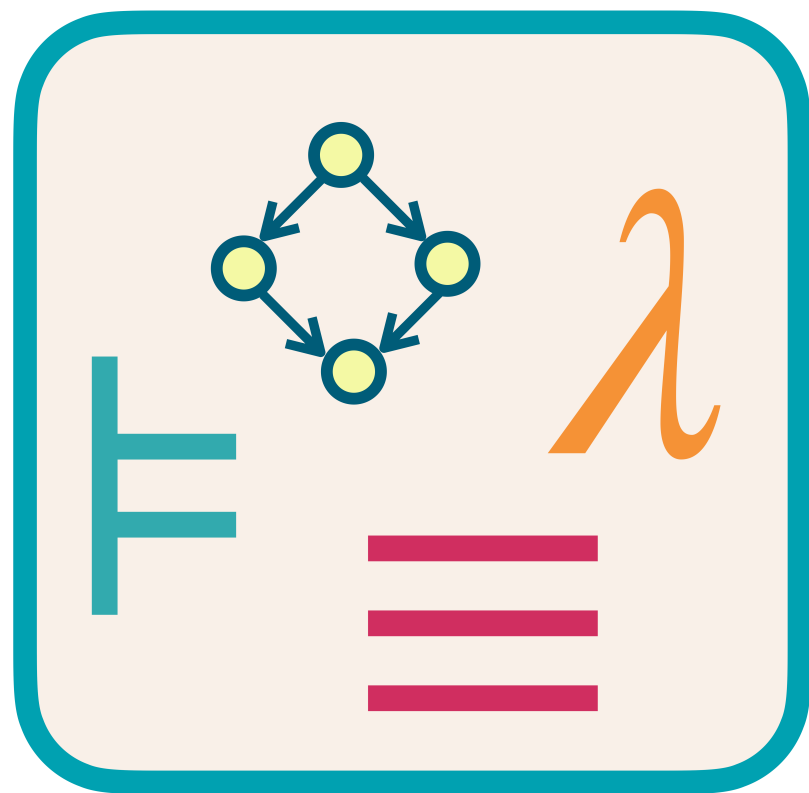
MPP 2025/26 (0077A, 9CFU)

Models for Programming Paradigms

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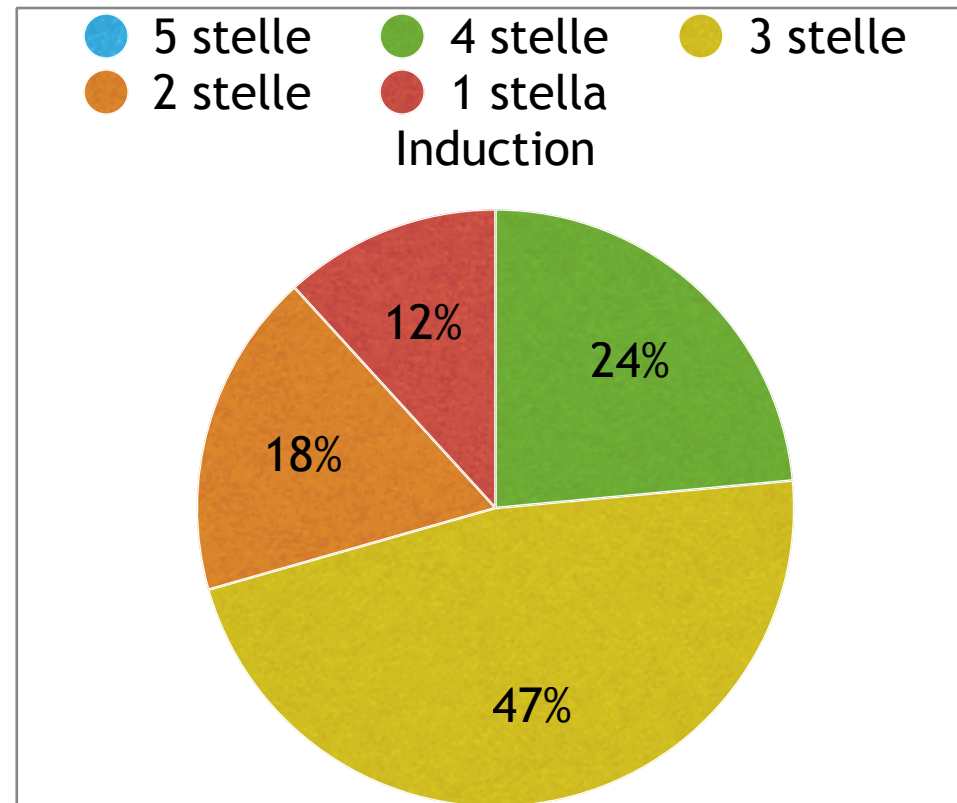
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05a - Induction

From your forms



(over 17 answers)

Induction everywhere

How to

prove an existential statement?

there is a program that diverges

$$\exists x. P(x)$$

exhibit a witness

$$\exists n \in \mathbb{N}. n^2 \leq n$$

$$n = 0$$

disprove a universal statement?

there is a program that converges
not all programs diverge

$$\neg \forall x. P(x) \equiv \exists x. \neg P(x)$$

exhibit a counterexample to P

$$\neg \forall n \in \mathbb{N}. n^2 \leq n$$

$$n = 2$$

prove a universal statement?

all programs diverge

$$\forall x. P(x)$$

use induction!

What is common to

natural numbers

lists

trees

grammar languages

terms of a signature

theorems of a logic system

derivations

computations

generated by
finite applications
of some given rules

base cases

inductive cases

What is common to

	base case	inductive case
natural numbers	0	succ
lists	nil	cons
trees	nil	node
grammar languages	productions with only terminal symbols	productions with non terminal symbols
terms of a signature	constants	operators
theorems of a logic system	axioms	inference rules
derivations	axioms	inference rules
computations	single step	concatenation

A famous proof

Every non prime number greater than 1 can be written as the product of two or more prime numbers

base case ($n = 2$): 2 is prime

inductive case: taken a generic n , we assume the property holds for all numbers from 2 to n and prove it holds for $n + 1$:

- if $n + 1$ is prime we are done;
- otherwise, let $n + 1 = a \cdot b$ for some $1 < a, b \leq n$. By inductive hypothesis, a and b can be written as product of prime numbers. Let $a = p_1 \cdots p_k$ and $b = q_1 \cdots q_h$. Then $n + 1 = p_1 \cdots p_k \cdot q_1 \cdots q_h$ can be written as the product of $k + h$ primes.

A far less known proof

All cats are the same colour

base case ($n = 1$): trivial

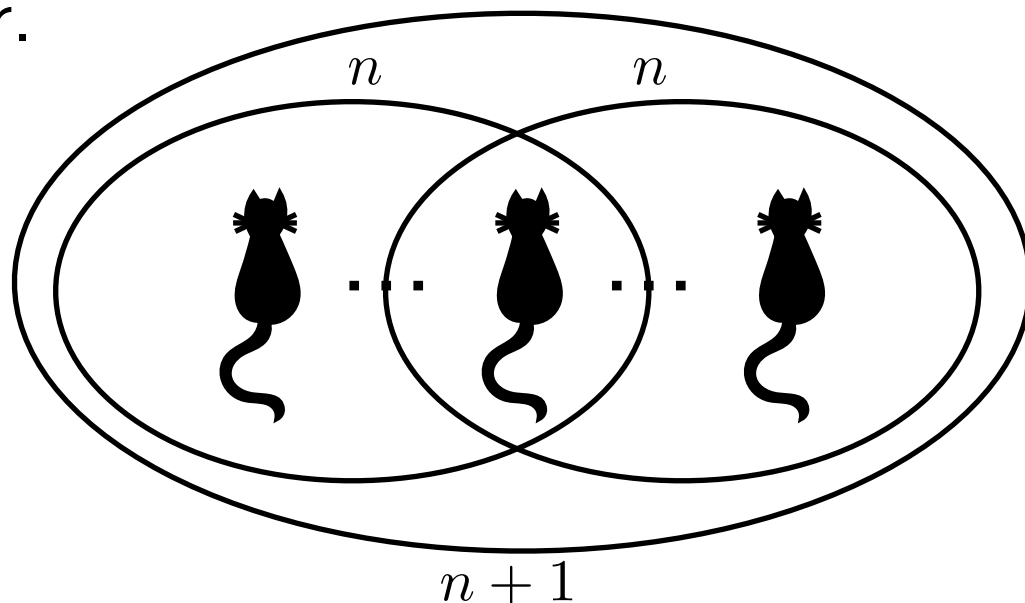
inductive case: taken a generic n , we assume the property holds for all groups with $k \leq n$ cats and prove it holds for any group with $n + 1$ cats as well.

Take $n + 1$ cats and place them along a line (this is the hardest part of the proof!).

By inductive hypothesis, the first n cats are the same colour.

By inductive hypothesis, the last n cats are the same colour.

Since the cats in the middle of the line belongs to both groups, by transitivity all $n + 1$ cats are the same colour.



Well founded induction

Ingredients

a set of elements A (possibly infinite)

a predicate $P : A \rightarrow \mathbb{B}$

we want to prove $\forall a \in A. P(a)$

a binary relation of precedence $\prec \subseteq A \times A$

(not necessarily transitive)

$a \prec b$ reads a precedes b

also written $b \succ a$

also written $a \rightarrow b$ (graph notation)

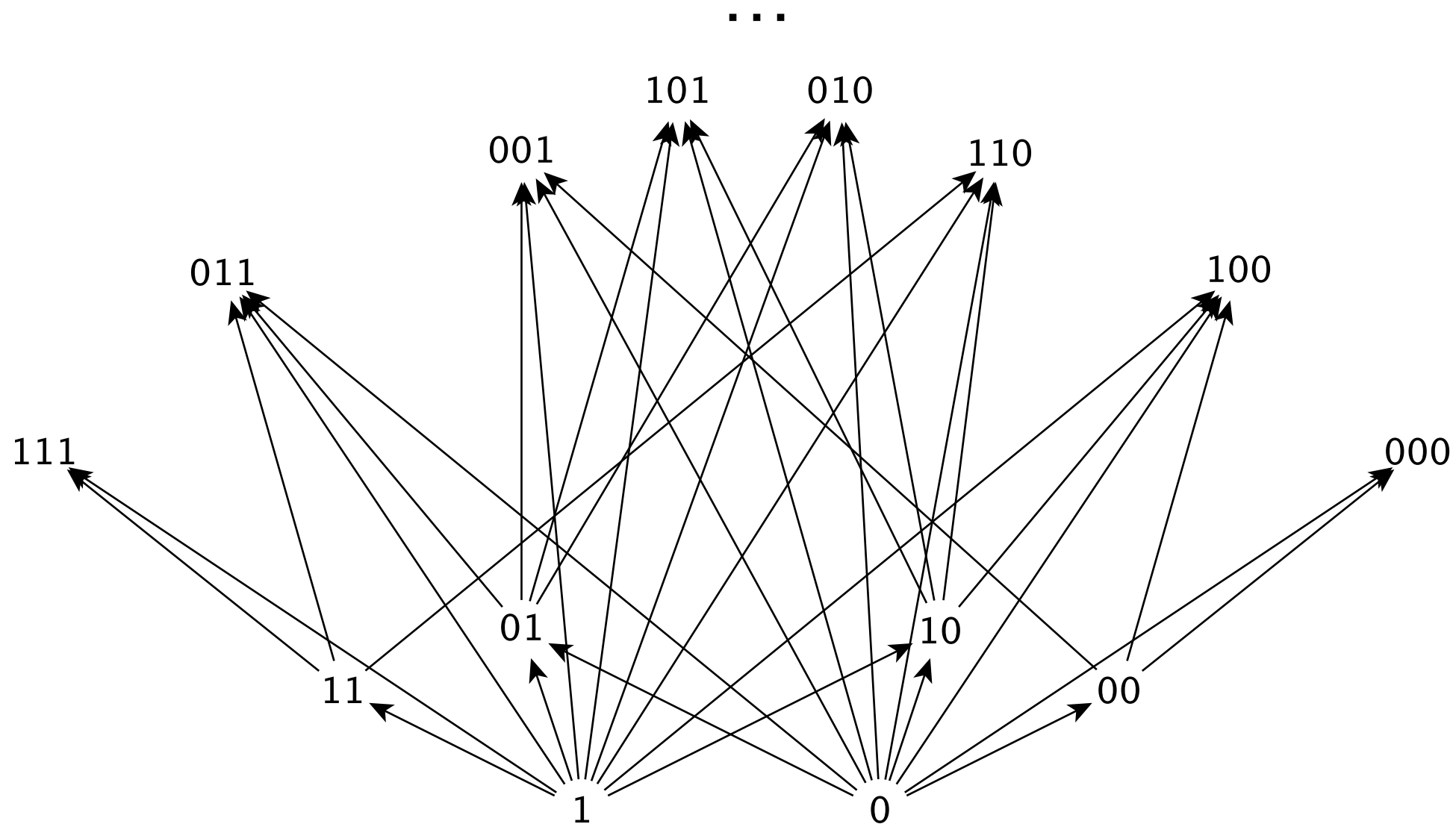
to use induction, we must guarantee to reach some base cases!

no infinite descending chain is allowed in $\prec$
(well-founded relation)

Graph of a relation

Example:

$A = \mathbb{B}^+$ $u \prec w$ if u appears in w (with $u \neq \epsilon$ and $u \neq w$)



Infinite descending chain



Infinite descending chain

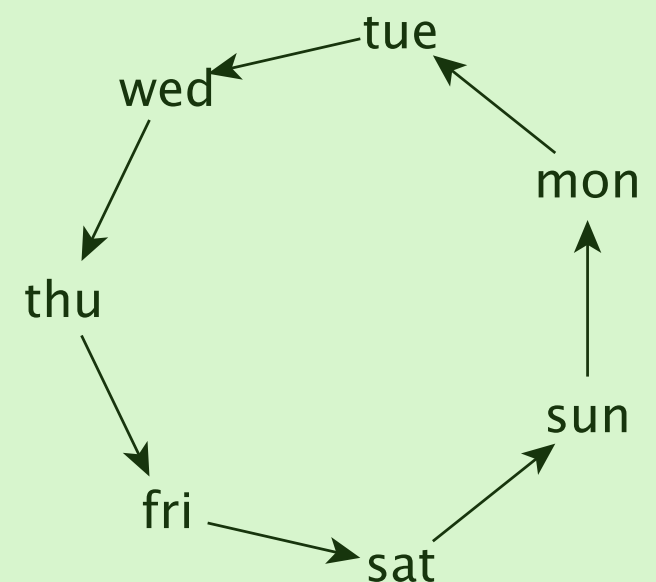
an infinite sequence $\{a_i\}_{i \in \mathbb{N}}$ of elements in A
such that $\forall i \in \mathbb{N}. a_i \succ a_{i+1}$

the sequence can also be seen as a function $a : \mathbb{N} \rightarrow A$
 $a(n) = a_n$

Example

$A = \{\text{mon, tue, wed, thu, fri, sat, sun}\}$

$a(n) = n\text{th day past}$



Well-founded relation

A relation is called **well-founded** if it has no infinite descending chain

$$\mathbb{N} \quad n \prec m \text{ if } m = n + 1 \quad \checkmark$$

$$\mathbb{Z} \quad n \prec m \text{ if } m = n + 1 \quad \times$$

$$\mathbb{N} \quad n \prec m \text{ if } n < m \quad \checkmark$$

$$\mathbb{Z} \quad n \prec m \text{ if } n < m \quad \times$$

$$\mathbb{N} \quad n \prec m \text{ if } n \leq m \quad \times$$

$$\mathbb{N} \quad n \prec m \text{ if } n = m \quad \times$$

Transitive closure

a binary relation $\prec \subseteq A \times A$

its **transitive closure** $\prec^+ \subseteq A \times A$

is the least relation generated by the following rules

$$\frac{a \prec b}{a \prec^+ b}$$

$$\frac{a \prec^+ b \quad b \prec^+ c}{a \prec^+ c}$$

by the first rule, it is obvious that $\prec \subseteq \prec^+$

it can be proved that $(\prec^+)^+ = \prec^+$

Transit. and refl. closure

a binary relation $\prec \subseteq A \times A$

its **transitive and reflexive closure** $\prec^* \subseteq A \times A$

is the least relation generated by the following rules

$$\frac{a \in A}{a \prec^* a}$$

$$\frac{a \prec b}{a \prec^* b}$$

$$\frac{a \prec^* b \quad b \prec^* c}{a \prec^* c}$$

it is obvious that $\prec \subseteq \prec^+ \subseteq \prec^*$

it can be proved that $(\prec^*)^* = \prec^*$

Closures and paths

a binary relation $\prec \subseteq A \times A$

$a \prec^+ b$ iff there is a non-empty path from a to b in the graph of $\prec$

$$\exists k > 0, \{c_i\}_{i \in [0, k]}. a = c_0 \prec c_1 \prec \cdots \prec c_k = b$$

$a \prec^* b$ iff there is a possibly empty path from a to b in the graph of $\prec$

$$\exists k \geq 0, \{c_i\}_{i \in [0, k]}. a = c_0 \prec c_1 \prec \cdots \prec c_k = b$$

Closures

		$\prec^+$	$\prec^*$
$\mathbb{N}$	$n \prec m$ if $m = n + 1$	$n < m$	$n \leq m$
$\mathbb{Z}$	$n \prec m$ if $m = n + 1$	$n < m$	$n \leq m$
$\mathbb{N}$	$n \prec m$ if $n < m$	$n < m$	$n \leq m$
$\mathbb{N}$	$n \prec m$ if $n \leq m$	$n \leq m$	$n \leq m$
$\mathbb{N}$	$n \prec m$ if $n = m$	$n = m$	$n = m$

Get ready for theorems: proofs included

on the right, you see
one of the oldest
surviving fragments of
Euclid's Elements, a
textbook used for
millennia to teach
proof-writing
techniques
(source: *wikipedia*)



Theorem

A relation is well-founded iff its transitive closure is well-founded

$$\prec^+ \text{ w.f.} \Rightarrow \prec \text{ w.f.}$$

obvious:

any descending chain for $\prec$ is a descending chain for $\prec^+$
and thus it is finite because $\prec^+$ is w.f.

Theorem

A relation is well-founded iff its transitive closure is well-founded

$$\prec \text{ w.f.} \Rightarrow \prec^+ \text{ w.f.} \quad \equiv \quad \neg(\prec^+ \text{ w.f.}) \Rightarrow \neg(\prec \text{ w.f.})$$

by contraposition:

we assume $\prec^+$ is not w.f. and prove $\prec$ is not w.f.

take an infinite descending chain for $\prec^+$

$$a_0 \succ^+ a_1 \succ^+ a_2 \succ^+ \dots$$

$a \prec^+ b$ iff there is a non-empty path from a to b in the graph of $\prec$

$$a_0 \succ \dots \succ a_1 \succ \dots \succ a_2 \succ \dots$$

thus we get an infinite descending chain for $\prec$

Acyclic relation

a binary relation $\prec \subseteq A \times A$

$\prec$ has a cycle if $a \prec^+ a$ for some $a \in A$

We say that $\prec$ is **acyclic** if it has no cycle

note that $\prec$ is acyclic iff $\prec^+$ is such

Theorem

Well-founded relations are acyclic

by contraposition:

we prove that if $\prec$ has a cycle then it is not well-founded

take $a \in A$ such that $a \prec^+ a$

then we have an infinite descending chain for $\prec^+$

$$a \succ^+ a \succ^+ a \succ^+ \dots$$

therefore $\prec^+$ is not w.f.

by the previous theorem, $\prec$ is not w.f.

Theorem

If A is finite and $<$ acyclic, then $<$ is well-founded

the proof exploits the pigeonhole principle

Pigeonhole principle (aka drawer principle)

If n items are put into $m < n$ slots,
then at least one slot must contain more than one item



in the picture: ten pigeons and nine holes

Theorem

If A is finite and $\prec$ acyclic, then $\prec$ is well-founded

by contraposition:

we prove that if $\prec$ is not well-founded then it has a cycle

take an infinite descending chain for $\prec$

$$a_0 \succ a_1 \succ a_2 \succ \cdots$$

let $k = |A|$ and consider $a_0, \dots, a_k$ (they are $k + 1$ elements)

by the pigeonhole principle, $a_i = a_j$ for some $0 \leq i < j \leq k$

$$a_i \succ a_{i+1} \succ \cdots \succ a_{j-1} \succ a_j = a_i$$

thus $a_i \prec^+ a_i$ and $\prec$ has a cycle

Minimal elements

a binary relation $\prec \subseteq A \times A$

let $Q \subseteq A$ and $m \in Q$

m is **minimal** in Q if none of the elements in Q precedes m

$$\begin{aligned} & \forall x \in Q. x \not\prec m \\ & \equiv \neg \exists x \in Q. x \prec m \end{aligned}$$

Q has no minimal element means $\forall m \in Q. \exists x \in Q. x \prec m$

Minimal elements

minimal
element?

unique?

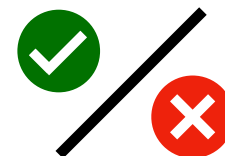
$\mathbb{N}, <$

$\emptyset \subset Q \subseteq \mathbb{N}$



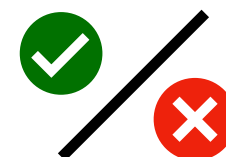
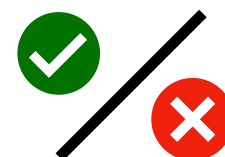
$\mathbb{Z}, <$

$\emptyset \subset Q \subseteq \mathbb{Z}$



$\wp(\mathbb{N}), \subset$

$\emptyset \subset Q \subseteq \wp(\mathbb{N})$



Lemma

$\prec$ is w.f. iff any non empty $Q \subseteq A$ has a minimal element
 $\equiv$

① $\prec$ has an infinite descending chain iff

② there is a (non empty) $Q \subseteq A$ with no minimal element

① $\Rightarrow$ ② Take an infinite descending chain $\{a_i\}_{i \in \mathbb{N}}$
the set $Q = \{a_i \mid i \in \mathbb{N}\}$ has no minimal element
(if it had one, say a_k , then $a_{k+1} \prec a_k$!)

Lemma

$\prec$ is w.f. iff any non empty $Q \subseteq A$ has a minimal element
 $\equiv$

① $\prec$ has an infinite descending chain iff

② there is a (non empty) $Q \subseteq A$ with no minimal element

② $\Rightarrow$ ① Take $\emptyset \subset Q \subseteq A$ with no minimal element

Since $Q \neq \emptyset$ we can pick $a_0 \in Q$

Since a_0 cannot be minimal, we can take $a_1 \in Q$ s.t. $a_1 \prec a_0$

Since a_1 cannot be minimal, we can take $a_2 \in Q$ s.t. $a_2 \prec a_1$

...

Since a_k cannot be minimal, we can take $a_{k+1} \in Q$ s.t. $a_{k+1} \prec a_k$

...

Theorem [w.f. induction]

Let $\prec \subseteq A \times A$ be w.f.

$$(\forall a \in A. P(a)) \quad \Leftrightarrow \quad (\forall a \in A. (\forall b \prec a. P(b)) \Rightarrow P(a))$$

Set $H(a) \triangleq \forall b \prec a. P(b)$

$S(a) \triangleq H(a) \Rightarrow P(a)$

$$\begin{array}{ccc} (\forall a \in A. P(a)) & \Leftrightarrow & (\forall a \in A. S(a)) \\ \textcircled{1} & & \textcircled{2} \end{array}$$

$\textcircled{1} \Rightarrow \textcircled{2}$

Assume $\forall a. P(a)$

Take a generic $a \in A$

$$S(a) \equiv (H(a) \Rightarrow P(a)) \equiv (\neg H(a) \vee P(a)) \equiv (\neg H(a) \vee \mathbf{tt}) \equiv \mathbf{tt}$$

Theorem [w.f. induction]

Let $\prec \subseteq A \times A$ be w.f.

$$S(a) \triangleq H(a) \Rightarrow P(a)$$

$$H(a) \triangleq \forall b \prec a. P(b)$$

$$\textcircled{1} (\forall a \in A. P(a)) \iff (\forall a \in A. S(a)) \textcircled{2}$$

$$\textcircled{2} \Rightarrow \textcircled{1} \equiv \neg \textcircled{1} \Rightarrow \neg \textcircled{2} \quad \text{Assume } \exists a \in A. \neg P(a)$$

Take $Q \triangleq \{q \in A \mid \neg P(q)\} \neq \emptyset$

Since $\prec$ is w.f., then Q has a minimal element $m \in Q$

Obviously $\neg P(m)$ (because $m \in Q$)

Since m is minimal, $\forall b \prec m. b \notin Q$

$$\text{i.e. } \forall b \prec m. P(b) \equiv H(m)$$

$$\text{Thus } H(m) \wedge \neg P(m) \equiv \neg(H(m) \Rightarrow P(m)) \equiv \neg S(m)$$

$$\text{i.e. } \exists a \in A. \neg S(a)$$

w.f. induction principle

a w.f. relation $\prec \subseteq A \times A$

$$\frac{\forall a \in A. ((\forall b \prec a. P(b)) \Rightarrow P(a))}{\forall a \in A. P(a)}$$

Advantage: when proving $P(a)$ for a generic a ,
we can exploit the assumption $\forall b \prec a. P(b)$!

Weak mathematical induction

$$\frac{\forall a \in A. ((\forall b \prec a. P(b)) \Rightarrow P(a))}{\forall a \in A. P(a)}$$

$$A = \mathbb{N}$$

$\prec = \{(n, n+1) \mid n \in \mathbb{N}\}$ (immediate precedence relation)

- if $a = 0$, then there is no $b \prec 0$, hence $(\forall b \prec 0. P(b)) \equiv \mathbf{tt}$ and $((\forall b \prec 0. P(b)) \Rightarrow P(0)) \equiv \mathbf{tt} \Rightarrow P(0) \equiv P(0)$
- if $a = n + 1$, then there is only one b such that $b \prec n + 1$, namely $b = n$ then $((\forall b \prec n + 1. P(b)) \Rightarrow P(n + 1)) \equiv P(n) \Rightarrow P(n + 1)$

Weak mathematical induction principle

$$\frac{P(0) \quad \forall n \in \mathbb{N}. (P(n) \Rightarrow P(n+1))}{\forall n \in \mathbb{N}. P(n)}$$

Weak: we can exploit $P(n)$,
for proving $P(n+1)$!

Weak induction: example

$n^3 - n$ is always a multiple of 3

Prove that: $\forall n \in \mathbb{N}. \exists k \in \mathbb{N}. n^3 - n = 3k$

Let $P(n) \triangleq \exists k \in \mathbb{N}. n^3 - n = 3k$

Prove $P(0) \triangleq \exists k \in \mathbb{N}. 0^3 - 0 = 3k$ trivial: take $k = 0$

Prove $\forall n \in \mathbb{N}. (P(n) \Rightarrow P(n+1))$ Take a generic $n \in \mathbb{N}$

Assume $P(n) \triangleq \exists k_1 \in \mathbb{N}. n^3 - n = 3k_1$

Prove $P(n+1) \triangleq \exists k \in \mathbb{N}. (n+1)^3 - (n+1) = 3k$

Observe $(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1$

$$= (n^3 - n) + 3n^2 + 3n$$

$$\text{(by } P(n)) = 3k_1 + 3n^2 + 3n$$

$$= 3(k_1 + n^2 + n)$$

Take $k = k_1 + n^2 + n$

Strong mathematical induction principle

$$\frac{\forall a \in A. ((\forall b \prec a. P(b)) \Rightarrow P(a))}{\forall a \in A. P(a)}$$

$$A = \mathbb{N}$$

$\prec = <$ (strictly-less-than relation)

- if $a = 0$, as before, then there is no $b \prec 0$, hence $((\forall b \prec 0. P(b)) \Rightarrow P(0)) \equiv P(0)$
- if $a = n + 1$, then $(\forall b \prec n + 1. P(b)) \equiv P(0) \wedge P(1) \wedge \cdots \wedge P(n)$ and $((\forall b \prec n + 1. P(b)) \Rightarrow P(n + 1)) \equiv (P(0) \wedge \cdots \wedge P(n)) \Rightarrow P(n + 1)$

Strong mathematical induction

$$\frac{P(0) \quad \forall n \in \mathbb{N}. ((P(0) \wedge \cdots \wedge P(n)) \Rightarrow P(n+1))}{\forall n \in \mathbb{N}. P(n)}$$

Strong: we can exploit more hypotheses than $P(n)$,
for proving $P(n+1)$!

Strong induction: example

Prove that: $\forall n \geq 8. \exists a, b \in \mathbb{N}. n = 3a + 5b$

Let $P(n) \triangleq \exists a, b \in \mathbb{N}. n = 3a + 5b$

Prove $P(8) \triangleq \exists a, b \in \mathbb{N}. 8 = 3a + 5b$ trivial: take $a = 1, b = 1$

Prove $\forall n \geq 8. (P(8) \wedge \dots \wedge P(n) \Rightarrow P(n+1))$

Assume $P(8) \wedge \dots \wedge P(n)$

Prove $P(n+1) \triangleq \exists a, b \in \mathbb{N}. n+1 = 3a + 5b$

Observe $n+1 = (n+1-3) + 3 = (n-2) + 3$

$$\begin{aligned} & \text{(by } P(n-2)) = (3a_1 + 5b_1) + 3 \\ & = 3(a_1 + 1) + 5b_1 \end{aligned}$$

Take $a = a_1 + 1, b = b_1$

but... did we miss something?

Prove $P(9)$ and $P(10)$!

Structural induction

Immediate subterms

a signature $\{\Sigma_n\}_{n \in \mathbb{N}}$

Take $A = T_\Sigma$ (closed terms)

$\prec = \{(t_i, f(t_1, \dots, t_n)) \mid f \in \Sigma_n, i \in [1, n]\}$
(immediate subterm relation)

Example

$\Sigma_0 = \{0\}$ $\Sigma_1 = \{\text{succ}\}$ $\Sigma_2 = \{\text{plus}\}$

$0 \prec \text{succ}(0) \prec \text{plus}(0, \text{succ}(0))$

$0 \prec \text{plus}(0, \text{succ}(0))$

$0 \not\prec \text{plus}(\text{succ}(0), \text{succ}(0))$

Lemma

$$T_\Sigma, \prec \text{ is w.f.}$$

Let $depth : T_\Sigma \rightarrow \mathbb{N}$ defined as:

$$\begin{aligned} depth(c) &\stackrel{\Delta}{=} 1 && \text{if } c \in \Sigma_0 \\ depth(f(t_1, \dots, t_n)) &\stackrel{\Delta}{=} 1 + \max_{i \in [1, n]} depth(t_i) && \text{if } f \in \Sigma_n \end{aligned}$$

By definition, if $t \prec t'$ then $depth(t) < depth(t')$

Any descending chain in $\prec$ induces a descending chain in $<$

Since $<$ is w.f., so is $\prec$

Structural induction principle

$$\frac{\forall n \in \mathbb{N}. \forall f \in \Sigma_n. \forall t_1, \dots, t_n \in T_\Sigma. (P(t_1) \wedge \dots \wedge P(t_n)) \Rightarrow P(f(t_1, \dots, t_n))}{\forall t \in T_\Sigma. P(t)}$$

Corollary

$$T_{\Sigma}, \prec^+ \text{ is w.f.}$$

Because $\prec^+$ is the transitive closure of a w.f. relation

Example

$$\Sigma_0 = \{0\} \quad \Sigma_1 = \{\text{succ}\} \quad \Sigma_2 = \{\text{plus}\}$$

$$0 \prec^+ \text{succ}(0) \prec^+ \text{plus}(0, \text{succ}(0))$$

$$0 \prec^+ \text{plus}(0, \text{succ}(0))$$

$$0 \prec^+ \text{plus}(\text{succ}(0), \text{succ}(0))$$

Termination of arithmetic expressions

$a ::= x \mid n \mid a \text{ op } a$

$x \in \text{Ide} \quad \text{op} \in \{+, \times, -\}$

$n \in \mathbb{Z} \quad \mathbb{M} \triangleq \{\sigma \mid \sigma : \text{Ide} \rightarrow \mathbb{Z}\}$

$$\frac{}{\langle x, \sigma \rangle \longrightarrow \sigma(x)} \quad \frac{}{\langle n, \sigma \rangle \longrightarrow n} \quad \frac{\langle a_0, \sigma \rangle \longrightarrow n_0 \quad \langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow n_0 \text{ op } n_1}$$

$$P(a) \triangleq \forall \sigma \in \mathbb{M}. \exists m \in \mathbb{Z}. \langle a, \sigma \rangle \longrightarrow m$$

$$\forall a. P(a) ?$$

Structural induction principle

$$\forall x \in \text{Ide. } P(x)$$

$$\forall n \in \mathbb{Z}. P(n)$$

$$\forall a_0, a_1. P(a_0) \wedge P(a_1) \Rightarrow P(a_0 \text{ op } a_1)$$

$$\forall a. P(a)$$

Base case

$\forall x \in \text{Ide. } P(x)$ Take a generic $x \in \text{Ide}$

We want to prove $P(x) \triangleq \forall \sigma. \exists m. \langle x, \sigma \rangle \longrightarrow m$

Take a generic $\sigma \in \mathbb{M}$ and consider the goal $\langle x, \sigma \rangle \longrightarrow m$

the only
variable

By rule $\frac{}{\langle x, \sigma \rangle \longrightarrow \sigma(x)}$ we have $\langle x, \sigma \rangle \longrightarrow m \nwarrow_{[m=\sigma(x)]} \square$

And we are done (taking $m = \sigma(x)$)

Base case

$\forall n \in \mathbb{Z}. P(n)$ Take a generic $n \in \mathbb{Z}$

We want to prove $P(n) \triangleq \forall \sigma. \exists m. \langle n, \sigma \rangle \longrightarrow m$

Take a generic $\sigma \in \mathbb{M}$ and consider the goal $\langle n, \sigma \rangle \longrightarrow m$

the only
variable

By rule $\frac{}{\langle n, \sigma \rangle \longrightarrow n}$ we have $\langle n, \sigma \rangle \longrightarrow m \nwarrow_{[m=n]} \square$

And we are done (taking $m = n$)

Inductive case

$\forall a_0, a_1. P(a_0) \wedge P(a_1) \Rightarrow P(a_0 \text{ op } a_1)$ Take generic a_0, a_1

We assume $P(a_0) \triangleq \forall \sigma. \exists m_0. \langle a_0, \sigma \rangle \longrightarrow m_0$

$P(a_1) \triangleq \forall \sigma. \exists m_1. \langle a_1, \sigma \rangle \longrightarrow m_1$

We want to prove $P(a_0 \text{ op } a_1) \triangleq \forall \sigma. \exists m. \langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m$

Inductive case (ctd)

Take a generic $\sigma \in \mathbb{M}$ and consider the goal $\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m$

By rule $\frac{\langle a_0, \sigma \rangle \longrightarrow n_0 \quad \langle a_1, \sigma \rangle \longrightarrow n_1}{\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow n_0 \text{ op } n_1}$ we have

$$\langle a_0 \text{ op } a_1, \sigma \rangle \longrightarrow m \nwarrow_{[m=m_0 \text{ op } m_1]} \langle a_0, \sigma \rangle \longrightarrow m_0, \langle a_1, \sigma \rangle \longrightarrow m_1$$

By inductive hypotheses, there are m_0, m_1 s.t.

$$\langle a_0, \sigma \rangle \longrightarrow m_0 \text{ and } \langle a_1, \sigma \rangle \longrightarrow m_1$$

And we are done (taking $m = m_0 \text{ op } m_1$)