

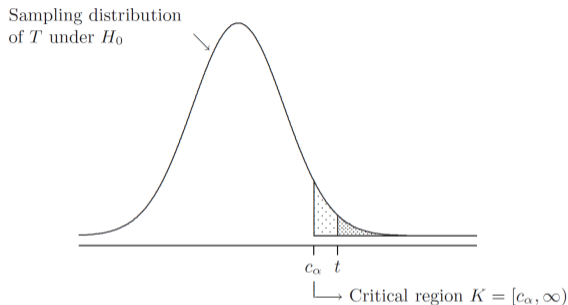
Statistical Methods for Data Science

Lesson 22 - Multiple comparisons. Fitting distributions.

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Critical values and p-values



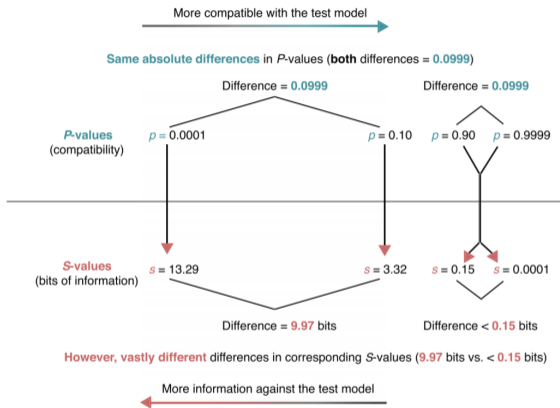
- *Critical region K* : the set of values that reject H_0 in favor of H_1 at significance level α
- *Critical values*: values on the boundary of the critical region
- *p-value*: the probability of obtaining test results at least as extreme as the results actually observed, under the assumption that H_0 is true
- $t \in K$ iff $p\text{-value} \leq \alpha$

Misues of p -values

Misinterpretations of p -values, Greenland et al., 2016

- ~~The p value is the probability that the null hypothesis is true, or the probability that the alternative hypothesis is false.~~ A p -value indicates the degree of compatibility between a dataset and a particular hypothetical explanation
- ~~The 0.05 significance level is the one to be used:~~ No, it is merely a convention. There is no reason to consider results on opposite sides of any threshold as qualitatively different.
- ~~A large p value is evidence in favor of the test hypothesis:~~ A p -value cannot be said to favor the test hypothesis except in relation to those hypotheses with smaller p -values
- ~~If you reject the test hypothesis because $p \leq 0.05$, the chance you are in error is 5%:~~ No, the chance is either 100% or 0%. The 5% refers only to how often you would reject it, and therefore be in error.

s-values



- Shannon information value or surprisal value (s -value) is $-\log_2 p$
- $p = 0.05 \Rightarrow s = 4.3$ - no more surprising than getting all heads on 4 fair coin tosses.
- $p = 0.005 \Rightarrow p = 7.64$ - no more surprising than getting all heads on 8 fair coin tosses.

The multiple comparisons problem

- Single test $H_0 : \theta = 0$, with significance level $\alpha = 0.05$ [false positive rate]
 - ▶ test is called *significant* when we reject H_0
 - ▶ α is Type I error, probability of rejecting H_0 when it is true
- Multiple tests, say $m = 20$
 - ▶ E.g., $H_0^i : \theta_i = 0$ for $i = 1, \dots, m$ where θ_i is the **expectation of a subpopulation**
- What is the probability of rejecting at least one H_0^i when all of them are true?

$$P(\text{at least one reject}) = P(\cup_{i=1}^m \{p_i \leq \alpha\}) = 1 - P(\cap_{i=1}^m \{p_i > \alpha\}) = 1 - (1 - \alpha)^m$$

and then $1 - (0.95)^{20} \approx 0.64$

Family-wise error rate (FWER)

The FWER is the probability of making at least one Type I error in a family of n tests. If the tests are independent:

$$\alpha_{FWER} = 1 - (1 - \alpha)^m$$

If the test are dependent: $\alpha_{FWER} \leq m \cdot \alpha$

Multiple comparisons: corrections

- *Bonferroni correction* (most conservative one):

$$\alpha = \frac{\alpha_{FWER}}{m}$$

Hence, $p < \alpha$ iff $p \cdot m < \alpha_{FWER}$

- *Šidák correction* (exact for independent tests):

$$\alpha = 1 - (1 - \alpha_{FWER})^{1/m}$$

Hence, $p < \alpha$ iff $1 - (1 - p)^m < \alpha_{FWER}$

See R script

False Discovery Rate and q -values

		True state of nature	
		H_0 is true	H_1 is true
Our decision on the basis of the data	Reject H_0	False Positive	True Positive
	Not reject H_0	True Negative	False Negative

- False Positive Rate: $FPR = FP / (FP + TN)$
 - ▶ Corrections control for FPR since $FWER = P(FP > 0 | H_0^i \ i = 1, \dots, m)$
- Drawback: acting on α increases $FNR = FN / (FN + TP)$
- False Discovery Rate: $FDR = FP / (FP + TP)$
 - ▶ $FDR = 0.05$ means 5% of rejected H_0 's are actually true
- q -value is $P(H_0 | T \geq t)$
 - ▶ FDR can be controlled by requiring $q \leq \text{threshold}$

$$[p = P(T \geq t | H_0)]$$

See R script

Distribution fitting

- Dataset x_1, \dots, x_n realization of $X_1, \dots, X_n \sim F$
- What is a plausible F ?
- Parametric approaches:
 - ▶ Assume $F = F(\lambda)$ for some family F , and estimate λ as $\hat{\lambda}$
 - Maximum Likelihood Estimation (point estimate):

$$\hat{\lambda} = \operatorname{argmax}_{\lambda} L(\lambda)$$

- Parametric bootstrap (p -value):

$$T_{ks} = \sup_{a \in \mathbb{R}} |F_n^*(a) - F_{\hat{\lambda}^*}(a)|$$

- Non-parametric approaches:
 - ▶ Empirical distribution
 - ▶ Kernel Density Estimation
- Goodness of fit: how good is F in fitting the data?

Goodness of fit

- Loss functions (to be minimized)

- ▶ Akaike information criterion (AIC), balances model fit against model simplicity

$$AIC(F(\lambda)) = 2|\lambda| - 2\ell(\lambda)$$

- ▶ Bayesian information criterion (BIC), stronger balances over model simplicity

$$BIC(F(\lambda)) = |\lambda| \log n - 2\ell(\lambda)$$

- Statistics (continuous data):

- ▶ **KS test** $H_0 : X \sim F$ $H_1 : X \not\sim F$ with Kolmogorov-Smirnov (KS) statistic:

$$D = \sup_{a \in \mathbb{R}} |F_n(a) - F(a)| \sim K$$

- ▶ **LR test** $H_0 : X \sim F_1$ $H_1 : X \sim F_2$ with the likelihood-ratio test:

$$\lambda_{LR} = \log \frac{L(F_1(\lambda_1))}{L(F_2(\lambda_2))} = \ell(F_1(\lambda_1)) - \ell(F_2(\lambda_2)) \quad \text{with} \quad -2\lambda_{LR} \sim \chi^2$$

See R script

Chi-square distribution

The Chi-square distribution with k degrees of freedom $\chi^2(k)$ has density:

$$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}$$

Let $X_1, \dots, X_k \sim N(0, 1)$. Then $Y = \sum_{i=1}^k X_i^2 \sim \chi^2(k)$

- Statistics (discrete data):

- ▶ **Pearson's Chi-Square test** $H_0 : X \sim F$ $H_1 : X \not\sim F$ with χ^2 statistic:

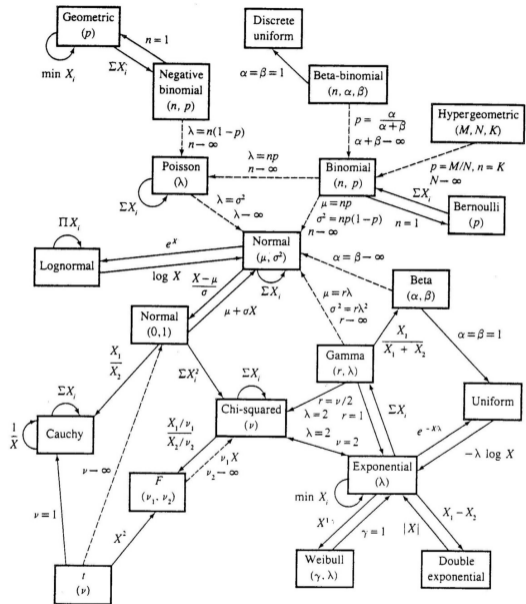
$$\chi^2 = \sum_{N_i > 0} \frac{(N_i - n_i)^2}{n_i} = n \cdot \sum_{N_i > 0} \frac{(N_i/n - p(i))^2}{p(i)} \sim \chi^2(df)$$

where N_i number of observations of value i , $n_i = n \cdot p(i)$ expected number of observations (rescaled), and $df = |\{i \mid N_i > 0\}| - 1$ is the number of observed values minus 1.

$\chi^2 = \infty$ if for some i : $n_i = 0$

See R script

Common distributions



Relationships among common distributions. Solid lines represent transformations and special cases, dashed lines represent limits. Adapted from Leemis (1986).

Comparing two datasets

- Dataset x_1, \dots, x_n realization of $X_1, \dots, X_n \sim F_1$
- Dataset y_1, \dots, y_m realization of $Y_1, \dots, Y_m \sim F_2$
- $H_0 : F_1 = F_2$ $H_1 : F_1 \neq F_2$
- Continuous data: KS statistics

$$D = \sup_{a \in \mathbb{R}} |F_1(a) - F_2(a)| \sim K$$

- Discrete data: χ^2 statistics

$$\chi^2 = \sum_{R_i > 0 \vee S_i > 0} \frac{(\sqrt{\frac{m}{n}} R_i - \sqrt{\frac{n}{m}} S_i)^2}{R_i + S_i} \sim \chi^2(df)$$

where R_i (resp., S_i) is the number of observations in x_1, \dots, x_n (resp., y_1, \dots, y_m) of value i , $df = |\{i \mid R_i > 0 \vee S_i > 0\}| - 1$

See R script