Statistical Methods for Data Science

Lesson 04 - Recalls on calculus

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J. Ward, J. Abdey. Mathematics and Statistics. *University of London, 2013. Chapters 1-8 of Part 1.*

- Errata-corrige at pag. 30: $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot b}{b \cdot d}$ and $\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d - c \cdot b}{b \cdot d}$

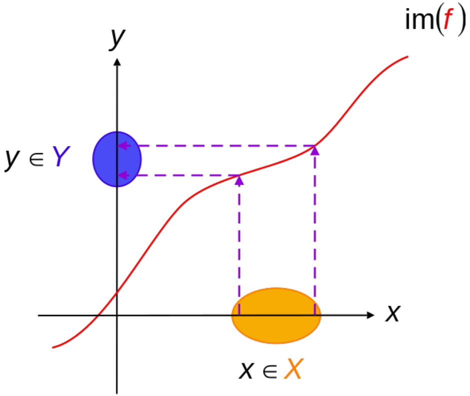
Sets and functions

- Numerical sets
  - $\mathbb{N} = \{0, 1, 2, \ldots\}$ *[Natural numbers]*
  - $\mathbb{Z} = \mathbb{N} \cup \{-1, -2, \ldots\}$ *[Integers]*
  - $\mathbb{Q} = \{{}^{m}/_{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$ *[Rationals]*
  - $\mathbb{R} = \{$ fractional numbers with possibly infinitely many digits $\} \supseteq \mathbb{Q}$ *[Reals]*
  - $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ *[Irrationals]*
    - y such that $y \cdot y = 2$ belongs to $\mathbb{I}$
- Functions
  - $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}$ *[Cartesian product]*
  - $f : \mathbb{R} \to \mathbb{R}$ is a subset $f \subseteq \mathbb{R} \times \mathbb{R}$ such that $(x, y_0), (x, y_1) \in f$ implies $y_0 = y_1$ *[Functions]*
    - usually written $f(x) = y$ for $(x, y) \in f$
    - $f(x) = v$ for all x *[Constant functions]*
    - $f(x) = a \cdot x + b$ for fixed a, b *[Linear functions]*
    - $f(x) = a \cdot x^2 + b$ for fixed a, b *[Quadratic functions]*
    - $f(x) = \sum_{i=0}^{n} a_i \cdot x^i$ for fixed $a_0, \ldots, a_n$ *[Polinomials]*

See R script

Functions

- $dom(f) = \{x \in \mathbb{R} \mid \exists\ y \in \mathbb{R}.(x, y) \in f\}$ *[Domain or Support]*
- $im(f) = \{y \in \mathbb{R} \mid \exists\ x \in \mathbb{R}.(x, y) \in f\}$ *[Co-domain or Image]*
- $f^{-1} = \{(y, x) \mid (x, y) \in f\}$ *[Inverse function]*
  - f^{-1} is a function iff f is injective
  - $f(x) = y$ iff $f^{-1}(y) = x$
  - $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$
- Examples
  - $\sqrt{y} = x$ iff $x^2 = y$ over $x \geq 0$
  - $\sqrt[n]{y} = x$ iff $x^n = y$ over $x \geq 0$ *[positive root]*



Powers and logarithms

Power laws

The power laws state that

$$a^n \cdot a^m = a^{n+m} \qquad \frac{a^n}{a^m} = a^{n-m} \qquad (a^n)^m = a^{nm}$$

provided that both sides of these expressions exist. In particular, we have

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}.$$

If it exists, we also define the *positive* nth root of a, written $\sqrt[n]{a}$, to be $a^{\frac{1}{n}}$.

- $log_a(y) = x$ iff $a^x = y$ for $a \neq 1, x > 0$ *[Logarithms]*
- for $n/m \in \mathbb{Q}$: $\quad a^{n/m} \overset{\text{def}}{=} (a^n)^{1/m}$
- what is a^x for $x \in \mathbb{I}$?

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots = \sum_{n \geq 0} \frac{x^n}{n!}$$

and $a^x = (e^{log_e(a)})^x = e^{x \cdot log_e(a)}$

See R script

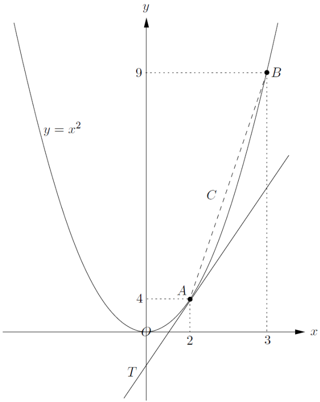
Gradient and derivatives

- The **gradient** of a straight line is a measure of how 'steep' the line is.

$$y = a \cdot x + b$$

  a is the gradient and b the intercept (at $x = 0$)
- For $y = f(x) = x^2$?
  - Tangent at $x = a$ is $y = m \cdot x + b$
  - $m = \frac{f(a+\delta)-f(a)}{\delta} = \frac{2 \cdot a \cdot \delta + \delta^2}{\delta} = 2 \cdot a$ for $\delta \to 0$
  - $b = 2 \cdot a - a^2$ because $m \cdot a + b = a^2$
- More in general?
  - For $y = f(x)$, $m = f'(x)$
  - $f'()$ is called the **derivative** of $f()$, also written $\frac{\delta f}{\delta x}$ or $\frac{df}{dx}$
  - Not all functions are differentiable!

See R script or **this Colab Notebook**



Derivatives

Standard derivatives

- If k is a constant, then $f(x) = k$ gives $f'(x) = 0$.
- If $k \neq 0$ is a constant, then $f(x) = x^k$ gives $f'(x) = kx^{k-1}$.
- $f(x) = \mathrm{e}^x$ gives $f'(x) = \mathrm{e}^x$.
- $f(x) = \ln x$ gives $f'(x) = \dfrac{1}{x}$.

- Constant multiple rule:

$$\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{df}{dx}(x)$$

- Sum rule:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx}(x) + \frac{dg}{dx}(x)$$

Derivatives

- Product rule:

$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{df}{dx}(x) \cdot g(x) + f(x) \cdot \frac{dg}{dx}(x)$$

- Quotient rule:

$$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \left[\frac{df}{dx}(x) \cdot g(x) - f(x) \cdot \frac{dg}{dx}(x)\right] \cdot \frac{1}{g(x)^2}$$

- Chain rule:

$$\frac{d}{dx}[f(g(x))] = \frac{df}{dg}(g(x)) \cdot \frac{dg}{dx}(x)$$

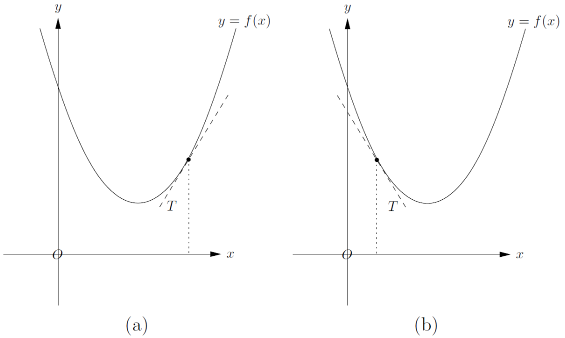
- $\frac{d}{dx} e^{-x} = \dots$
- Inverse rule:

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{\frac{df}{dx}(f^{-1}(x))}$$

- $\frac{d}{dx} log\, x = \dots$

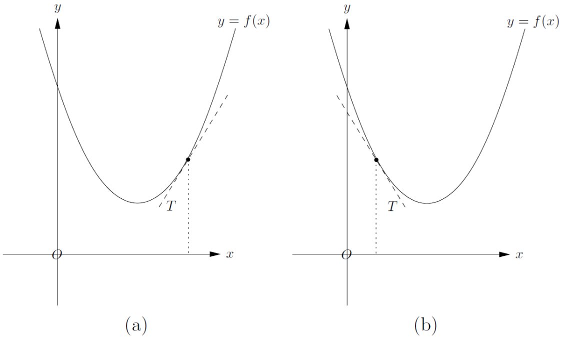
See R script or this Colab Notebook

Optimization



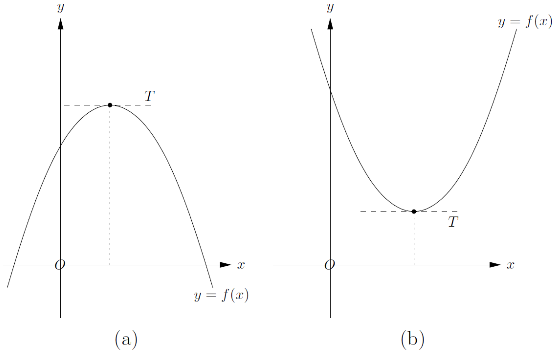
- $f'(x) > 0$ implies $f()$ is increasing at x
- $f'(x) < 0$ implies $f()$ is decreasing at x
- $f'(x) = 0$ we cannot say

Optimization



- $f'(x) > 0$ implies $f()$ is increasing at x
- $f'(x) < 0$ implies $f()$ is decreasing at x
- $f'(x) = 0$ we cannot say *[Stationary point]*

Optimization - second derivatives

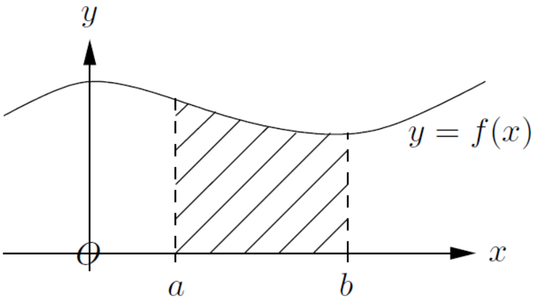


- $f''(x) < 0$ implies $f(x)$ is a maximum
- $f''(x) > 0$ implies $f(x)$ is a minimum
- $f'(x) = 0$ we cannot say *[Maximum, minimum, or point of inflection]*

See this Colab Notebook

Integration

- Given $f'(x)$, what is $f(x)$?
- Integration is the inverse of differentiation
- Geometrical meaning:



Integration

Key concepts in integration

If $F(x)$ is a function whose derivative is the function $f(x)$, then we have

$$\int f(x)\,\mathrm{d}x = F(x) + c,$$

where c is an arbitrary constant. In particular, we call the

- function, $f(x)$, the *integrand* as it is what we are integrating,
- function, $F(x)$, **an** *antiderivative* as its derivative is $f(x)$,
- constant, c, a *constant of integration* which is completely arbitrary,[†] and
- integral, $\int f(x)\,\mathrm{d}x$, an *indefinite integral* since, in the result, c is arbitrary.

- Definite integrals over an interval $[a, b]$:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Integration

Standard integrals

- If $k \neq -1$ is a constant, then $\int x^k \, \mathrm{d}x = \frac{x^{k+1}}{k+1} + c$.

In particular, if $k = 0$, we have $\int 1 \, \mathrm{d}x = \int x^0 \, \mathrm{d}x = x + c$.

- $\int x^{-1} \, \mathrm{d}x = \ln |x| + c$.
- $\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + c$.

- Constant multiple rule:

$$\int [k \cdot f(x)] dx = k \cdot \int f(x) dx$$

- Sum rule:

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

See R script

Integration by parts

- From the product rule of derivatives:

$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{df}{dx}(x) \cdot g(x) + f(x) \cdot \frac{dg}{dx}(x)$$

- take the inverse of both sides:

$$f(x) \cdot g(x) = \int f'(x) \cdot g(x)dx + \int f(x) \cdot g'(x)dx$$

- and then:

$$\int f(x) \cdot g'(x)dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)dx$$

- $\int \lambda x e^{-\lambda x} dx = \ldots = -e^{-\lambda x}(x + 1/\lambda)$
  - consider $f(x) = x$ and $g'(x) = \lambda e^{-\lambda x}$
  - $g(x) = -e^{-\lambda x}$ and $f'(x) = 1$

Integration by change of variable

- Change of variable rule:

$$\int f(y)dy =_{y=g(x)} \int f(g(x))g'(x)dx$$

- $\int \frac{log x}{x} dx = \int y dy = {}^{y^2}/_2$ for $y = log x$ hence, $\int \frac{log x}{x} dx = (log x)^2$
  - consider $f(y) = y$ and $g(x) = log x$