Master Program in *Data Science and Business Informatics*

Statistics for Data Science

Lesson 11 - Moments. Functions of random variables

Salvatore Ruggieri

Department of Computer Science
University of Pisa, Italy
salvatore.ruggieri@unipi.it

Moments

- Let X be a continuous random variable with density function $f(x)$
- k^{th} *moment* of X, if it exists, is:

$$E[X^k] = \int_{-\infty}^{\infty} x^k f(x) dx$$

- $\mu = E[X]$ is the first moment of X
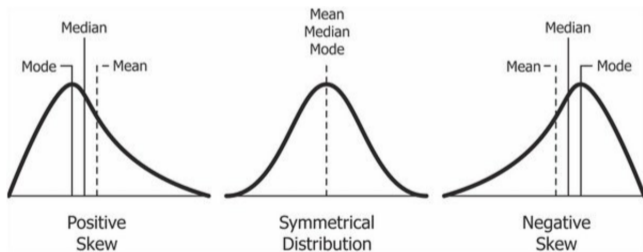- k^{th} *central moment* of X is:

$$\mu_k = E[(X - \mu)^k] = \int_{-\infty}^{\infty} (x - \mu)^k f(x) dx$$

- $\sigma = \sqrt{E[(X - \mu)^2]}$ standard deviation is the square root of the second central moment
- k^{th} *standardized moment* of X is:

$$\tilde{\mu}_k = \frac{\mu_k}{\sigma^k} = E\left[(\frac{X - \mu}{\sigma})^k\right]$$

Skewness

- $\tilde{\mu}_1 = {}^{E[(X-\mu)]}/_\sigma = 0$ since $E[X - \mu] = 0$
- $\tilde{\mu}_2 = {}^{E[(X-\mu)^2]}/_{\sigma^2} = 1$ since $\sigma^2 = E[(X - \mu)^2]$
- $\tilde{\mu}_3 = {}^{E[(X-\mu)^3]}/_{\sigma^3}$ *[(Pearson's moment) coefficient of skewness]*
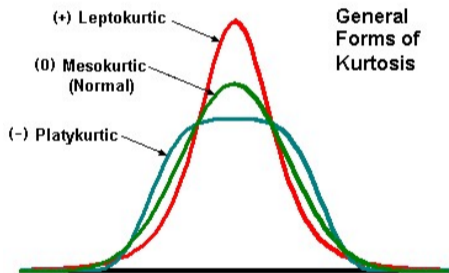- Skewness indicates direction and magnitude of a distribution's deviation from symmetry

- E.g., for $X \sim Exp(\lambda)$, $\tilde{\mu}_3 = 2$ **Prove it!**

Kurtosis

- $\tilde{\mu}_4 = E[(\frac{X-\mu}{\sigma})^4]$ *[(Pearson's moment) coefficient of kurtosis]*
- For $X \sim N(\mu, \sigma)$, $\tilde{\mu}_4 = 3$ *$\tilde{\mu}_4 - 3$ is called kurtosis in excess*
- Kurtosis is a measure of the dispersion of X around the two values $\mu \pm \sigma$

- $\tilde{\mu}_4 > 3$ *Leptokurtic* (slender) distribution has *fatter* tails. May have outlier problems.
- $\tilde{\mu}_4 < 3$ *Platykurtic* (broad) distribution has *thinner* tails

See R script

Functions of random variables: expectation

- $V = \pi H R^2$ be the volume of a vase of height H and radius R
- $g(H, R) = \pi H R^2$ is a random variable (function of random variables)
- $P_V(V = 3) = P_{HR}(\pi H R^2 = 3)$
- How to calculate $E[V]$ for $H \perp\!\!\!\perp R$?

$$E[V] = E[\pi H R^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi h r^2 f_H(h) f_R(r) \, dh \, dr$$

TWO-DIMENSIONAL CHANGE-OF-VARIABLE FORMULA. Let X and Y be random variables, and let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function. If X and Y are *discrete* random variables with values $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, respectively, then

$$\mathrm{E}[g(X, Y)] = \sum_i \sum_j g(a_i, b_j) \mathrm{P}(X = a_i, Y = b_j).$$

If X and Y are *continuous* random variables with joint probability density function f, then

$$\mathrm{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

Linearity of expectations

Theorem. For X and Y random variables, and $s, t \in \mathbb{R}$:

$$E[rX + sY + t] = rE[X] + sE[Y] + t$$

Proof. (discrete case)

$$E[rX + Ys + t] = \sum_a \sum_b (ra + sb + t)P(X = a, Y = b)$$

$$= \left(r \sum_a \sum_b aP(X = a, Y = b) \right) + \left(s \sum_a \sum_b bP(X = a, Y = b) \right) + \left(t \sum_a \sum_b P(X = a, Y = b) \right)$$

$$= \left(r \sum_a aP(X = a) \right) + \left(s \sum_b bP(Y = b) \right) + t = rE[X] + sE[Y] + t$$

Corollary. $E[a_0 + \sum_{i=1}^{n} a_i X_i] = a_o + \sum_{i=1}^{n} a_i E[X_i]$

Corollary. $X \leq Y$ implies $E[X] \leq E[Y]$
Proof. $Z = Y - X \geq 0$ implies $E[Z] = E[Y] - E[X] \geq 0$, i.e., $E[Y] \geq E[X]$.

Applications

- Expectation of some discrete distributions
  - $X \sim Ber(p)$ $E[X] = p$
  - $X \sim Bin(n, p)$ $E[X] = n \cdot p$
    - Because $X = \sum_{i=1}^{n} X_i$ for $X_1, \ldots, X_n \sim Ber(p)$
  - $X \sim Geo(p)$ $E[X] = \frac{1}{p}$
  - $X \sim NBin(n, p)$ $E[X] = \frac{n \cdot (1-p)}{p}$
    - Because $X = \sum_{i=1}^{n} X_i - n$ for $X_1, \ldots, X_n \sim Geo(p)$
- Expectation of some continuous distributions
  - $X \sim Exp(\lambda)$ $E[X] = 1/\lambda$
  - $X \sim Erl(n, \lambda)$ $E[X] = \frac{n}{\lambda}$
    - Because $X = \sum_{i=1}^{n} X_i$ for $X_1, \ldots, X_n \sim Exp(\lambda)$

Expectation of product and quotients

Theorem. For $X \perp\!\!\!\perp Y$, we have: $E[XY] = E[X]E[Y]$ **Prove it!**

> PROPAGATION OF INDEPENDENCE. Let $X_1, X_2, \ldots, X_n$ be independent random variables. For each i, let $h_i : \mathbb{R} \to \mathbb{R}$ be a function and define the random variable
>
> $$Y_i = h_i(X_i).$$
>
> Then $Y_1, Y_2, \ldots, Y_n$ are also independent.

Corollary. For $X \perp\!\!\!\perp Y$ and $Y \geq 0$, we have: $E[X/Y] \geq E[X]/E[Y]$
Proof. $X \perp\!\!\!\perp Y$ implies $X \perp\!\!\!\perp 1/Y$. By theorem above:

$$E[X/Y] = E[X \cdot 1/Y] = E[X]E[1/Y]$$

and then by Jensen's inequality $E[X/Y] \geq E[X]/E[Y]$ since $1/y$ is convex for $y \geq 0$. $\qquad \square$

Exercise at home. Show that $E[X/Y] = E[X]/E[Y]$ is a false claim.

Law of iterated expectations

> ### Conditional expectation
>
> $$E[X|Y = b] = \sum_i a_i p_{X|Y}(a_i|b) \qquad E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

- **Def.** $E_Y[E[X|Y]] = \sum_j E[X|Y = b_j] p_Y(b_j)$ and $E_Y[E[X|Y]] = \int_{-\infty}^{\infty} E[X|Y = y] f_Y(y) dy$

- **Theorem.** (Law of iterated expectations)

$$E_Y[E[X|Y]] = E[X]$$

Proof. (X, Y discrete random variables)

$$E_Y[E[X|Y]] = \sum_j \sum_i a_i p_{X|Y}(a_i|b_j) p_Y(b_j) = \sum_j \sum_i a_i p_{XY}(a_i, b_j) = \sum_i a_i p_X(a_i) = E[X]$$
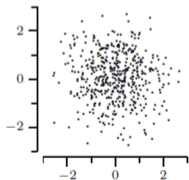
Variance of the sum and Covariance

$$Var(X + Y) = E[(X + Y - E[X + Y])^2] = E[((X - E[X]) + (Y - E[Y]))^2]$$
$$= E[(X - E[X])^2] + E[(Y - E[Y])^2] + 2E[(X - E[X])(Y - E[Y])]$$
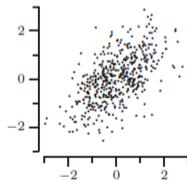$$= Var(X) + Var(Y) + 2Cov(X, Y)$$

Covariance

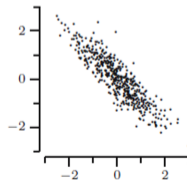The *covariance* $Cov(X, Y)$ of two random variables X and Y is the number:

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$

Uncorrelated **Positively correlated** **Negatively correlated**

Covariance

Theorem. $Cov(X, Y) = E[XY] - E[X]E[Y]$ **Prove it!**

- If X and Y are independent, $Cov(X, Y) = 0$ and $Var(X + Y) = Var(X) + Var(Y)$
- But there are X and Y uncorrelated (ie., $Cov(X, Y) = 0$) that are dependent!
- Variances of some discrete distributions
  - $X \sim Ber(p)$ $Var(X) = p(1 - p)$
  - $X \sim Bin(n, p)$ $Var(X) = np(1 - p)$
    - $\square$ Because $X = \sum_{i=1}^{n} X_i$ for $X_1, \ldots, X_n \sim Ber(p)$ and independent
  - $X \sim Geo(p)$ $Var(X) = \frac{1-p}{p^2}$
  - $X \sim NBin(n, p)$ $Var(X) = n\frac{1-p}{p^2}$
    - $\square$ Because $X = \sum_{i=1}^{n} X_i - n$ for $X_1, \ldots, X_n \sim Geo(p)$ and independent
- Variances of some continuous distributions
  - $X \sim Exp(\lambda)$ $Var(X) = 1/\lambda^2$
  - $X \sim Erl(n, \lambda)$ $Var(X) = \frac{n}{\lambda^2}$
    - $\square$ Because $X = \sum_{i=1}^{n} X_i$ for $X_1, \ldots, X_n \sim Exp(\lambda)$ and independent

Covariance

COVARIANCE UNDER CHANGE OF UNITS. Let X and Y be two random variables. Then

$$\operatorname{Cov}(rX + s, tY + u) = rt \operatorname{Cov}(X, Y)$$

for all numbers $r, s, t,$ and u.

- Hence, $Var(rX + sY + t) = r^2 Var(X) + s^2 Var(Y) + 2rs Cov(X, Y)$
- Covariance depends on the units of measure!

Correlation coefficient

> DEFINITION. Let X and Y be two random variables. The *correlation coefficient* $\rho(X, Y)$ is defined to be 0 if $\text{Var}(X) = 0$ or $\text{Var}(Y) = 0$, and otherwise
> $$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}.$$

- Correlation coefficient is *dimensionless* (not affected by change of units)
  - E.g., if X and Y are in Km, then $Cov(X, Y)$, $Var(X)$ and $Var(Y)$ are in Km^2
  $$-1 \leq \rho(X, Y) \leq 1$$

- The bounds are derived from the **Schwarz's inequality**:
$$E[|XY|] \leq \sqrt{E[X^2]}\sqrt{E[Y^2]}$$

Proof. For any $u, w \in \mathbb{R}$, we have $2|uw| \leq u^2 + w^2$. Therefore, $2|UW| \leq U^2 + W^2$ for r.v.'s U and V. By defining $U = X/\sqrt{E[X^2]}$ and $W = Y/\sqrt{E[Y^2]}$ (*), we have $2XY/\sqrt{E[X^2]}\sqrt{E[Y^2]} \leq X^2/E[X^2] + Y^2/E[Y^2]$. Taking the expectations, we conclude: $2E[XY]/\sqrt{E[X^2]}\sqrt{E[Y^2]} \leq 2$.   (*) The case $E[X^2] = 0$ or $E[Y^2] = 0$ is left as an exercise.   □

Sum of independent random variables

- For $X \sim F_X$ and $Y \sim F_Y$, let $Z = X + Y$. We know

$$E[Z] = E[X] + E[Y] \qquad Var(Z) = Var(X) + Var(Y) + 2Cov(X, Y)$$

- What is the distribution function of Z when $X \perp\!\!\!\perp Y$?

- Examples:
  - For $X \sim Bin(n, p)$ and $Y \sim Bin(m, p)$, $Z \sim Bin(n + m, p)$
  - For $X \sim Geo(p)$ (days radio 1 breaks) and $Y \sim Geo(p)$ (days radio 2 breaks):

$$p_Z(X + Y = k) = \sum_{l=1}^{k-1} p_X(l) \cdot p_Y(k - l) = (k - 1)p^2(1 - p)^{k-2}$$

- See Lesson 04 and Lesson 08 for convolution formulas

> ADDING TWO INDEPENDENT DISCRETE RANDOM VARIABLES. Let X
> and Y be two independent discrete random variables, with probability mass functions p_X and p_Y. Then the probability mass function
> p_Z of $Z = X + Y$ satisfies
>
> $$p_Z(c) = \sum_j p_X(c - b_j)p_Y(b_j),$$

Sum of two Normal random variables

Theorem. If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ and $X \perp\!\!\!\perp Y$, then:

$$Z = X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

Proof. See [T, Sect. 11.2] □

- In general: $Z = aX + bY + c \sim N(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$

- The converse of the theorem also holds: [**Lévy-Cramér theorem**]

  ▸ If $X \perp\!\!\!\perp Y$ and $Z = X + Y$ is normally distributed, then X and Y follow a normal distribution.

Extremes of independent random variables

THE DISTRIBUTION OF THE MAXIMUM. Let $X_1, X_2, \ldots, X_n$ be n independent random variables with the same distribution function F, and let $Z = \max\{X_1, X_2, \ldots, X_n\}$. Then

$$F_Z(a) = (F(a))^n.$$

- $P(Z \leq a) = P(X_1 \leq a, \ldots, X_n \leq a) = \prod_{i=1}^{n} P(X_i \leq a) = ((F(a))^n$
- Example: maximum water level over 365 days

THE DISTRIBUTION OF THE MINIMUM. Let $X_1, X_2, \ldots, X_n$ be n independent random variables with the same distribution function F, and let $V = \min\{X_1, X_2, \ldots, X_n\}$. Then

$$F_V(a) = 1 - (1 - F(a))^n.$$

- $P(V \leq a) = 1 - P(X_1 > a, \ldots, X_n > a) = 1 - \prod_{i=1}^{n}(1 - P(X_i \leq a) = 1 - ((1 - F(a))^n$

Product and quotient of independent random variables

PRODUCT OF INDEPENDENT CONTINUOUS RANDOM VARIABLES. Let X and Y be two independent continuous random variables with probability densities f_X and f_Y. Then the probability density function f_Z of $Z = XY$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y\left(\frac{z}{x}\right) f_X(x)\frac{1}{|x|}\,\mathrm{d}x$$

for $-\infty < z < \infty$.

QUOTIENT OF INDEPENDENT CONTINUOUS RANDOM VARIABLES. Let X and Y be two independent continuous random variables with probability densities f_X and f_Y. Then the probability density function f_Z of $Z = X/Y$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(zx)f_Y(x)|x|\,\mathrm{d}x$$

for $-\infty < z < \infty$.

- $X, Y \sim N(0, 1)$ independent, $Z = X/Y \sim Cau(0, 1)$ where:

$$f_Z(x) = \frac{1}{\pi(1 + x^2)}$$

Kullback-Leibler divergence

KL divergence

For X, Y discrete random variables with p.m.f. p_X and p_Y:

$$D(X \parallel Y) = \sum_a p_X(a) \log \frac{p_X(a)}{p_Y(a)} = H(X, Y) - H(X)$$

where $H(X) = -\sum_a p_X(a) \log p_X(a)$ and $H(X; Y) = -\sum_a p_X(a) \log p_Y(a)$

- Measure how distribution of Y (model) can reconstruct the distribution of X (data)
  - Also called: relative entropy or information gain of X w.r.t. Y
  - $H(X)$ is the entropy of X, and $H(X, Y)$ is the **cross entropy** of X w.r.t Y
  - $H(X; Y)$ is the "information" or "uncertainty" or "loss" when using Y to encode X
- Properties
  - $D(X \parallel Y) = 0$ iff $P(X \neq Y) = 0$, $\quad D(X \parallel Y) \neq D(Y \parallel X)$, and
  - $D(X \parallel Y) \geq 0$ **[Gibbs' inequality]**
- For X, Y continuous: $D(X \parallel Y) = \int_{-\infty}^{\infty} f_X(x) \log \frac{f_X(x)}{f_Y(x)} dx$ **See R script**

Mutual information

Mutual information

For X, Y discrete random variables with p.m.f. p_X and p_Y and joint p.m.f. p_{XY}:

$$I(X, Y) = D(p_{XY} \parallel p_X p_Y) = \sum_{a,b} p_{XY}(a, b) \log \frac{p_{XY}(a, b)}{p_X(a) p_Y(b)} = H(X) + H(Y) - H((X, Y))$$

where $H(X) = -\sum_a p_X(a) \log p_X(a)$ and $H((X, Y)) = -\sum_{a,b} p_{XY}(a, b) \log p_{XY}(a, b)$

- MI measures how dependent two distributions are
  - Measure how product of marginals can reconstruct the distribution joint distribution
- Properties
  - $I(X, Y) = I(Y, X)$, and $I(X, Y) \geq 0$
  - $I(X, Y) = 0$ iff $X \perp\!\!\!\perp Y$
  - $NMI = \frac{I(X,Y)}{\min\{H(X), H(Y)\}} \in [0, 1]$ *[Normalized mutual information]*
- For X, Y continuous: $I(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) \log \frac{f_{XY}(x,y)}{f_X(x) f_Y(y)} dx dy$ **See R script**

Optional reference

📄 Kevin P. Murphy (2022)
Probabilistic Machine Learning: An Introduction
Chapter 6: Information Theory
online book