

Master Program in *Data Science and Business Informatics*

Statistics for Data Science

Lesson 13 - Power laws and Zipf's law

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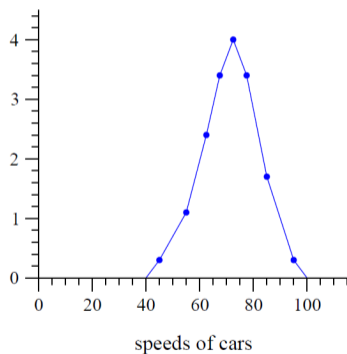
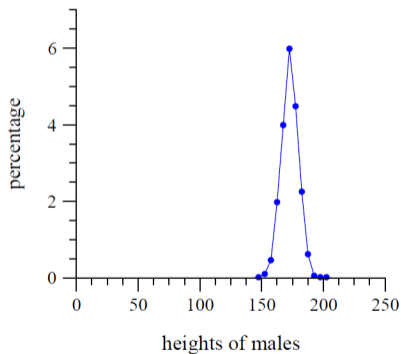
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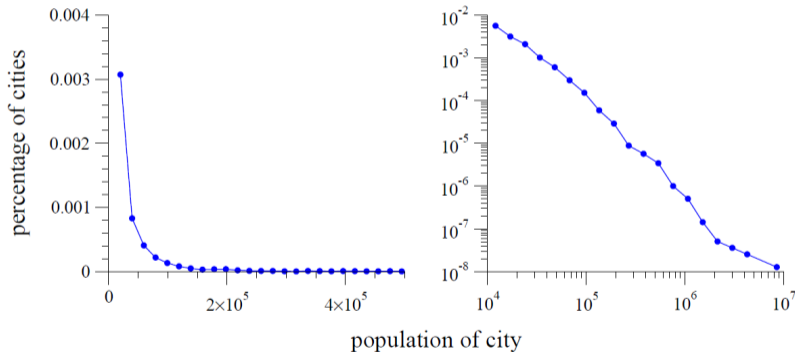
Scaled distributions

- Many of the things that scientists measure have a typical size or “scale” — a typical value around which individual measurements are centered



Scale-free distributions

- But not all things we measure are peaked around a typical value. Some vary over an enormous dynamic range.



Look at [Figure 4 of Newman's paper](#)

Continuous power-law

Power-law

A continuous random variable X has the *power-law distribution*, if for some $\alpha > 1$ its density function is given by

$$p(x) = C \cdot x^{-\alpha} \quad \text{for } x \geq x_{min}$$

We denote this distribution by $Pow(x_{min}, \alpha)$.

- C is called the **intercept**, and α the **exponent**.
- Passing to the logs:

$$\log p(x) = -\alpha \cdot \log(x) + \log C$$

linearity in log-log scale plots!

See R script

Scale-free distributions

$$p(bx) = g(b)p(x)$$

- Measuring in cm, inches, Km, or miles does not change the form of the distribution (up to some constant)!
- For a power-law $p(x) = Cx^{-\alpha}$

$$p(bx) = b^{-\alpha} Cx^{-\alpha}$$

hence, $g(b) = b^{-\alpha}$

- Actually, power-laws are the only scale-free distributions!
 - ▶ see Eq. 30-34 of [Newman's paper](#) for a proof

Intercept

- What is the constant C ?

$$1 = \int_{x_{min}}^{\infty} C \cdot x^{-\alpha} dx = \frac{C}{-\alpha + 1} [x^{-\alpha+1}]_{x_{min}}^{\infty} \stackrel{(*)}{=} \frac{C}{-\alpha + 1} (0 - x_{min}^{-\alpha+1}) = \frac{C}{\alpha - 1} x_{min}^{-\alpha+1}$$

(*) Finite only for $\alpha > 1$, because:

- ▶ for $\alpha < 1$: $\lim_{x \rightarrow \infty} x^{-\alpha+1} = \infty$
 - ▶ for $\alpha = 1$: denominator $-\alpha + 1$ is 0
- Therefore:

$$C = (\alpha - 1) / x_{min}^{-\alpha+1} \quad (1)$$

and, in summary:

$$p(x) = \frac{(\alpha - 1)}{x_{min}} \left(\frac{x}{x_{min}} \right)^{-\alpha}$$

- Let's compute:

$$P(X > x) = \int_x^{\infty} p(y) dy = C \int_x^{\infty} y^{-\alpha} dy = \frac{C}{-\alpha + 1} [y^{-\alpha+1}]_x^{\infty} = \frac{C}{\alpha - 1} x^{-\alpha+1}$$

and since $C = (\alpha - 1)/x_{min}^{-\alpha+1}$:

$$P(X > x) = \left(\frac{x}{x_{min}}\right)^{-\alpha+1} = \left(\frac{x}{x_{min}}\right)^{-(\alpha-1)}$$

- Same form as df (see Eq. 1) but with exponent $(\alpha - 1)$

See R script

Pareto distribution

- **Vilfredo Pareto** noticed that the number of people whose income exceeded level x (i.e., CCDF) is well approximated by C/x^β for some constants C and $\beta > 0$
 - ▶ It appears that for all countries $\beta \approx 1.5$.

Pareto distribution

A continuous random variable X has the *Pareto distribution*, if for some $\beta > 0$ its density function is given by

$$p(x) = C \cdot x^{-(\beta+1)} \quad \text{for } x \geq x_{min}$$

We denote this distribution by $Par(x_{min}, \beta)$.

- $Par(x_{min}, \beta) = Pow(x_{min}, \beta + 1)$ or $Pow(x_{min}, \alpha) = Par(x_{min}, \alpha - 1)$
- CCDF of $Par(x_{min}, \beta)$ is $(\frac{x}{x_{min}})^{-((\beta+1)-1)} = (\frac{x}{x_{min}})^{-\beta}$

See R script

Expectation and variance of a power-law

- What is the expectation of $X \sim \text{Pow}(x_{\min}, \alpha)$?

$$E[X] = \int_{x_{\min}}^{\infty} x \cdot p(x) dx = C \int_{x_{\min}}^{\infty} x^{-\alpha+1} dx = \frac{C}{-\alpha+2} [x^{-\alpha+2}]_{x_{\min}}^{\infty} \stackrel{(*)}{=} \frac{C}{\alpha-2} x_{\min}^{-\alpha+2}$$

(*) Finite only for $\alpha > 2$, because:

$$\lim_{x \rightarrow \infty} x^{-\alpha+2} = \infty \text{ for } \alpha \leq 2$$

and since $C = (\alpha - 1)/x_{\min}^{-\alpha+1}$:

$$E[X] = \frac{\alpha - 1}{\alpha - 2} x_{\min}$$

- ▶ For $1 < \alpha \leq 2$, there is no expectation: the mean of a dataset has no reliable value!
- $\text{Var}(X)$ finite only for $\alpha > 3$
 - ▶ For $2 < \alpha \leq 3$, there is no variance: the empirical variance of a dataset has no reliable value!

Discrete power-law

Discrete power-law

A discrete random variable X has the *power-law distribution*, if for some $\alpha > 1$ its p.m.f. function is given by

$$p(k) = C \cdot k^{-\alpha} \quad \text{for } k = k_{min}, k_{min} + 1, \dots$$

We denote this distribution by $Pow(k_{min}, \alpha)$.

- Population of cities, number of books sold, number of citations, etc.
- Since $1 = \sum_{k=k_{min}}^{\infty} C \cdot k^{-\alpha}$, we have

$$C = \frac{1}{\sum_{k=k_{min}}^{\infty} k^{-\alpha}} = \frac{1}{\zeta(\alpha, k_{min})}$$

where $\zeta(\alpha, k_{min}) = \sum_{k=k_{min}}^{\infty} k^{-\alpha}$

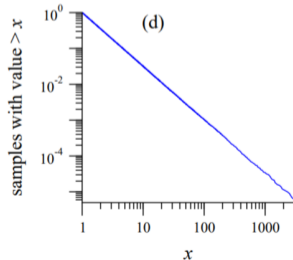
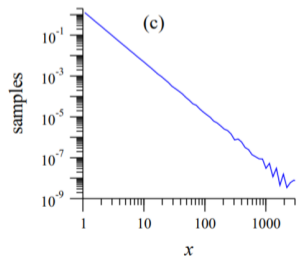
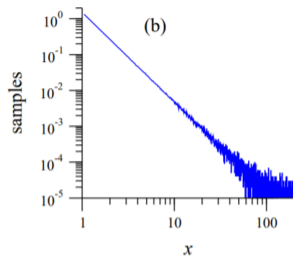
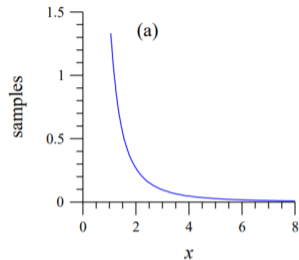
- Special case: $\zeta(\alpha) = \zeta(\alpha, 1) = \sum_{k=1}^{\infty} k^{-\alpha}$

[Hurwitz zeta-function]

[Riemann zeta-function]

See R script

Logarithmic binning vs CCDF



See R script

Zipf's law

Zipf's law distribution

A discrete random variable R has the *Zipf's law distribution*, if for some $\alpha > 1$ its p.m.f. function is given by

$$p(r) = C \cdot r^{-\alpha} \quad \text{for } r = 1, 2, \dots, N$$

We denote this distribution by $\text{Zipf}(\alpha)$.

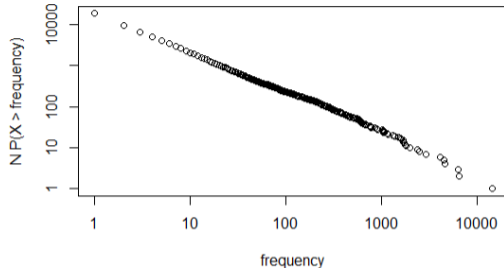
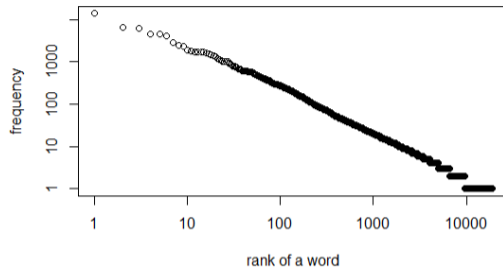
- Since $\sum_{r=1}^N C \cdot r^{-\alpha} = 1$, we have:

$$C = \frac{1}{\sum_{r=1}^N r^{-\alpha}} = \frac{1}{\zeta(\alpha) - \zeta(\alpha, N+1)}$$

- Read $p(r)$ as the probability of an event based its rank
 - ▶ e.g., prob. of occurrence of a word in a book given the word rank, prob. of occurrence of an inhabitant of a city given the city rank
 - *Contrast to discrete power laws*: prob. of words with a given number of occurrences, prob. of cities with a given number of inhabitants
 - ▶ If V the total number of words/inhabitants, $V \cdot p(r)$ is the frequency/population of the word/city of rank r . Alternatively, if v is the population of the city $p(r) = v/V$

See R script

Zipf's law



Left: (rank-frequency plot) frequency of words based on rank

Right: number of words with a given minimum frequency

[Zipf's law]

[CCDF of a Power-law]

From power-law to Zipf's law and vice-versa

- $\Omega = \{\omega_1, \dots, \omega_N\}$, ω_i is a city with n_i inhabitants, for a total of N cities and $V = \sum_{i=1}^N n_i$ inhab.
- $P_1(\omega_i) = 1/N$ and $X(\omega_i) = n_i$ is the population of the city ω_i
 - ▶ e.g., $X(\omega_{Tokyo}) = 37,115,035$ for the city of **Tokyo** (world's most populated city)
 - ▶ $p_X(k) = P_X(X = k) = P_1(\{\omega \in \Omega \mid X(\omega) = k\}) =$ fraction of cities with k inhabitants
- $P_2(\omega_i) = n_i/V$ and $R(\omega_i) =$ rank of the city ω_i w.r.t. city population
 - ▶ e.g., $R(\omega_{Tokyo}) = 1$ for the city of Tokyo
 - ▶ $p_R(r) = P_R(R = r) = X(\omega_{r-th})/V$ where ω_{r-th} is the r^{th} largest city
- Assume $X \sim Pow(x_{min}, \alpha)$, where x_{min} is the smallest population of a city, e.g., $x_{min} = 1$
 - ▶ $P(X > k) \propto k^{-(\alpha-1)}$, (prop. to the) fraction of cities with more than k inhabitants
[\propto reads "proportional to" up to multip./additive constants]
 - ▶ $N \cdot k^{-(\alpha-1)}$, (prop. to the) number of cities with more than k inhabitants
 - ▶ if $X(\omega) = k$ then $R(\omega) \propto N \cdot X(\omega)^{-(\alpha-1)} + 1$, where $+1$ to add ω itself
 - ▶ In summary $R(\omega) \propto X(\omega)^{-(\alpha-1)}$, or, by inverting, $X(\omega) \propto R(\omega)^{-\frac{1}{\alpha-1}}$, and then:
$$p_R(r) = P_R(R = r) = \frac{X(\omega_{r-th})}{V} \propto X(\omega_{r-th}) \propto r^{-\beta} \quad \text{where } \beta = \frac{1}{\alpha-1}$$

i.e., $R \sim Zipf(\beta)$ (the r^{th} most populated city has population proportional to $r^{-\beta}$)

See R script