
- Errata-corrige at page 30: \( \frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot b}{b \cdot d} \) and \( \frac{a}{b} - \frac{c}{d} = \frac{a \cdot d - c \cdot b}{b \cdot d} \)
Sets and functions

• Numerical sets
  ▶ $\mathbb{N} = \{0, 1, 2, \ldots\}$
  ▶ $\mathbb{Z} = \mathbb{N} \cup \{-1, -2, \ldots\}$
  ▶ $\mathbb{Q} = \{m/n \mid m, n \in \mathbb{Z}, n \neq 0\}$
  ▶ $\mathbb{R} = \{\text{fractional numbers with possibly infinitely many digits}\}$ $\supseteq \mathbb{Q}$
  ▶ $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$
    □ $y$ such that $y \cdot y = 2$ belongs to $\mathbb{I}$

• Functions
  ▶ $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$
  ▶ $f : \mathbb{R} \to \mathbb{R}$ is a subset $f \subseteq \mathbb{R} \times \mathbb{R}$ such that $(x, y_0), (x, y_1) \in f$ implies $y_0 = y_1$
    □ usually written $f(x) = y$ for $(x, y) \in f$
    □ $f(x) = v$ for all $x$
    □ $f(x) = a \cdot x + b$ for fixed $a, b$
    □ $f(x) = a \cdot x^2 + b \cdot x + c$ for fixed $a, b, c$
    □ $f(x) = \sum_{i=0}^{n} a_i \cdot x^i$ for fixed $a_0, \ldots, a_n$

See R script
Functions

- $\text{dom}(f) = \{ x \in \mathbb{R} | \exists y \in \mathbb{R}. (x, y) \in f \}$
- $\text{im}(f) = \{ y \in \mathbb{R} | \exists x \in \mathbb{R}. (x, y) \in f \}$
- $f^{-1} = \{(y, x) | (x, y) \in f\}$
  - $f^{-1}$ is a function iff $f$ is injective
  - $f^{-1}(y) = x$ iff $f(x) = y$
  - $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$

- Examples
  - $\sqrt{y} = x$ iff $x^2 = y$ over $x \geq 0$
  - $\sqrt[3]{y} = x$ iff $x^n = y$ over $x \geq 0$  [positive root]
Powers and logarithms

\[ \log_a(y) = x \ \text{iff} \ a^x = y \ \text{for} \ a \neq 1, \ x > 0 \]

for \( n/m \in \mathbb{Q} \): \( a^{n/m} \overset{\text{def}}{=} \sqrt[m]{a^n} \)

what is \( a^x \) for \( x \in \mathbb{I} \)?

\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n \geq 0} \frac{x^n}{n!} \]

and \( a^x = (e^{\log_e(a)})^x = e^{x \cdot \log_e(a)} \)

\( X \sim \text{Poi}(\mu), \quad \sum_{k=0}^{\infty} \frac{\mu^k}{k!} e^{-\mu} = e^{-\mu} \sum_{k=0}^{\infty} \frac{\mu^k}{k!} = e^{-\mu} \cdot e^\mu = 1 \)

See R script
Limits

For a function $f()$, and $a \in \mathbb{R} \cup \{-\infty, \infty\}$

$$\lim_{x \to a} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to a$$

if $f(x)$ can be made as close to $L$ as desired, by making $x$ close enough, but not equal, to $a$.

- Example: $\lim_{x \to 0} \frac{2\cdot x + x^2}{x} = 2$
- A function $f()$ is called continuous at $c$, if $\lim_{x \to c} f(x) = f(c)$

- The limit may not exist, e.g., $\lim_{x \to 0} \frac{1}{x}$
Gradient and derivatives

- The **gradient** is a measure of how ‘steep’ a function is.
  - For \( f(x) = mx + b \), \( m \) is the (constant!) gradient and \( b \) the intercept (i.e., \( f(x) \) at \( x = 0 \))
- For \( f(x) = x^2 \)
  - Tangent at \( x = a \) is \( y = mx + b \) where:
    - \( m = \lim_{\delta \to 0} \frac{f(a+\delta)-f(a)}{\delta} = 2a \) for \( \delta \to 0 \)
    - since \( f(a) = m \cdot a + b \), we have \( b = f(a) - m \cdot a = -a^2 \)
- In general, for \( f(x) \)?
  - Since \( m \) depends on \( a \), we write \( m \) as \( f'(a) \)
  - \( f'(a) = \lim_{\delta \to 0} \frac{f(a+\delta)-f(a)}{\delta} \) is called the **derivative** of \( f() \),
  - \( f'(x) \) also written \( \frac{df}{dx} \) or \( \frac{\delta f}{\delta x} \)
  - Not all functions are differentiable!

**See R script** or **this Colab Notebook**
Derivatives

Standard derivatives

- If $k$ is a constant, then $f(x) = k$ gives $f'(x) = 0$.
- If $k \neq 0$ is a constant, then $f(x) = x^k$ gives $f'(x) = kx^{k-1}$.
- $f(x) = e^x$ gives $f'(x) = e^x$.
- $f(x) = \ln x$ gives $f'(x) = \frac{1}{x}$.

- Constant multiple rule:
  \[
  \frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{df}{dx}(x)
  \]

- Sum rule:
  \[
  \frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx}(x) + \frac{dg}{dx}(x)
  \]
Derivatives

• Product rule:

\[ \frac{d}{dx}[f(x) \cdot g(x)] = \frac{df}{dx}(x) \cdot g(x) + f(x) \cdot \frac{dg}{dx}(x) \]

• Quotient rule:

\[ \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \left[ \frac{df}{dx}(x) \cdot g(x) - f(x) \cdot \frac{dg}{dx}(x) \right] \cdot \frac{1}{g(x)^2} \]

• Chain rule:

\[ \frac{d}{dx}[f(g(x))] = \frac{df}{dg}(g(x)) \cdot \frac{dg}{dx}(x) \]

• \( \frac{d}{dx}e^{-x} = \ldots \)

• Inverse rule:

\[ \frac{d}{dx}[f^{-1}(x)] = \frac{1}{\frac{df}{dx}(f^{-1}(x))} \]

\[ \frac{d}{dx}\log x = \ldots \]

See R script or this Colab Notebook
• $f'(x) > 0$ implies $f()$ is increasing at $x$
• $f'(x) < 0$ implies $f()$ is decreasing at $x$
• $f'(x) = 0$ we cannot say
Optimization - second derivatives

- $f''(x) < 0$ implies $f(x)$ is a maximum
- $f''(x) > 0$ implies $f(x)$ is a minimum
- $f''(x) = 0$ we cannot say

See this Colab Notebook
Integration

- Given $f(x)$, what is $F(x)$ such that $f(x) = \frac{d}{dx}F(x)$? i.e., such that $F'(x) = f(x)$
- Quick answer: $F(x) = \int_{-\infty}^{x} f(t)dt$
  - Integration is the inverse of differentiation
- Geometrical definition of integrals:
  - $\int_{a}^{b} f(x)dx$ is the area below $f(x)$
  - defined as approximation of domain partitioning (Riemann–Darboux integrals) or image partitioning (Lebesgue integrals). For $f(x)$ continuous, the two integrals do coincide.
Integration

Key concepts in integration

If $F(x)$ is a function whose derivative is the function $f(x)$, then we have

$$\int f(x) \, dx = F(x) + c,$$

where $c$ is an arbitrary constant. In particular, we call the

- function, $f(x)$, the *integrand* as it is what we are integrating,
- function, $F(x)$, an *antiderivative* as its derivative is $f(x)$,
- constant, $c$, a *constant of integration* which is completely arbitrary,† and
- integral, $\int f(x) \, dx$, an *indefinite integral* since, in the result, $c$ is arbitrary.

- Definite integrals over an interval $[a, b]$:

$$\int_a^b f(x) \, dx = [F(x)]_a^b = F(b) - F(a)$$
Integration

- **Constant multiple rule:**
  \[ k \int f(x) \, dx = \int k f(x) \, dx \]

- **Sum rule:**
  \[ \int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx \]

**Standard integrals**

- If \( k \neq -1 \) is a constant, then \( \int x^k \, dx = \frac{x^{k+1}}{k+1} + c \).

  In particular, if \( k = 0 \), we have \( \int 1 \, dx = \int x^0 \, dx = x + c \).

- \( \int x^{-1} \, dx = \ln |x| + c \).

- \( \int e^x \, dx = e^x + c \).

See R script
Integration by parts

- From the product rule of derivatives:
  \[
  \frac{d}{dx}[f(x) \cdot g(x)] = \frac{df}{dx}(x) \cdot g(x) + f(x) \cdot \frac{dg}{dx}(x)
  \]

- take the inverse of both sides:
  \[
  f(x) \cdot g(x) = \int f'(x) \cdot g(x) \, dx + \int f(x) \cdot g'(x) \, dx
  \]

- and then:
  \[
  \int f(x) \cdot g'(x) \, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) \, dx
  \]

- \[
  \int \lambda xe^{-\lambda x} \, dx = \ldots = -e^{-\lambda x}(x + 1/\lambda)
  \]
  - consider \( f(x) = x \) and \( g'(x) = \lambda e^{-\lambda x} \)
  - \( g(x) = -e^{-\lambda x} \) and \( f'(x) = 1 \)
Integration by change of variable

• Change of variable rule:

\[ \int f(y) \, dy =_{y=g(x)} \int f(g(x))g'(x) \, dx \]

• \[ \int \frac{\log x}{x} \, dx = \int y \, dy = \frac{y^2}{2} \] for \( y = \log x \) \hspace{1em} \text{hence,} \hspace{1em} \int \frac{\log x}{x} \, dx = \frac{(\log x)^2}{2} \\
\quad \text{consider} \ f(y) = y \ \text{and} \ g(x) = \log x \]
Functions of two or more variables

- Symmetry of second derivatives

\[
\frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \frac{\partial}{\partial x} f(x, y)
\]

- Leibniz integral rule

\[
\frac{d}{dx} \int_{a}^{b} f(x, y) dy = \int_{a}^{b} \frac{d}{dx} f(x, y) dy
\]

- Gradient (pronounced “del”)

\[
\nabla f(x, y) = \left( \frac{\partial}{\partial x} f(x, y), \frac{\partial}{\partial y} f(x, y) \right)
\]

- Hessian matrix (2 × 2 case):

\[
H_2(x, y) = \begin{pmatrix}
\frac{\partial}{\partial x} \frac{\partial}{\partial x} f(x, y) & \frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x, y) \\
\frac{\partial}{\partial y} \frac{\partial}{\partial x} f(x, y) & \frac{\partial}{\partial y} \frac{\partial}{\partial y} f(x, y)
\end{pmatrix}
\]

[Schwarz’s theorem]

[direction and rate of fastest increase]

[Generalize the second derivative test for max/min]
Feyman’s trick

\[ F(t) = \int_0^\infty e^{-tx} \, dx = \left[ -\frac{e^{-tx}}{t} \right]_0^\infty = \frac{1}{t} \]

- using Leibniz integral rule

\[ \frac{d}{dt} F(t) = \frac{d}{dt} \int_0^\infty e^{-tx} \, dx = \int_0^\infty \frac{d}{dt} e^{-tx} \, dx = -\int_0^\infty xe^{-tx} \, dx = -\frac{1}{t^2} \]

- Taking further derivatives yields:

\[ \int_0^\infty x^{n-1} e^{-tx} \, dx = \frac{(n-1)!}{t^n} \]

and for \( t = 1 \):

\[ \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx = (n-1)! \]

[\text{Euler’s } \Gamma(n)]

See R script