

Master Program in *Data Science and Business Informatics*

# Statistics for Data Science

Lesson 04 - Discrete random variables

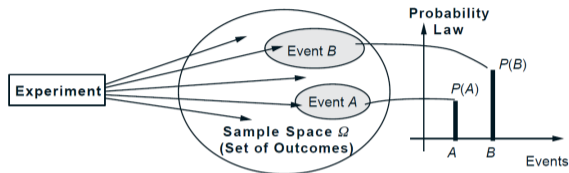
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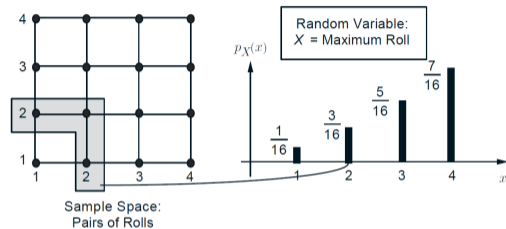
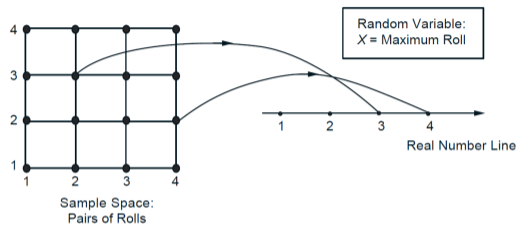
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# Experiments



- **Experiment:** roll two independent 4 sided die.
- We are interested in probability of the *maximum of the two rolls*.
- Modeling so far
  - ▶  $\Omega = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), \dots, (4, 4)\}$
  - ▶  $A = \{\text{maximum roll is 2}\} = \{(1, 2), (2, 1), (2, 2)\}$
  - ▶  $P(A) = P(\{(1, 2), (2, 1), (2, 2)\}) = 3/16$

# Random variables

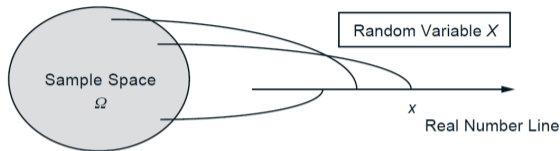


- Modeling  $X : \Omega \rightarrow \mathbb{R}$

- ▶  $X((a, b)) = \max(a, b)$
- ▶  $A = \{\text{maximum roll is 2}\} = \{(a, b) \in \Omega \mid X((a, b)) = 2\} = X^{-1}(2)$
- ▶  $P(A) = P(X^{-1}(2)) = \frac{3}{16}$
- ▶ We write  $P_X(X = 2) \stackrel{\text{def}}{=} P(X^{-1}(2))$

**Induced probability**

# (Discrete) Random variables



- A random variable is a function  $X : \Omega \rightarrow \mathbb{R}$ 
  - ▶ it transforms  $\Omega$  into a more tangible sample space  $\mathbb{R}$ 
    - from  $(a, b)$  to  $\min(a, b)$
  - ▶ it decouples the details of a specific  $\Omega$  from the probability of events of interest
    - from  $\Omega = \{H, T\}$  or  $\Omega = \{\text{good}, \text{bad}\}$  or  $\Omega = \dots$  to  $\{0, 1\}$
  - ▶ it is not 'random' nor 'variable'

DEFINITION. Let  $\Omega$  be a sample space. A *discrete random variable* is a function  $X : \Omega \rightarrow \mathbb{R}$  that takes on a finite number of values  $a_1, a_2, \dots, a_n$  or an infinite number of values  $a_1, a_2, \dots$

# (Machine Learning) Binary Classifiers

- $\Omega = \{f, m\} \times \mathbb{N} \times \{+, -\}$
- Features as Random Variables:
  - ▶  $G((g, a, c)) = g$  gender,  $G = f$  is  $G^{-1}(f) = \{\omega \in \Omega \mid \omega = (f, -, -)\}$
  - ▶  $A((g, a, c)) = a$  gender,  $A = a$  is  $A^{-1}(a) = \{\omega \in \Omega \mid \omega = (-, a, -)\}$
  - ▶  $Y((g, a, c)) = y$  gender,  $Y = c$  is  $Y^{-1}(c) = \{\omega \in \Omega \mid \omega = (-, -, c)\}$
- Binary Classifier as Random Variable:  $\hat{Y} : \{f, m\} \times \mathbb{N} \rightarrow \{+, -\}$  predicted class
  - ▶  $\hat{Y} = +$  is  $\{\omega \in \Omega \mid \hat{Y}(G(\omega), A(\omega)) = +\}$ , e.g, predicted Covid-19 positive
  - ▶  $\hat{Y} = -$  is  $\{\omega \in \Omega \mid \hat{Y}(G(\omega), A(\omega)) = -\}$ , e.g., predicted Covid-19 negative
- $P(Y = \hat{Y})$ , i.e.,  $P(Y = + \cap \hat{Y} = +) + P(Y = - \cap \hat{Y} = -)$  *[True Accuracy]*
- $P(Y = + \mid \hat{Y} = +)$  *[True Precision]*
- $P(\hat{Y} = + \mid Y = +)$  *[True Recall]*
- **Such probabilities are unknown!** They can only be estimated on a sample (*test set*)

# Probability Mass Function (PMF)

DEFINITION. The *probability mass function*  $p$  of a discrete random variable  $X$  is the function  $p : \mathbb{R} \rightarrow [0, 1]$ , defined by

$$p(a) = P(X = a) \quad \text{for } -\infty < a < \infty.$$

- Support of  $X$  is  $\{a \in \mathbb{R} \mid P(X = a) > 0\} = \{a_1, a_2, \dots\}$ 
  - ▶  $p(a_i) > 0$  for  $i = 1, 2, \dots$
  - ▶  $p(a_1) + p(a_2) + \dots = 1$
  - ▶  $p(a) = 0$  if  $a \notin \{a_1, a_2, \dots\}$

# Cumulative Distribution Function (CDF) and CCDF

DEFINITION. The *distribution function*  $F$  of a random variable  $X$  is the function  $F : \mathbb{R} \rightarrow [0, 1]$ , defined by

$$F(a) = P(X \leq a) \quad \text{for } -\infty < a < \infty.$$

- $F(a) = P(X \in \{a_i \mid a_i \leq a\}) = P(X \leq a) = \sum_{a_i \leq a} p(a_i)$
- if  $a \leq b$  then  $F(a) \leq F(b)$
- $P(a < X \leq b) = F(b) - F(a) = \sum_{a < a_i \leq b} p(a_i)$

[Non-decreasing]

## Complementary cumulative distribution function (CCDF)

$$\bar{F}(a) = P(X > a) = 1 - P(X \leq a) = 1 - F(a)$$

- $\bar{F}(a) = P(X \in \{a_i \mid a_i > a\}) = P(X > a) = \sum_{a_i > a} p(a_i)$

**See R script**

$$X \sim U(m, M)$$

### Uniform discrete distribution

A discrete random variable  $X$  has the *uniform distribution* with parameters  $m, M \in \mathbb{Z}$  such that  $m \leq M$ , if its pmf is given by

$$p(a) = \frac{1}{M - m + 1} \quad \text{for } a = m, m + 1, \dots, M$$

We denote this distribution by  $U(m, M)$ .

- **Intuition:** all integers in  $[m, M]$  have equal chances of being observed.

$$F(a) = \frac{\lfloor a \rfloor - m + 1}{M - m + 1} \quad \text{for } m \leq a \leq M$$

See R script



### Benford's law

A discrete random variable  $X$  has the *Benford's distribution*, if its pmf is given by

$$p(a) = \log_{10} \left( 1 + \frac{1}{a} \right) \quad \text{for } a = 1, 2, \dots, 9$$

We denote this distribution by *Ben*.

- Plausible and empirically adequate model for to the frequency distribution of leading digits in many real-life numerical datasets.
- See [Wikipedia](#) for its interesting history and applications!

**See R script**

# $X \sim \text{Ber}(p)$

DEFINITION. A discrete random variable  $X$  has a **Bernoulli distribution** with parameter  $p$ , where  $0 \leq p \leq 1$ , if its probability mass function is given by

$$p_X(1) = P(X = 1) = p \quad \text{and} \quad p_X(0) = P(X = 0) = 1 - p.$$

We denote this distribution by  $\text{Ber}(p)$ .

- $X$  models success/failure in tossing a coin (H, T), testing for a disease (infected, not infected), membership in a set (member, non-member), etc.
- $p_X$  is the *pmf* (to distinguish from parameter  $p$ )
- Alternative definition:  $p_X(a) = p^a \cdot (1 - p)^{1-a}$  for  $a \in \{0, 1\}$

**See R script**

### Identically distributed random variables

Two random variables  $X$  and  $Y$  are said *identically distributed* (in symbols,  $X \sim Y$ ), if  $F_X = F_Y$ , i.e.,

$$F_X(a) = F_Y(a) \quad \text{for } a \in \mathbb{R}$$

- Identically distributed does **not** mean equal
- Toss a fair coin
  - ▶ let  $X$  be 1 for  $H$  and 0 for  $T$
  - ▶ let  $Y$  be  $1 - X$
- $X \sim \text{Ber}(0.5)$  and  $Y \sim \text{Ber}(0.5)$
- Thus,  $X \sim Y$  but are clearly always different.

# Joint p.m.f.

- For a same  $\Omega$ , several random variables can be defined
  - ▶ Random variables related to the same experiment often influence one another
  - ▶  $\Omega = \{(i, j) \mid i, j \in 1, \dots, 6\}$  rolls of two dies
    - $X((i, j)) = i + j$  and  $Y((i, j)) = \max(i, j)$
    - $P(X = 4, Y = 3) = P(X^{-1}(4) \cap Y^{-1}(3)) = P(\{(3, 1), (1, 3)\}) = 2/36$
  - ▶  $\Omega = \{f, m\} \times \mathbb{N} \times \{+, -\}$  (testing for Covid-19 - multivariate)
    - $G((g, a, c)) = 0$  if  $g = f$  and 1 otherwise       $A((g, a, c)) = a$
    - $Y((g, a, c)) = 0$  if  $c = -$  and 1 otherwise
- In general:

$$P_{XY}(X = a, Y = b) = P(\{\omega \in \Omega \mid X(\omega) = a \text{ and } Y(\omega) = b\}) = P(X^{-1}(a) \cap Y^{-1}(b))$$

DEFINITION. The *joint probability mass function*  $p$  of two discrete random variables  $X$  and  $Y$  is the function  $p : \mathbb{R}^2 \rightarrow [0, 1]$ , defined by

$$p(a, b) = P(X = a, Y = b) \quad \text{for } -\infty < a, b < \infty.$$

# Joint and marginal p.m.f.

- **Joint distribution function**  $F : \mathbb{R} \times \mathbb{R} \rightarrow [0, 1]$ :

$$F_{XY}(a, b) = P(X \leq a, Y \leq b) = \sum_{a_i \leq a, b_i \leq b} p(a_i, b_i)$$

- By generalized additivity, the **marginal p.m.f.**'s can be derived: [Tabular method]

$$p_X(a) = P_X(X = a) = \sum_b P_{XY}(X = a, Y = b) \quad p_Y(b) = P_Y(Y = b) = \sum_a P_{XY}(X = a, Y = b)$$

and the marginal distribution function of  $X$  as:

$$F_X(a) = P_X(X \leq a) = \lim_{b \rightarrow \infty} F_{XY}(a, b) \quad F_Y(b) = P_Y(Y \leq b) = \lim_{a \rightarrow \infty} F_{XY}(a, b)$$

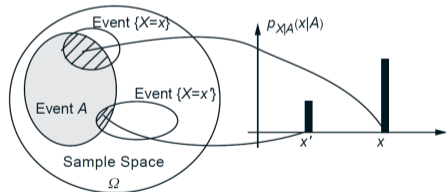
- Deriving the joint p.m.f. from marginal p.m.f.'s is not always possible!
  - ▶ **Exercise at home.** Prove it (hint: find two joint p.m.f.'s with the same marginals)
- Deriving the joint p.m.f. from marginal p.m.f.'s is possible for independent events!
  - ▶  $\Omega = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ ,  $X((a, b)) = a$ ,  $Y((a, b)) = b$
  - ▶  $P(X = 1, Y = 2) = 1/16 = 1/4 \cdot 1/4 = P(X = 1) \cdot P(Y = 2)$

# Conditional distribution

## Conditional distribution

Consider the joint distribution  $P_{XY}$  of  $X$  and  $Y$ . The conditional distribution of  $X$  given  $Y \in B$  with  $P_Y(Y \in B) > 0$ , is the function  $F_{X|Y \in B} : \mathbb{R} \rightarrow [0, 1]$ :

$$F_{X|Y \in B}(a) = P_{X|Y}(X \leq a | Y \in B) = \frac{P_{XY}(X \leq a, Y \in B)}{P_Y(Y \in B)} \quad \text{for } -\infty < a < \infty$$



- Distribution of  $X$  after knowing  $Y \in B$ .
- Chain rule:  $P_{XY}(X \leq a, Y \in B) = P_{X|Y}(X \leq a | Y \in B)P_Y(Y \in B)$
- What if the distribution does not change w.r.t. the prior  $P_X$ ?

# Independence of two random variables

## Independence $X \perp\!\!\!\perp Y$

A random variable  $X$  is independent from a random variable  $Y$ , if for all  $P_Y(Y \leq b) > 0$ :

$$P_{X|Y}(X \leq a | Y \leq b) = P_X(X \leq a) \quad \text{for } -\infty < a < \infty$$

- Properties
  - ▶  $X \perp\!\!\!\perp Y$  iff  $P_{XY}(X \leq a, Y \leq b) = P_X(X \leq a) \cdot P_Y(Y \leq b)$  for  $-\infty < a, b < \infty$
  - ▶  $X \perp\!\!\!\perp Y$  iff  $Y \perp\!\!\!\perp X$  *[Symmetry]*
- For  $X, Y$  **discrete** random variables:
  - ▶  $X \perp\!\!\!\perp Y$  iff  $P_{XY}(X = a, Y = b) = P_X(X = a) \cdot P_Y(Y = b)$  for  $-\infty < a, b < \infty$
  - ▶ **Exercise at home.** Prove it!
  - ▶  $X \perp\!\!\!\perp Y$  iff  $P_{XY}(X \in \mathcal{A}, Y \in \mathcal{B}) = P_X(X \in \mathcal{A}) \cdot P_Y(Y \in \mathcal{B})$  for  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}$
  - ▶ **Exercise at home.** Prove it!

**See R script**

# Sum of independent discrete random variables

ADDING TWO INDEPENDENT DISCRETE RANDOM VARIABLES. Let  $X$  and  $Y$  be two independent discrete random variables, with probability mass functions  $p_X$  and  $p_Y$ . Then the probability mass function  $p_Z$  of  $Z = X + Y$  satisfies

$$p_Z(c) = \sum_j p_X(c - b_j)p_Y(b_j),$$

where the sum runs over all possible values  $b_j$  of  $Y$ .

- **Proof (sketch).**  $P(Z = c) = \sum_j P(Z = c|Y = b_j) \cdot P(Y = b_j) = \sum_j P(X = c - b_j|Y = b_j) \cdot P(Y = b_j) = \sum_j P(X = c - b_j)P(Y = b_j)$



# Independence of multiple random variables

## Independence (factorization formula)

Random variables  $X_1, \dots, X_n$  are independent, if:

$$P_{X_1, \dots, X_n}(X_1 \leq a_1, \dots, X_n \leq a_n) = \prod_{i=1}^n P_{X_i}(X_i \leq a_i) \quad \text{for } -\infty < a_1, \dots, a_n < \infty$$

- $X_1, \dots, X_n$  **discrete** random variables are independent iff:

$$P_{X_1, \dots, X_n}(X_1 = a_1, \dots, X_n = a_n) = \prod_{i=1}^n P_{X_i}(X_i = a_i) \quad \text{for } -\infty < a_1, \dots, a_n < \infty$$

- **Definition:**  $X_1, \dots, X_n$  are **i.i.d.** (independent and identically distributed) if  $X_1, \dots, X_n$  are independent and  $X_i \sim F$  for  $i = 1, \dots, n$  for some distribution  $F$

# $X \sim \text{Bin}(n, p)$

DEFINITION. A discrete random variable  $X$  has a *binomial distribution* with parameters  $n$  and  $p$ , where  $n = 1, 2, \dots$  and  $0 \leq p \leq 1$ , if its probability mass function is given by

$$p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \dots, n.$$

We denote this distribution by  $\text{Bin}(n, p)$ .

- $X$  models the number of successes in  $n$  Bernoulli trials (How many H's when tossing  $n$  coins?)
- **Intuition:** for  $X_1, X_2, \dots, X_n$  such that  $X_i \sim \text{Ber}(p)$  and independent (**i.i.d.**):

$$X = \sum_{i=1}^n X_i \sim \text{Bin}(n, p)$$

- $p^k \cdot (1-p)^{n-k}$  is the probability of observing first  $k$  H's and then  $n-k$  T's
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  number of ways to choose the first  $k$  variables *[Binomial coefficient]*
- $p_X(k)$  computationally expensive to calculate (no closed formula, but approximation/bounds)
- **Exercise at home.** Prove  $X_1 + X_2 \sim \text{Bin}(2, p)$  using the sum of independent random variables.

**See R script**

$$X \sim \text{Bin}(n, p)$$

DEFINITION. A discrete random variable  $X$  has a **binomial distribution** with parameters  $n$  and  $p$ , where  $n = 1, 2, \dots$  and  $0 \leq p \leq 1$ , if its probability mass function is given by

$$p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \dots, n.$$

We denote this distribution by  **$\text{Bin}(n, p)$** .

- **Exercise:** there are  $c$  bikes shared among  $n$  persons. Assuming that each person needs a bike with probability  $p$ , what is the probability that all bikes will be in use?

$$P(X = c) = \binom{n}{c} p^c \cdot (1-p)^{n-c} = \text{dbinom}(c-1, n, p)$$

# $X \sim \text{Geo}(p)$

DEFINITION. A discrete random variable  $X$  has a *geometric distribution* with parameter  $p$ , where  $0 < p \leq 1$ , if its probability mass function is given by

$$p_X(k) = P(X = k) = (1 - p)^{k-1} p \quad \text{for } k = 1, 2, \dots$$

We denote this distribution by  $\text{Geo}(p)$ .

- $X$  models the number of Bernoulli trials before a success (how many tosses to have a H?)
- **Intuition:** for  $X_1, X_2, \dots$  such that  $X_i \sim \text{Ber}(p)$  i.i.d.:

$$X = \min_i (X_i = 1) \sim \text{Geo}(p)$$

- $\bar{F}(a) = P(X > a) = (1 - p)^{\lfloor a \rfloor}$
- $F(a) = P(X \leq a) = 1 - \bar{F}(a) = 1 - (1 - p)^{\lfloor a \rfloor}$

See R script

# You cannot always loose

- H is 1, T is 0,  $0 < p < 1$
- $B_n = \{\text{T in the first } n\text{-th coin tosses}\}$
- $P(\cap_{n \geq 1} B_i) = ?$
- $X \sim \text{Geom}(p)$
- $P(B_n) = P(X > n) = (1 - p)^n$
- $P(\cap_{n \geq 1} B_n) = \lim_{n \rightarrow \infty} P(B_n) = \lim_{n \rightarrow \infty} (1 - p)^n = 0$
- $P(\cap_{n \geq 1} B_n) = \lim_{n \rightarrow \infty} P(B_n)$  for  $B_n$  non-increasing

*[ $\sigma$ -additivity, see Lesson 01]*

# But if you lost so far, you can lose again

## Memoryless property

For  $X \sim \text{Geo}(p)$ , and  $n, k = 0, 1, 2, \dots$

$$P(X > n + k | X > k) = P(X > n)$$

### Proof

$$\begin{aligned} P(X > n + k | X > k) &= \frac{P(\{X > n + k\} \cap \{X > k\})}{P(\{X > k\})} \\ &= \frac{P(\{X > n + k\})}{P(\{X > k\})} \\ &= \frac{(1 - p)^{n+k}}{(1 - p)^k} \\ &= (1 - p)^n = P(X > n) \end{aligned}$$

$$X \sim NBin(n, p)$$

### Negative binomial (or Pascal distribution)

A discrete random variable  $X$  has a negative binomial with parameters  $n$  and  $p$ , where  $n = 0, 1, 2, \dots$  and  $0 < p \leq 1$ , if its probability mass function is given by

$$p_X(k) = P(X = k) = \binom{k+n-1}{k} (1-p)^k \cdot p^n \quad \text{for } k = 0, 1, 2, \dots$$

- $X$  models the number of failures before the  $n$ -th success in Bernoulli trials (how many T's to have  $n$  H's?)
- **Intuition:** for  $X_1, X_2, \dots, X_n$  such that  $X_i \sim Geo(p)$  i.i.d.:

$$X = \sum_{i=1}^n X_i - n \sim NBin(n, p)$$

- $(1-p)^k \cdot p^n$  is the probability of observing first  $k$  T's and then  $n$  H's
- $\binom{k+n-1}{k} = \frac{(k+n-1)!}{k!(n-1)!}$  number of ways to choose the first  $k$  variables among  $k+n-1$  (the last one must be a success!)

See R script

# $X \sim Poi(\mu)$

DEFINITION. A discrete random variable  $X$  has a *Poisson distribution* with parameter  $\mu$ , where  $\mu > 0$  if its probability mass function  $p$  is given by

$$p(k) = P(X = k) = \frac{\mu^k}{k!} e^{-\mu} \quad \text{for } k = 0, 1, 2, \dots$$

We denote this distribution by  $Pois(\mu)$ .

- $X$  models the number of events in a fixed interval if these events occur with a known constant mean rate  $\mu$  and independently of the last event
  - ▶ telephone calls arriving in a system
  - ▶ number of patients arriving at an hospital
  - ▶ customers arriving at a counter
- $\mu$  denotes the mean number of events
- $Bin(n, \mu/n)$  is the number of successes in  $n$  trials, assuming  $p = \mu/n$ , i.e.,  $p \cdot n = \mu$
- When  $n \rightarrow \infty$ :  $Bin(n, \mu/n) \rightarrow Poi(\mu)$  [Law of rare events]
  - ▶ Number of typos in a book, number of cars involved in accidents, etc.

See R script



# The discrete Bayes' rule

**BAYES' RULE.** Suppose the events  $C_1, C_2, \dots, C_m$  are disjoint and  $C_1 \cup C_2 \cup \dots \cup C_m = \Omega$ . The conditional probability of  $C_i$ , given an arbitrary event  $A$ , can be expressed as:

$$P(C_i | A) = \frac{P(A | C_i) \cdot P(C_i)}{P(A | C_1)P(C_1) + P(A | C_2)P(C_2) + \dots + P(A | C_m)P(C_m)}.$$

- **Definition.** Conditional p.m.f. of  $X$  given  $Y = b$  with  $P_Y(Y = b) > 0$

$$p_{X|Y}(a|b) = \frac{p_{XY}(a, b)}{p_Y(b)} \quad \text{i.e.,} \quad P_{X|Y}(X = a | Y = b) = \frac{P_{XY}(X = a, Y = b)}{P_Y(Y = b)}$$

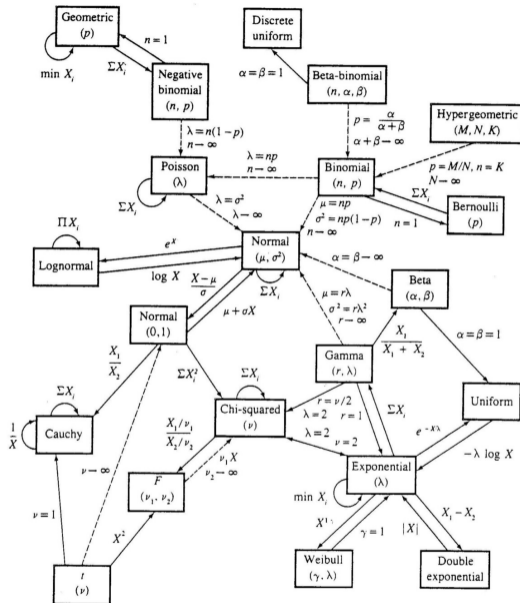
- Discrete Bayes' rule:

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)} = \frac{p_{Y|X}(y|x)p_X(x)}{\sum_{a \in \text{dom}(X)} p_{Y|X}(y|a)p_X(a)}$$

- **Exercise at home.** A machine fails after  $n$  days with a p.m.f.  $X \sim \text{Geo}(p)$ .  $p$  is known to be either  $p = 0.1$  or  $0.05$  with equal probability. What can we say about the distribution of  $p$  given  $n$ ? Code your solution in R.

# Common distributions

- Probability distributions at Wikipedia
- Probability distributions in R
-  C. Forbes, M. Evans, N. Hastings, B. Peacock (2010) Statistical Distributions, 4th Edition Wiley



Relationships among common distributions. Solid lines represent transformations and special cases, dashed lines represent limits. Adapted from Leemis (1986).