Es1

Ex 1

$$T = \{ \sigma \mid \exists k \geq 0 \ \sigma(x) = \sigma(y) = k \}$$

① Let $\prec \subseteq T \times T$ be the well-founded relation

$$\sigma \prec \sigma' \overset{def}{\iff} \sigma(x) < \sigma'(x)$$

we prove the property $P(\sigma) \overset{def}{=} \langle w, \sigma \rangle \to \sigma[0/x, 0/y]$

by well-founded induction on T

- $\sigma(x) = \sigma(y) = 0$ $\quad \langle w, \sigma \rangle \to \sigma' \nwarrow \langle \neg(x=0) \vee \neg(y=0), \sigma \rangle \to false \nwarrow \square$ $\quad \sigma' = \sigma$

  we conclude by noting that $\sigma = \sigma[0/x, 0/y]$

- $\sigma(x) = \sigma(y) = k > 0$ $\quad \langle w, \sigma \rangle \to \sigma' \nwarrow \langle \neg(x=0) \vee \neg(y=0), \sigma \rangle \to true$

  $\langle x := x-1 ; y := y-1, \sigma \rangle \to \sigma''$

  $\langle w, \sigma'' \rangle \to \sigma'$

  $\nwarrow^{*} \ \sigma'' = \sigma[\sigma(x)-1/x, \sigma(y)-1/y]$

  $\langle w, \sigma[\sigma(x)-1/x, \sigma(y)-1/y] \rangle \to \sigma'$

  but $\sigma[\sigma(x)-1/x, \sigma(y)-1/y] \prec \sigma$ therefore by inductive hypothesis

  we have $\sigma' = \sigma''[0/x, 0/y] = \sigma[0/x, 0/y]$

② we recall the rule for divergence

$$\frac{\sigma \in S \quad \forall \sigma' \in S \ (\langle c, \sigma' \rangle \to \sigma'' \Rightarrow \sigma'' \in S) \quad \forall \sigma' \in S \ (\langle b, \sigma' \rangle \to true)}{\langle \text{while } b \text{ do } c, \sigma \rangle \not\to}$$

Let $S = \overline{T} = \{ \sigma \mid \sigma(x) \neq \sigma(y) \vee \exists k < 0 \ \sigma(x) = \sigma(y) = k \}$ and take $\sigma \in S$

a) $\sigma \in S$

b) take $\sigma' \in S$ $\quad \langle x := x-1 ; y := y-1, \sigma' \rangle \to \sigma'[\sigma'(x)-1/x, \sigma'(y)-1/y]$

then $\sigma'' = \sigma'[\sigma'(x)-1/x, \sigma'(y)-1/y] \in S$ (if $\sigma'(x) \neq \sigma'(y)$ then $\sigma''(x) \neq \sigma''(y)$,

and if $\sigma'(x) = \sigma'(y) < 0$ then $\sigma''(x) = \sigma''(y) < 0$

c) $\langle \neg(x=0) \vee \neg(y=0), \sigma' \rangle \to true$ for any $\sigma' \in S$

(because any σ such that $\sigma(x) = \sigma(y) = 0$ belongs to T)

Text

Es 3

$$\mathrm{rec}\ \underbrace{f}_{int\to int\to\tau}.\ \lambda\underbrace{x}_{int}.\ \lambda\underbrace{y}_{int}.\ \mathrm{if}\ \overbrace{\underbrace{x}_{int}\times\underbrace{y}_{int}}^{int}\ \mathrm{then}\ \overbrace{x}^{int}\ \mathrm{else}\ \overbrace{\underbrace{(\underbrace{f}_{int\to int\to\tau}\ \underbrace{x}_{int})}_{int\to\tau}\underbrace{(\underbrace{(\underbrace{f}_{int\to int\to\tau}\ \underbrace{x}_{int})}_{int\to\tau}\ \underbrace{y}_{int})}_{\tau=int}}^{int}$$

$$int \qquad int \qquad int\to int\to int$$

$$\hat{t} = int\to int\to int$$

$$\mathrm{rec}\ f.\ \lambda x.\ \lambda y.\ \mathrm{if}\ x\times y\ \mathrm{then}\ x\ \mathrm{else}\ (fx)((fx)y) \to c$$

$$\lambda x.\ \lambda y.\ \mathrm{if}\ x\times y\ \mathrm{then}\ x\ \mathrm{else}\ (fx)((fx)y) \to c$$

$c = \lambda x.\lambda y.$

$$[\![\mathrm{rec}\ f.\ \lambda x.\ \lambda y.\ \mathrm{if}\ x\times y\ \mathrm{then}\ x\ \mathrm{else}\ (fx)((fx)y)]\!]\rho =$$

$$\mathrm{fix}\ \lambda d.\lfloor\lambda d'.\lfloor\lambda d''.[\![\mathrm{if}\ x\times y\ \mathrm{then}\ x\ \mathrm{else}\ (fx)((fx)y)]\!]\rho''\rfloor\rfloor$$

where $\rho'' = \rho[d/f, d'/x, d''/y]$

$$= \mathrm{fix}\ \lambda d.\lfloor\lambda d'.\lfloor\lambda d''.\ \mathrm{Cond}(d'\times d'', d', \mathrm{let}\ \varphi\Leftarrow z.\ \varphi(w))\rfloor\rfloor$$

where $z = [\![f x]\!]\rho'' = \mathrm{let}\ \varphi'\Leftarrow d.\ \varphi'(d')$

and $w = [\![(fx)y]\!]\rho'' = \mathrm{let}\ \varphi''\Leftarrow z.\ \varphi''(d'')$