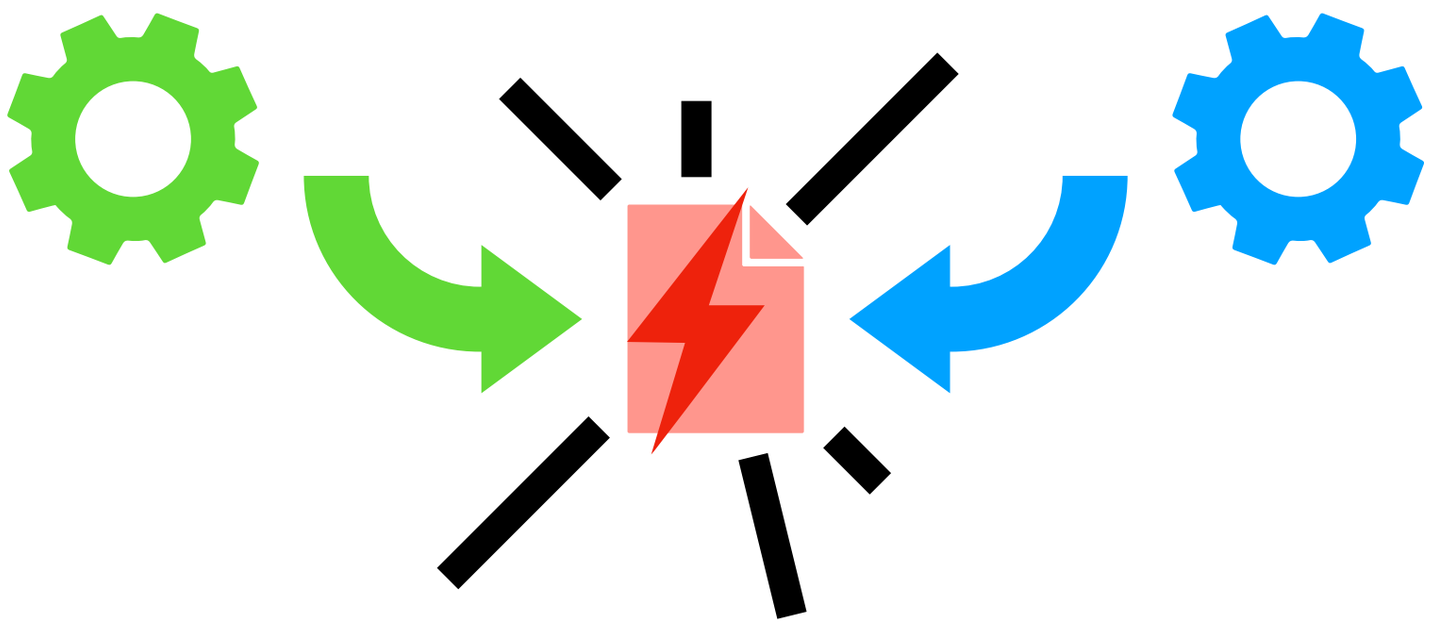
Linguaggi di Programmazione



Roberta Gori

Bisimilarita' come punto fisso

CCS: sintassi

$$
\begin{array}{lcll}
p, q & ::= & \mathbf{nil} & \text{processo inattivo} \\
 & | & x & \text{variabile di processo (per la ricorsione)} \\
 & | & \mu.p & \text{prefisso azione} \\
 & | & p\backslash\alpha & \text{canale ristretto} \\
 & | & p[\phi] & \text{rietichettatura del canale} \\
 & | & p + q & \text{scelta nondeterministica (somma)} \\
 & | & p|q & \text{composizione parallela} \\
 & | & \mathbf{rec}\ x.\ p & \text{ricorsione}
\end{array}
$$

(gli operatori sono elencati in ordine di precedenza)

CCS semantica operazionale

Act) $$\frac{}{\mu.p \xrightarrow{\mu} p}$$

Res) $$\frac{p \xrightarrow{\mu} q \quad \mu \notin \{\alpha, \overline{\alpha}\}}{p\backslash\alpha \xrightarrow{\mu} q\backslash\alpha}$$

Rel) $$\frac{p \xrightarrow{\mu} q}{p[\phi] \xrightarrow{\phi(\mu)} q[\phi]}$$

SumL) $$\frac{p_1 \xrightarrow{\mu} q}{p_1 + p_2 \xrightarrow{\mu} q}$$

SumR) $$\frac{p_2 \xrightarrow{\mu} q}{p_1 + p_2 \xrightarrow{\mu} q}$$

ParL) $$\frac{p_1 \xrightarrow{\mu} q_1}{p_1 | p_2 \xrightarrow{\mu} q_1 | p_2}$$

Com) $$\frac{p_1 \xrightarrow{\lambda} q_1 \quad p_2 \xrightarrow{\overline{\lambda}} q_2}{p_1 | p_2 \xrightarrow{\tau} q_1 | q_2}$$

ParR) $$\frac{p_2 \xrightarrow{\mu} q_2}{p_1 | p_2 \xrightarrow{\mu} p_1 | q_2}$$

Rec) $$\frac{p\left[{}^{\mathbf{rec}\ x.\ p}/_{x}\right] \xrightarrow{\mu} q}{\mathbf{rec}\ x.\ p \xrightarrow{\mu} q}$$

bisimilarita' forte

$\mathcal{P}$ insiemi di processi $\mathbf{R} \subseteq \mathcal{P} \times \mathcal{P}$ una relazione binaria

$\mathbf{R}$ e' una bisimulazione forte se

$$\forall p, q.\ (p, q) \in \mathbf{R} \Rightarrow \begin{cases} \forall \mu, p'.\ p \xrightarrow{\mu} p' & \Rightarrow & \exists q'.\ q \xrightarrow{\mu} q' \wedge p'\ \mathbf{R}\ q' \\ \wedge & \text{Alice gioca} & \text{Bob replica} \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' & \Rightarrow & \exists p'.\ p \xrightarrow{\mu} p' \wedge p'\ \mathbf{R}\ q' \end{cases}$$

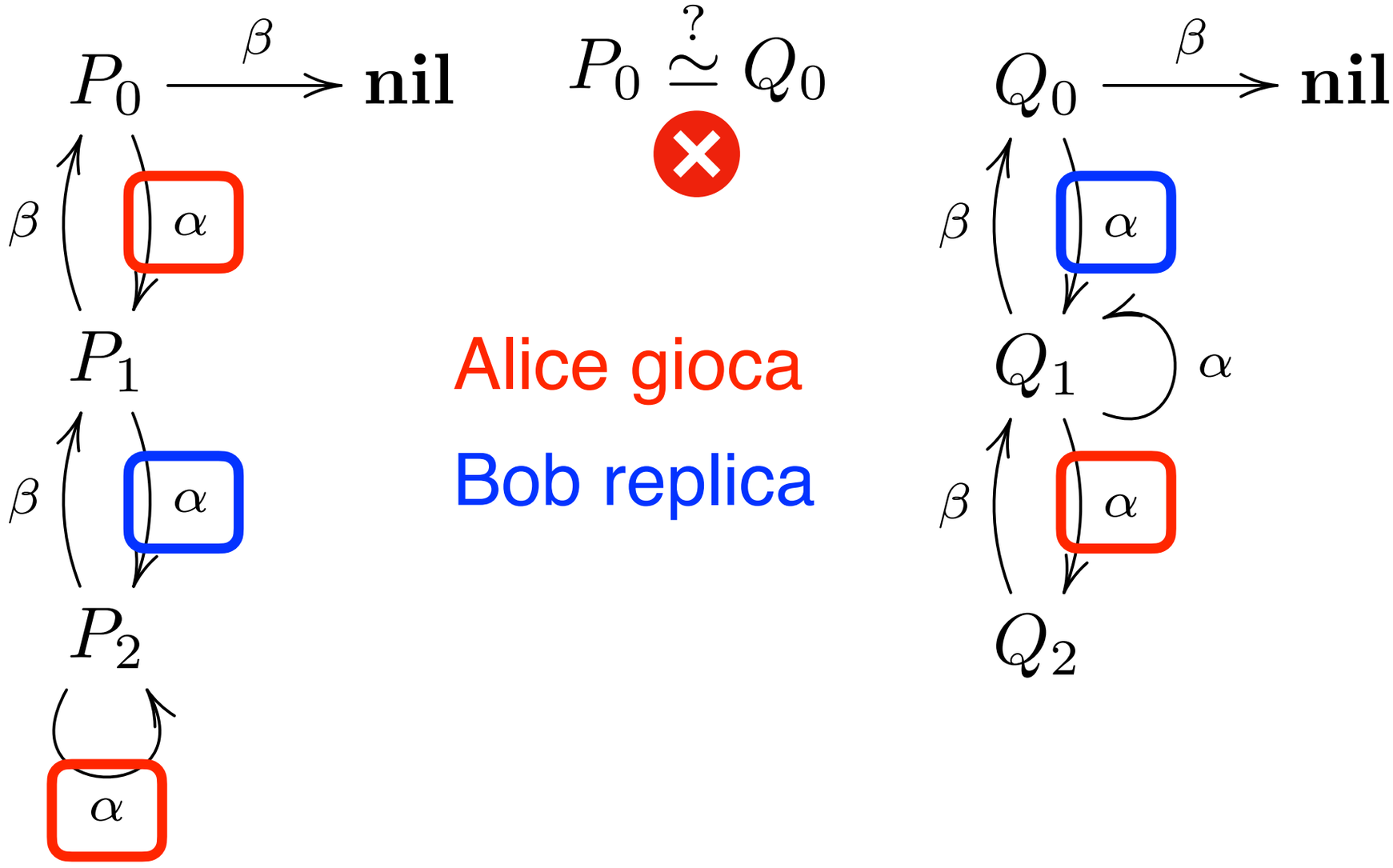
bisimilarita' forte $\simeq \triangleq \bigcup_{\mathbf{R}\ \text{s.b.}} \mathbf{R}$

e' un equivalenza

e' una bisimulazione forte

$$\forall p, q.\ p \simeq q \Leftrightarrow \begin{cases} \forall \mu, p'.\ p \xrightarrow{\mu} p' & \Rightarrow & \exists q'.\ q \xrightarrow{\mu} q' \wedge p' \simeq q' \\ \wedge & & \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' & \Rightarrow & \exists p'.\ p \xrightarrow{\mu} p' \wedge p' \simeq q' \end{cases}$$

il gioco della bisimulazione



CCS

Bisimilarita’ come punto fisso

Bisimulazione come post punto fisso

$$\forall p, q.\ (p,q) \in \mathbf{R} \Rightarrow \begin{cases} \forall \mu, p'.\ p \xrightarrow{\mu} p' & \Rightarrow & \exists q'.\ q \xrightarrow{\mu} q' \wedge p'\ \mathbf{R}\ q' \\ \wedge & & \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' & \Rightarrow & \exists p'.\ p \xrightarrow{\mu} p' \wedge p'\ \mathbf{R}\ q' \end{cases}$$

$\Phi : \wp(\mathcal{P} \times \mathcal{P}) \to \wp(\mathcal{P} \times \mathcal{P})$ mappa relazioni in relazioni

$$\Phi(\mathbf{R}) \triangleq \left\{ (p,q) \;\middle|\; \begin{array}{lcl} \forall \mu, p'.\ p \xrightarrow{\mu} p' & \Rightarrow & \exists q'.\ q \xrightarrow{\mu} q' \wedge p'\ \mathbf{R}\ q' \\ \wedge & & \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' & \Rightarrow & \exists p'.\ p \xrightarrow{\mu} p' \wedge p'\ \mathbf{R}\ q' \end{array} \right\}$$

$$\mathbf{R} \subseteq \Phi(\mathbf{R})$$

una bisimulazione forte

Bisimilarita' come punto fisso

$$\forall p, q.\ p \simeq q \Leftrightarrow \begin{cases} \forall \mu, p'.\ p \xrightarrow{\mu} p' & \Rightarrow & \exists q'.\ q \xrightarrow{\mu} q' \wedge p' \simeq q' \\ \wedge & & \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' & \Rightarrow & \exists p'.\ p \xrightarrow{\mu} p' \wedge p' \simeq q' \end{cases}$$

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$$\Phi(\mathbf{R}) \triangleq \left\{ (p,q) \;\middle|\; \begin{array}{lcl} \forall \mu, p'.\ p \xrightarrow{\mu} p' & \Rightarrow & \exists q'.\ q \xrightarrow{\mu} q' \wedge p'\ \mathbf{R}\ q' \\ \wedge & & \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' & \Rightarrow & \exists p'.\ p \xrightarrow{\mu} p' \wedge p'\ \mathbf{R}\ q' \end{array} \right\}$$

$$\simeq = \Phi(\simeq)$$

la bisimilarita' forte e' un punto fisso

Punto fisso: quale OPC?

Possiamo usare il teorema di Kleene?

Dominio $\wp(\mathcal{P} \times \mathcal{P})$

vogliamo trovare **la piu' grande** relazione,
non **la piu' piccola**

Idea: invertiamo l'ordine (inclusione)!

$(\wp(\mathcal{P} \times \mathcal{P}), \sqsubseteq)$

una relazione con più coppie *è*
più piccola di una con meno coppie

$$\mathbf{R} \sqsubseteq \mathbf{R}' \quad \Leftrightarrow \quad \mathbf{R}' \subseteq \mathbf{R}$$

$$\bot = \mathcal{P} \times \mathcal{P}$$

Minimo punto fisso...invertito

$\wp(\mathcal{P} \times \mathcal{P})$

$\mathbf{R} \sqsubseteq \mathbf{R}' \Leftrightarrow \mathbf{R}' \subseteq \mathbf{R}$

$\top = \emptyset$

pre-punti fissi $\Phi(\mathbf{R}) \sqsubseteq \mathbf{R}$

$(\mathbf{R} \subseteq \Phi(\mathbf{R}))$

↓piu' grande

bisimulazione forte

$\simeq = \Phi(\simeq)$

$\vdots$

minimo pre-punto fisso

bisimilarita' forte

↑piu' piccola

$\mathbf{R}_2 = \Phi(\mathbf{R}_1)$

$\mathbf{R}_1 = \Phi(\mathbf{R}_0)$

$\mathbf{R}_0 \quad \perp = \mathcal{P} \times \mathcal{P}$

Calcolare il punto fisso

possiamo riutilizzare il teorema del punto fisso di Kleene per calcolare $\simeq$?

$$\simeq \stackrel{?}{=} \bigsqcup \Phi^n(\mathcal{P} \times \mathcal{P})$$

intersezione

partiamo dalla relazione universale (tutte le coppie, una partizione unica)

tutti i processi sono equivalenti

applichiamo Φ per distinguere sempre più processi

$\mathbf{R}_1$ distinguibili in un passo

$\mathbf{R}_2$ distinguibili in due passi

$\vdots$

il numero di partizioni aumenta ad ogni passo

TH. Φ e' monotona

prova.

prendiamo $\mathbf{R}_1 \sqsubseteq \mathbf{R}_2$ dobbiamo provare $\Phi(\mathbf{R}_1) \sqsubseteq \Phi(\mathbf{R}_2)$

$\mathbf{R}_2 \subseteq \mathbf{R}_1$ $\Phi(\mathbf{R}_2) \subseteq \Phi(\mathbf{R}_1)$

prendiamo $(p, q) \in \Phi(\mathbf{R}_2)$ dobbiamo provare $(p, q) \in \Phi(\mathbf{R}_1)$

dal momento che $(p, q) \in \Phi(\mathbf{R}_2)$ abbiamo $\forall\, p \xrightarrow{\mu} p'$ con $\exists\, q \xrightarrow{\mu} q'$

e $(p', q') \in \mathbf{R}_2 \subseteq \mathbf{R}_1$ quindi, per def di Φ, abbiamo che $(p, q) \in \Phi(\mathbf{R}_1)$

Per $q \xrightarrow{\mu} q'$ e $p \xrightarrow{\mu} p'$ analogo al caso precedente

TH. Φ e' continua (su processi finitamente ramificati)

prova.

prendiamo una catena

$$\{\mathbf{R}_n\}_{n\in\mathbb{N}}$$

$$\mathbf{R}_0 \sqsubseteq \mathbf{R}_1 \sqsubseteq \cdots \sqsubseteq \mathbf{R}_n \sqsubseteq \ldots$$

$$\mathbf{R}_0 \sqsupseteq \mathbf{R}_1 \sqsupseteq \cdots \sqsupseteq \mathbf{R}_n \sqsupseteq \ldots$$

vogliamo provare $\Phi\left(\bigsqcup_{n\in\mathbb{N}} \mathbf{R}_n\right) = \bigsqcup_{n\in\mathbb{N}} \Phi(\mathbf{R}_n)$

$$\Phi\left(\bigsqcup_{n\in\mathbb{N}} \mathbf{R}_n\right) \sqsupseteq \bigsqcup_{n\in\mathbb{N}} \Phi(\mathbf{R}_n)$$

segue per la monotonia

$$\Phi\left(\bigsqcup_{n\in\mathbb{N}} \mathbf{R}_n\right) \sqsubseteq \bigsqcup_{n\in\mathbb{N}} \Phi(\mathbf{R}_n)$$

$$\Phi\left(\bigsqcap_{n\in\mathbb{N}} \mathbf{R}_n\right) \sqsupseteq \bigsqcap_{n\in\mathbb{N}} \Phi(\mathbf{R}_n)$$

prendiamo $(p,q) \in \bigsqcap_{n\in\mathbb{N}} \Phi(\mathbf{R}_n)$ vogliamo provare $(p,q) \in \Phi\left(\bigsqcap_{n\in\mathbb{N}} \mathbf{R}_n\right)$

$$\forall n.\ (p,q) \in \Phi(\mathbf{R}_n)$$

TH. Φ e' continua (su processi finitamente ramificati)

prova.

$$\forall n.\ (p,q) \in \Phi(\mathbf{R}_n) \ \Rightarrow\ (p,q) \in \Phi\left(\bigcap_{n\in\mathbb{N}} \mathbf{R}_n\right)$$

prendiamo $p \xrightarrow{\mu} p'$ vogliamo trovare $q \xrightarrow{\mu} q'$ con $(p',q') \in \bigcap_{n\in\mathbb{N}} \mathbf{R}_n$

dal momento che $\forall n.\ (p',q') \in \mathbf{R}_n$

$\forall n.\ (p,q) \in \Phi(\mathbf{R}_n)$ allora $\forall n.\ \exists q_n.\ q \xrightarrow{\mu} q_n$ con $(p',q_n) \in \mathbf{R}_n$

$$\mathbf{R}_0 \supseteq \mathbf{R}_1 \supseteq \cdots \supseteq \mathbf{R}_n \supseteq \ldots \qquad \forall k \leq n.\ (p',q_n) \in \mathbf{R}_k$$

q e' finitamente ramificato: $\{q' \mid q \xrightarrow{\mu} q'\}$ e' finito

quindi $\exists m \in \mathbb{N}$ tali che $\{n \mid q_n = q_m\}$ e' infinito

quindi $\forall n.\ (p',q_m) \in \mathbf{R}_n$ e prendiamo $q' = q_m$

$(p',q') \in \bigcap_{n\in\mathbb{N}} \mathbf{R}_n$

inverso, analogo al caso precedente

Bisimilarita' come punto fisso

$\mathcal{P}_f$ processi finitamente ramificati

$$\simeq = \bigsqcup_{n \in \mathbb{N}} \Phi^n(\mathcal{P}_f \times \mathcal{P}_f)$$

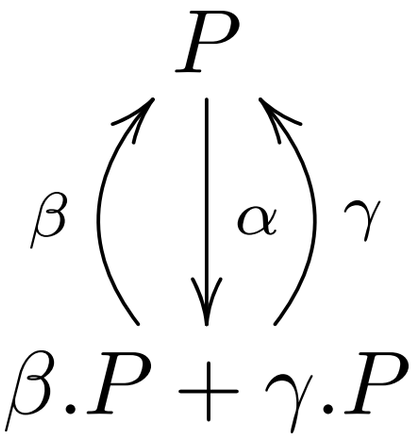
come facciamo a sapere che un processo è finitamente ramificato?

possiamo restringere la sintassi: processi guardati

Esempio

Guardato!

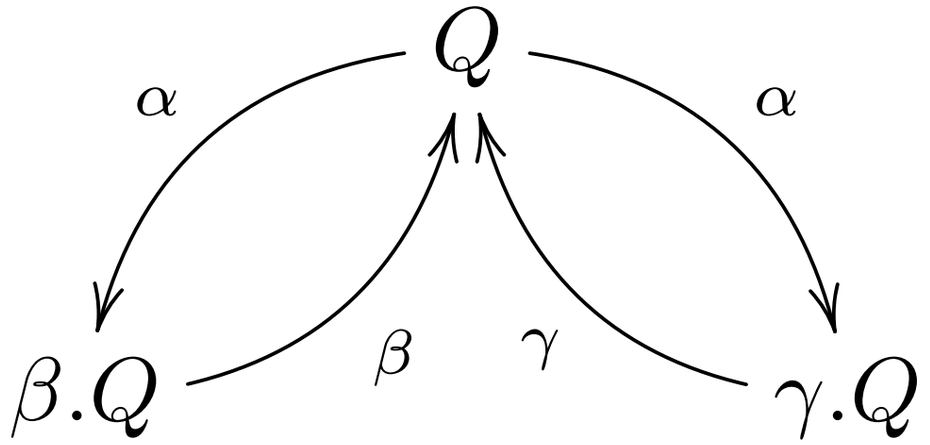
$$P \triangleq \alpha.(\beta.P + \gamma.P)$$



$$P \stackrel{?}{\simeq} Q$$

Guardato!

$$Q \triangleq \alpha.\beta.Q + \alpha.\gamma.Q$$

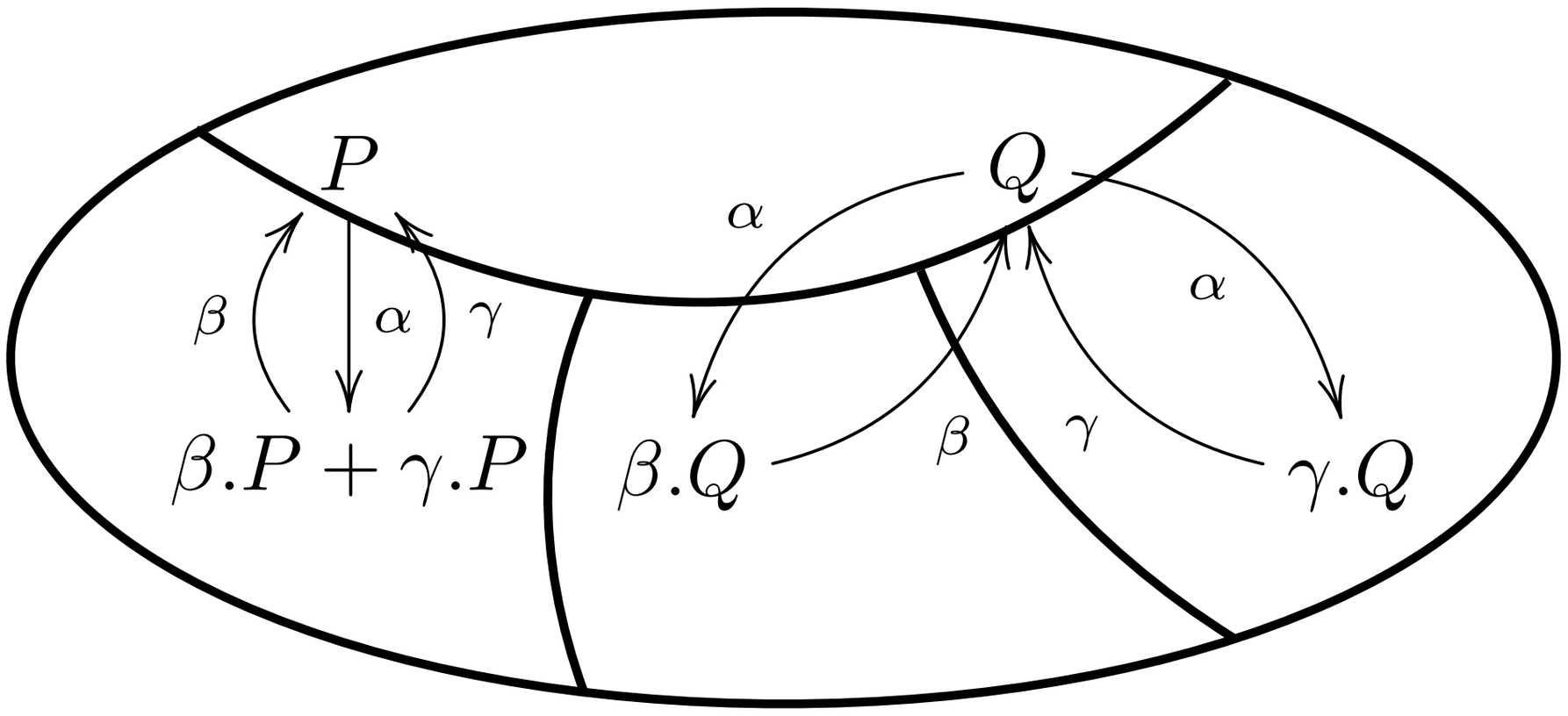


$$\mathbf{R}_0 = \{\ \{P\ ,\ Q\ ,\ \beta.P + \gamma.P\ ,\ \beta.Q\ ,\ \gamma.Q\}\ \}$$

Esempio

$P \triangleq \alpha.(\beta.P + \gamma.P)$ $\quad P \overset{?}{\simeq} Q \quad$ $Q \triangleq \alpha.\beta.Q + \alpha.\gamma.Q$

$$\mathbf{R}_0 = \{\ \{P\ ,\ Q\ ,\ \beta.P + \gamma.P\ ,\ \beta.Q\ ,\ \gamma.Q\}\ \}$$



$P, Q \xrightarrow{\alpha}$

$\beta.P + \gamma.P \xrightarrow{\beta,\gamma}$

$\beta.Q \xrightarrow{\beta}$

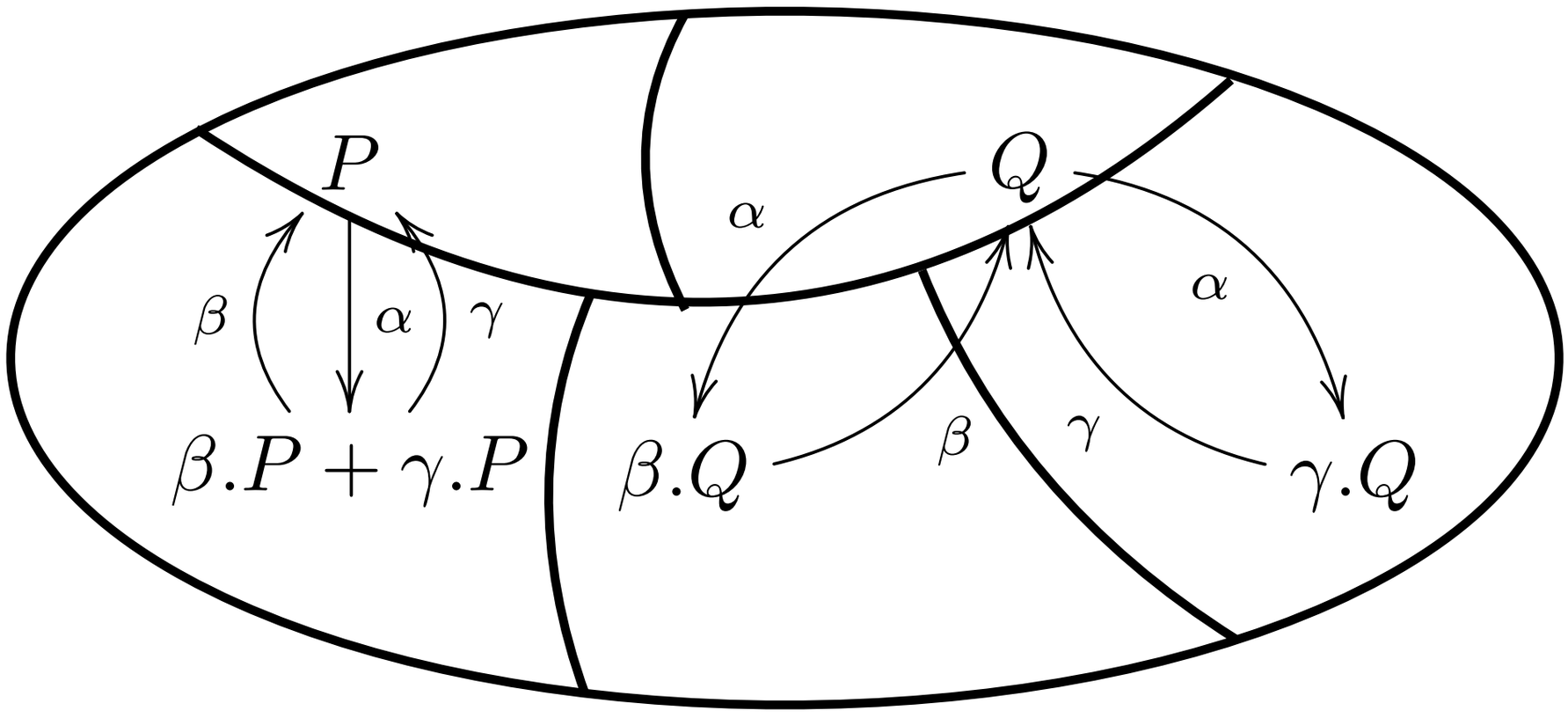
$\gamma.Q \xrightarrow{\gamma}$

Si devono distinguere i processi con capacità diverse

Esempio

$P \triangleq \alpha.(\beta.P + \gamma.P)$ $\quad P \stackrel{?}{\simeq} Q \quad$ $Q \triangleq \alpha.\beta.Q + \alpha.\gamma.Q$

$$\mathbf{R}_1 = \{\ \{P\ ,\ Q\}\ ,\ \{\beta.P + \gamma.P\}\ ,\ \{\beta.Q\}\ ,\ \{\gamma.Q\}\ \}$$



$P \xrightarrow{\alpha} [\beta.P + \gamma.Q]$

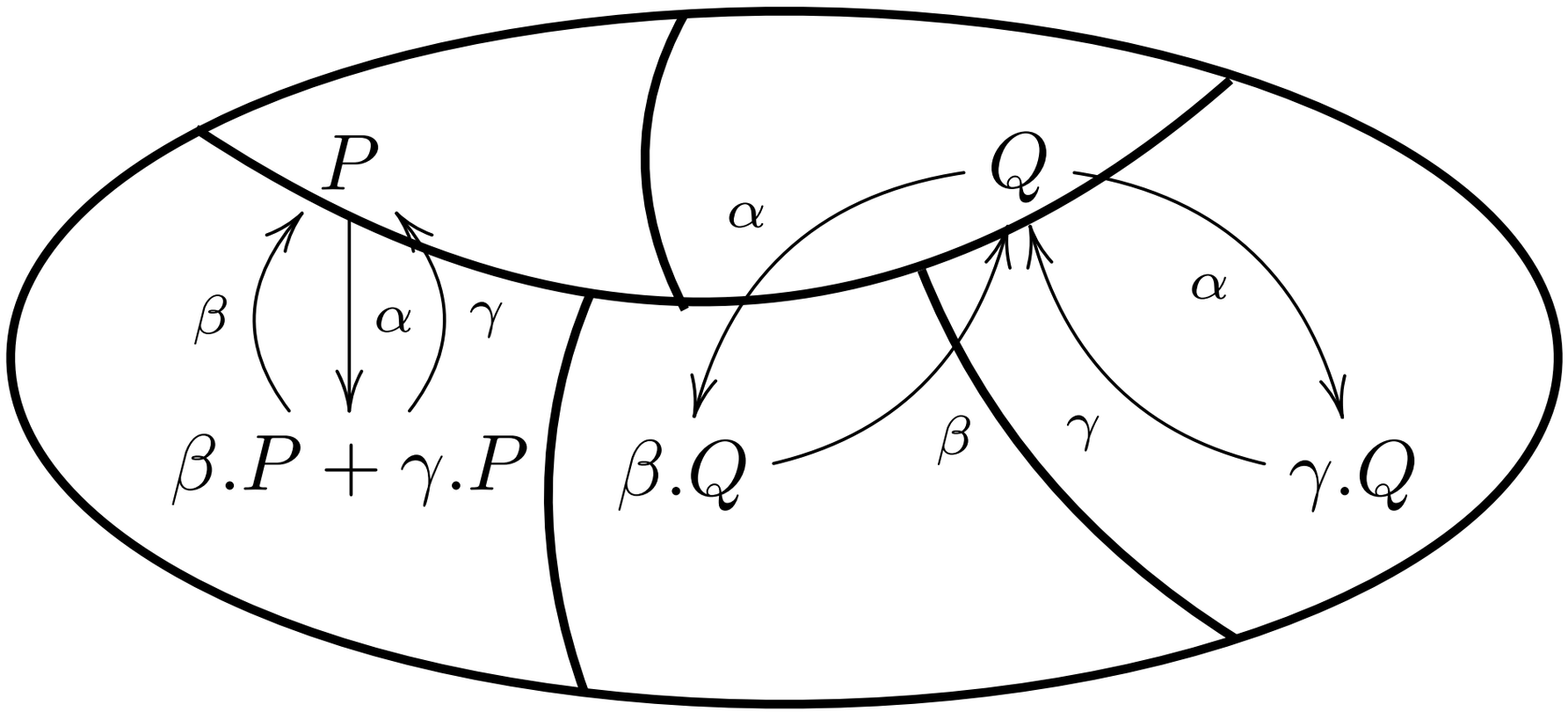
$Q \xrightarrow{\alpha} [\beta.Q], [\gamma.Q]$

Le transizioni α di P e Q finiscono in partizioni diverse

Esempio

$P \triangleq \alpha.(\beta.P + \gamma.P)$ $\qquad$ $P \stackrel{?}{\simeq} Q$ $\qquad$ $Q \triangleq \alpha.\beta.Q + \alpha.\gamma.Q$

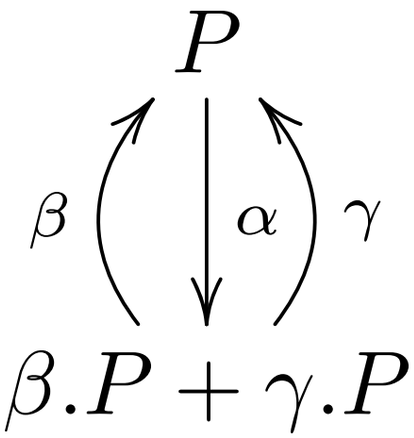
$$\mathbf{R}_2 = \{\ \{P\}\ ,\ \{Q\}\ ,\ \{\beta.P + \gamma.P\}\ ,\ \{\beta.Q\}\ ,\ \{\gamma.Q\}\ \}$$



Esempio

Guardato!

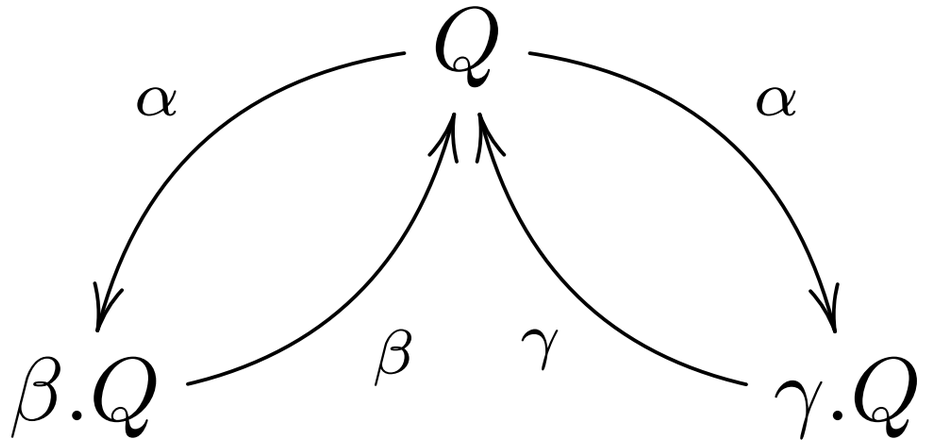
$$P \triangleq \alpha.(\beta.P + \gamma.P)$$



$$P \stackrel{?}{\simeq} Q$$

Guardato!

$$Q \triangleq \alpha.\beta.Q + \alpha.\gamma.Q$$



$$\mathbf{R}_0 = \{\ \{P\ ,\ Q\ ,\ \beta.P + \gamma.P\ ,\ \beta.Q\ ,\ \gamma.Q\}\ \}$$

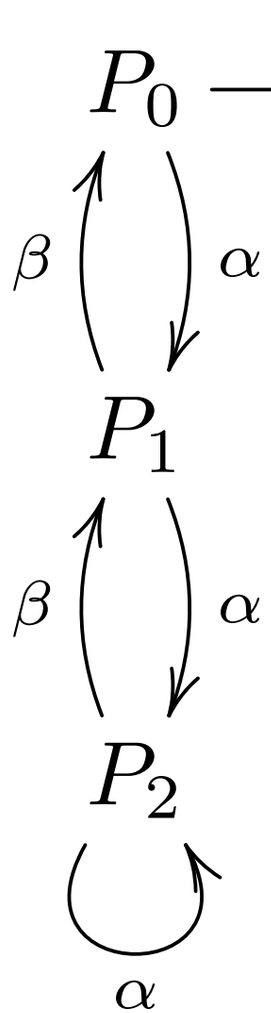
$$\mathbf{R}_1 = \{\ \{P\ ,\ Q\}\ ,\ \{\beta.P + \gamma.P\}\ ,\ \{\beta.Q\}\ ,\ \{\gamma.Q\}\ \}$$

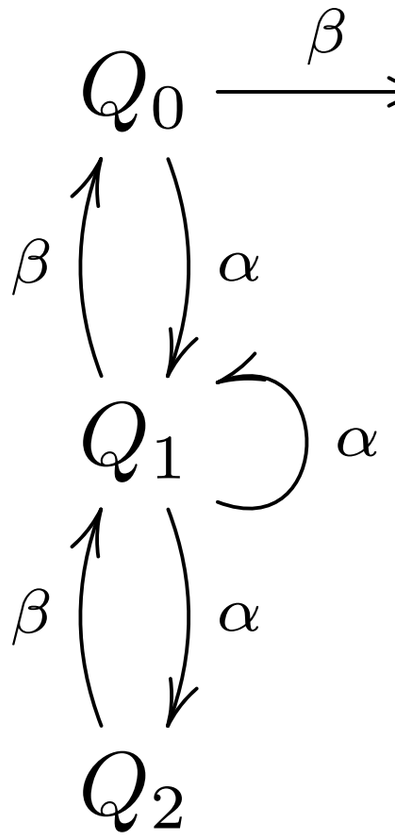
$$\mathbf{R}_2 = \{\ \{P\}\ ,\ \{Q\}\ ,\ \{\beta.P + \gamma.P\}\ ,\ \{\beta.Q\}\ ,\ \{\gamma.Q\}\ \}$$

Solo partizioni con un elemento, ci possiamo fermare $P \neq Q$

Esercizio

finitamente ramificato! $P_0 \stackrel{?}{\simeq} Q_0$ finitamente ramificato!





$\mathbf{nil} \not\to$

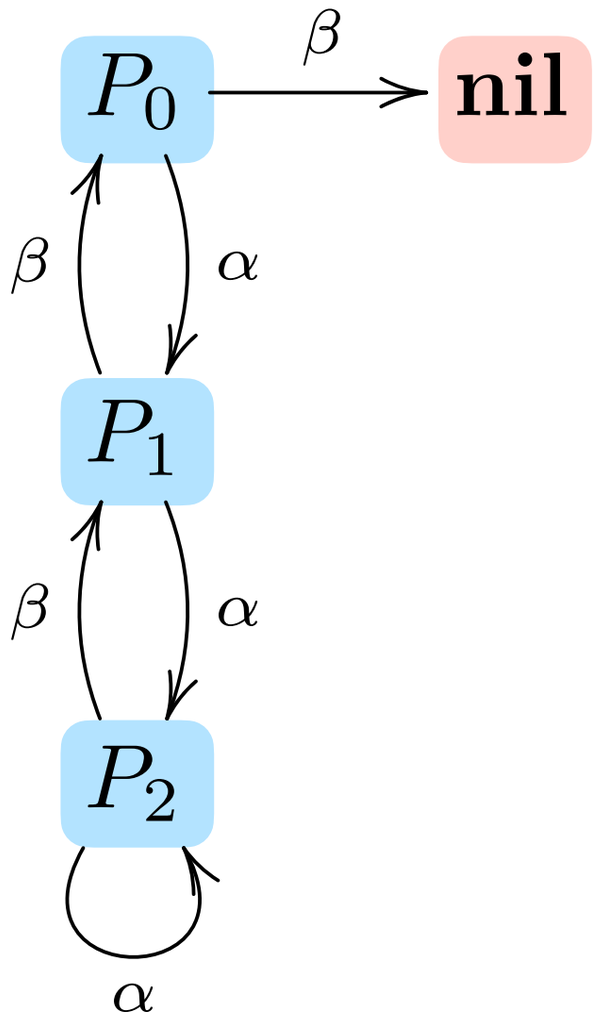
$Q_2 \xrightarrow{\beta}$

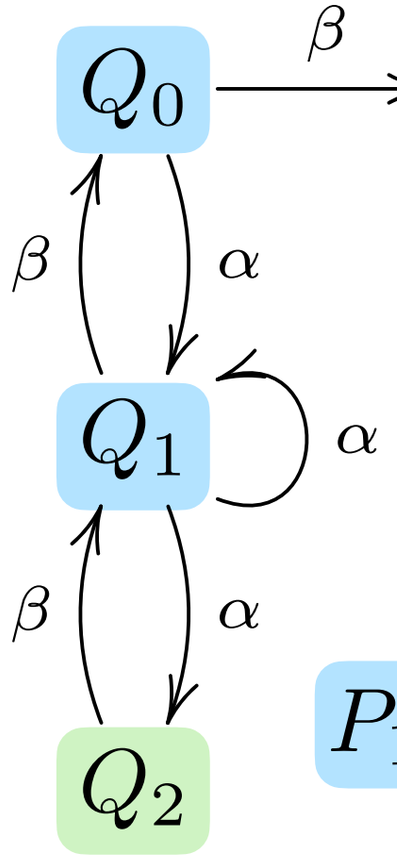
$P_0, Q_0, P_1, Q_1, P_2 \xrightarrow{\alpha,\beta}$

$\mathbf{R}_0 = \{\ \{P_0, Q_0, P_1, Q_1, P_2, Q_2, \mathbf{nil}\}\ \}$

Esercizio

$$P_0 \stackrel{?}{\simeq} Q_0$$





$P_0, Q_0 \xrightarrow{\beta} [\mathbf{nil}]$

$P_1, Q_1, P_2 \not\xrightarrow{\beta} [\mathbf{nil}]$

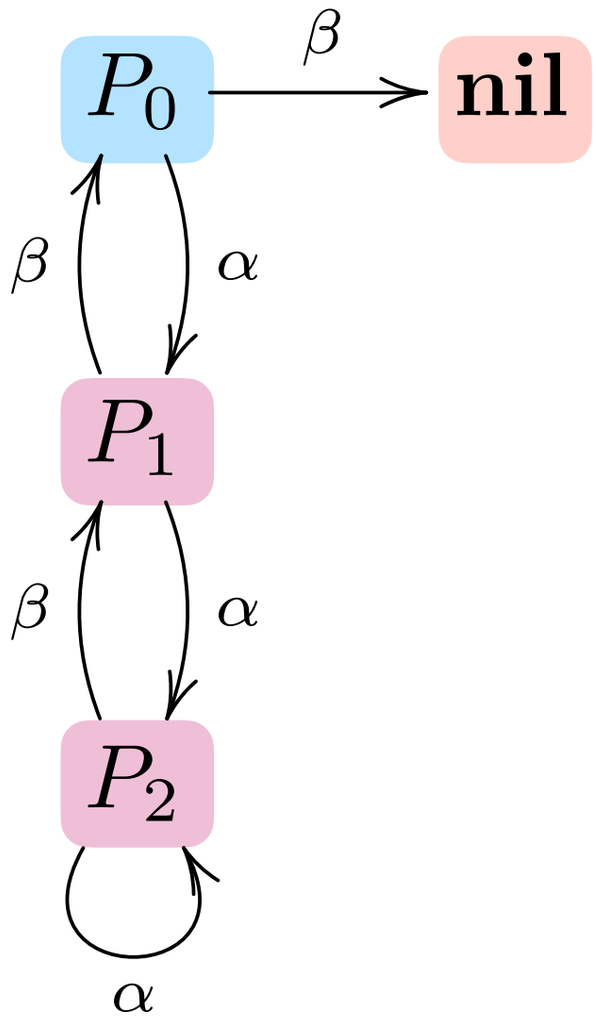
$Q_1 \xrightarrow{\alpha} [Q_2]$

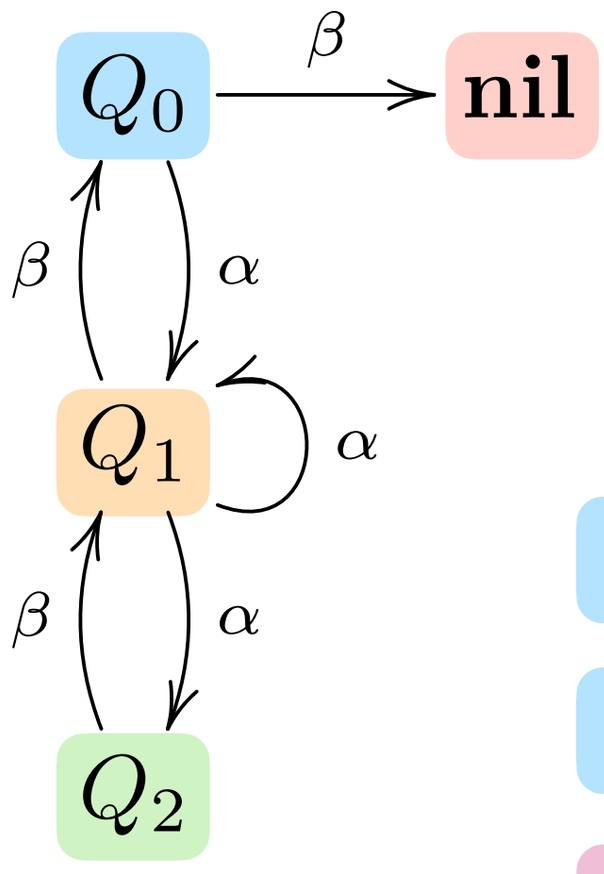
$P_1, P_2 \not\xrightarrow{\alpha} [Q_2]$

$$\mathbf{R}_1 = \{\ \{P_0, Q_0, P_1, Q_1, P_2\}\ ,\ \{Q_2\}\ ,\ \{\mathbf{nil}\}\ \}$$

Esercizio

$$P_0 \stackrel{?}{\simeq} Q_0$$





$P_0 \xrightarrow{\alpha} [P_1]$

$Q_0 \not\xrightarrow{\alpha} [P_1]$

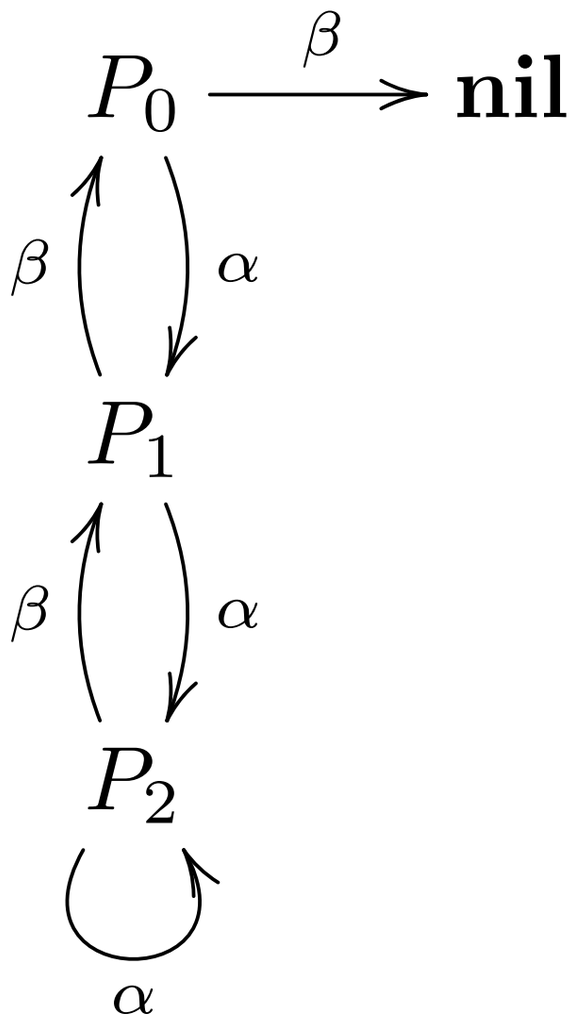
$P_1 \xrightarrow{\beta} [P_0]$

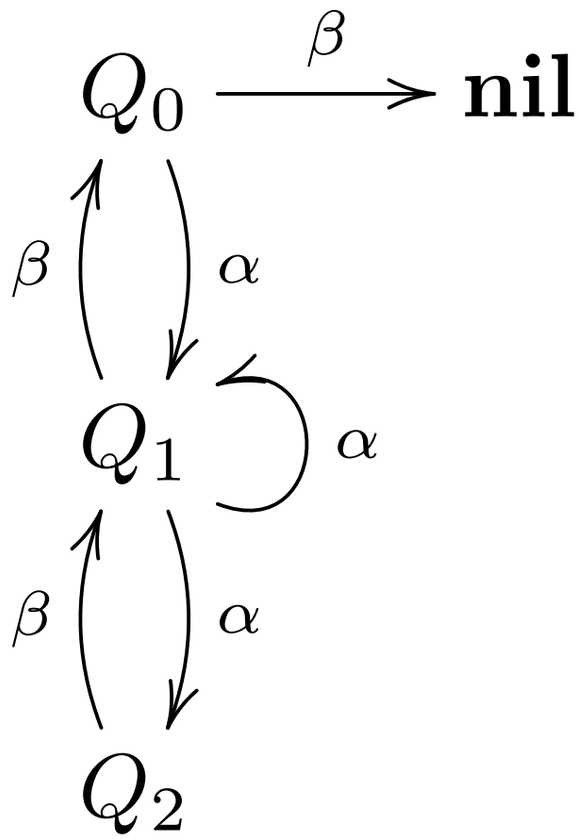
$P_2 \not\xrightarrow{\beta} [P_0]$

$$\mathbf{R}_2 = \{\ \{P_0, Q_0\}\ ,\ \{P_1, P_2\}\ ,\ \{Q_1\}\ ,\ \{Q_2\}\ ,\ \{\mathbf{nil}\}\ \}$$

Esercizio

$$P_0 \overset{?}{\simeq} Q_0$$



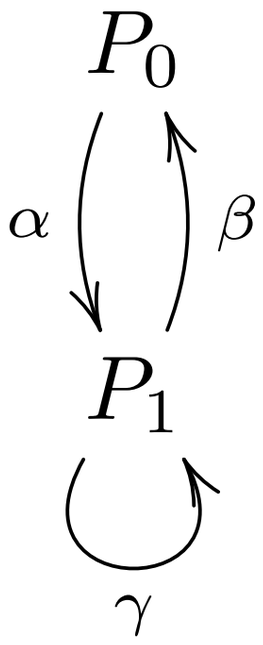


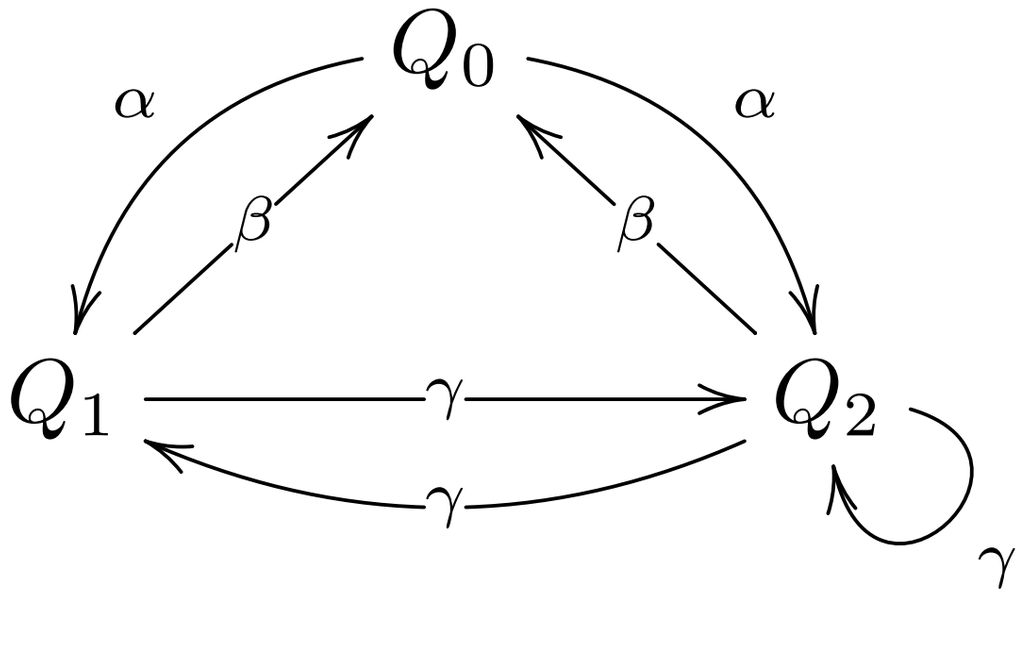
$$P_0 \not\simeq Q_0$$

$$\mathbf{R}_3 = \{\ \{P_0\}\ ,\ \{Q_0\}\ ,\ \{P_1\}\ ,\ \{P_2\}\ ,\ \{Q_1\}\ ,\ \{Q_2\}\ ,\ \{\mathbf{nil}\}\ \}$$

Esercizio

finitamente ramificato! $P_0 \overset{?}{\simeq} Q_0$ finitamente ramificato!





$\mathbf{R}_0 = \{\ \{P_0, Q_0, P_1, Q_1, Q_2\}\ \}$

$$P_0, Q_0 \xrightarrow{\alpha}$$
$$P_1, Q_1, Q_2 \xrightarrow{\beta,\gamma}$$

$\mathbf{R}_1 = \{\ \{P_0, Q_0\}\ ,\ \{P_1, Q_1, Q_2\}\ \}$

Non ci sono più ragioni per discriminare!

processi non guardati?

E il caso generale? (processi non guardati)

ogni power set ordinato per inclusione definisce un reticolo completo

Reticolo completo: $(D, \sqsubseteq)$ OP tale che

ogni $X \subseteq D$ ha un least upper bound $\bigsqcup X$

ogni $X \subseteq D$ ha un greatest lower bound $\bigsqcap X$

ha il bottom e top $\bot = \bigsqcap D$ $\top = \bigsqcup D$

TH. [Knaster-Tarski] $(D, \sqsubseteq)$ reticolo completo

$f : D \to D$ monotona

ha un least e un greatest fixpoint

$d_{\mathsf{min}} \triangleq \bigsqcap \{d \in D \mid f(d) \sqsubseteq d\}$ e' un least fixpoint

glb

pre punti fissi

$d_{\mathsf{max}} \triangleq \bigsqcup \{d \in D \mid d \sqsubseteq f(d)\}$ e' un greatest fixpoint

lub

post punti fissi

il minimo e il massimo punto fisso esistono... ma come calcolarli?