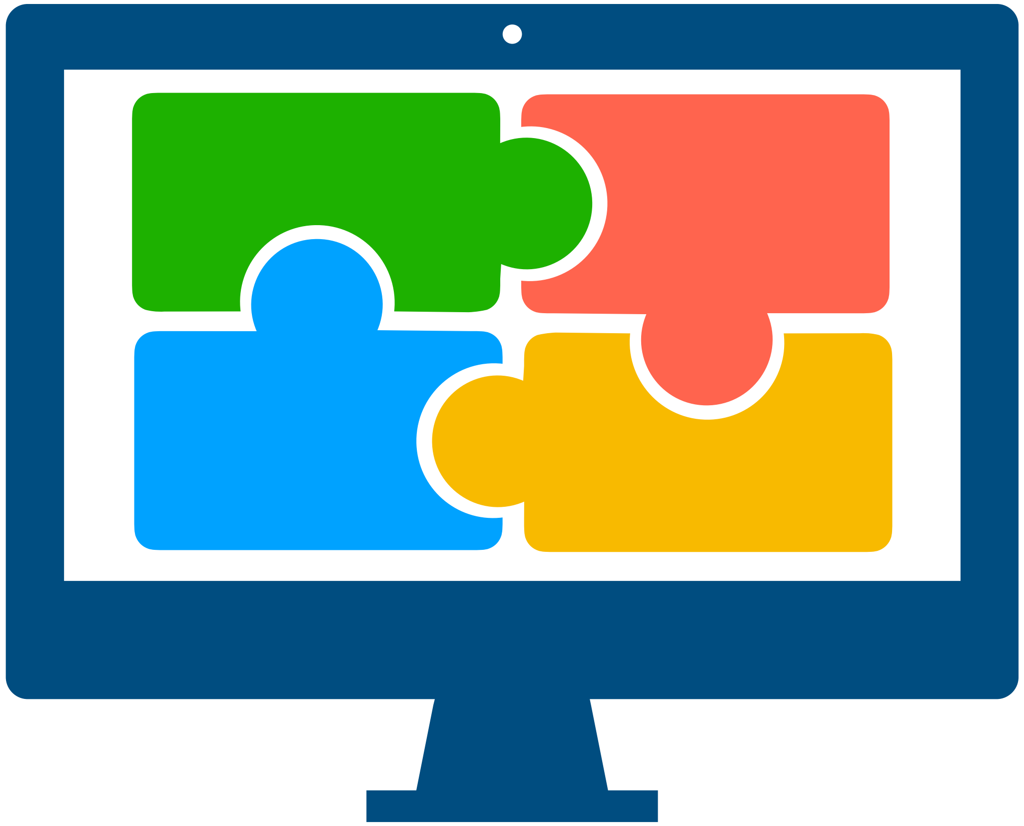

Linguaggi di Programmazione

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HOFL: Equivalenza semantica operazionale vs denotazionale?-cap 10

HOFL
Semantica operazionale vs denotazionale

Differenze

operazionale $t \to c$

termini tipabili e chiusi

senza environment

non e' una congruenza

termini canonici

denotazionale $[\![t]\!]\rho$

termini tipabili

con environment

e' una congruenza

oggetti matematici

$$\forall t, c. \quad t \to c \quad \stackrel{?}{\Leftrightarrow} \quad \forall \rho.\ [\![t]\!]\rho = [\![c]\!]\rho$$

$$t \to c \Rightarrow \forall \rho.\ [\![t]\!]\rho = [\![c]\!]\rho$$

$$(\forall \rho.\ [\![t]\!]\rho = [\![c]\!]\rho) \not\Rightarrow t \to c$$

c'è solo un tipo
per il quale vale
l'implicazione

Inconsistenza: esempio

$x : int \qquad c_0 = \lambda x.\, x + 0 \qquad c_1 = \lambda x.\, x$

sono gia' in forma canonica

$[\![c_0]\!]\,\rho = [\![c_1]\!]\,\rho \qquad c_0 \not\to c_1$

$$[\![c_0]\!]\,\rho = [\![\lambda x.\, x+0]\!]\,\rho = \lfloor \lambda d.\, d \underline{+}_{\perp} \lfloor 0 \rfloor \rfloor = \lfloor \lambda d.\, d \rfloor = [\![\lambda x.\, x]\!]\,\rho = [\![c_1]\!]\,\rho$$

Correttezza

TH. $t \to c \Rightarrow \forall \rho.\ [\![t]\!]\,\rho = [\![c]\!]\,\rho$

prova. per induzione sulle regole

$$P(t \to c) \stackrel{\text{def}}{=} \forall \rho.\ [\![t]\!]\,\rho = [\![c]\!]\,\rho$$

$$\frac{}{c \to c}$$

$$P(c \to c) \stackrel{\text{def}}{=} \forall \rho.\ [\![c]\!]\,\rho = [\![c]\!]\,\rho$$ ovvio

TH. $t \to c \Rightarrow \forall \rho.\ [\![t]\!]\,\rho = [\![c]\!]\,\rho$ (continua)

$$\frac{t_1 \to n_1 \quad t_2 \to n_2}{t_1 \text{ op } t_2 \to n_1 \ \underline{\text{op}}\ n_2}$$

assumiamo

$$P(t_1 \to n_1) \stackrel{\text{def}}{=} \forall \rho.\ [\![t_1]\!]\,\rho = [\![n_1]\!]\,\rho = \lfloor n_1 \rfloor$$

$$P(t_2 \to n_2) \stackrel{\text{def}}{=} \forall \rho.\ [\![t_2]\!]\,\rho = [\![n_2]\!]\,\rho = \lfloor n_2 \rfloor$$

proviamo $P(t_1 \text{ op } t_2 \to n_1 \ \underline{\text{op}}\ n_2) \stackrel{\text{def}}{=} \forall \rho.\ [\![t_1 \text{ op } t_2]\!]\,\rho = \left[\!\!\left[n_1 \ \underline{\text{op}}\ n_2 \right]\!\!\right] \rho$

$$\begin{aligned}
[\![t_1 \text{ op } t_2]\!]\,\rho &= [\![t_1]\!]\,\rho \ \underline{\text{op}}_{\perp}\ [\![t_2]\!]\,\rho && (\text{ per definizione di } [\![\cdot]\!]) \\
&= \lfloor n_1 \rfloor \ \underline{\text{op}}_{\perp}\ \lfloor n_2 \rfloor && (\text{ per ipotesi induttiva }) \\
&= \lfloor n_1 \ \underline{\text{op}}\ n_2 \rfloor && (\text{ per definizione di } \underline{\text{op}}_{\perp}) \\
&= \left[\!\!\left[n_1 \ \underline{\text{op}}\ n_2 \right]\!\!\right] \rho && (\text{ per definizione di } [\![\cdot]\!])
\end{aligned}$$

TH. $t \rightarrow c \Rightarrow \forall \rho.\ [\![t]\!]\,\rho = [\![c]\!]\,\rho$ (continua)

$$\frac{t \rightarrow 0 \quad t_0 \rightarrow c_0}{\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 \rightarrow c_0}$$

assumiamo

$$P(t \rightarrow 0) \stackrel{\text{def}}{=} \forall \rho.\ [\![t]\!]\,\rho = [\![0]\!]\,\rho = \lfloor 0 \rfloor$$

$$P(t_0 \rightarrow c_0) \stackrel{\text{def}}{=} \forall \rho.\ [\![t_0]\!]\,\rho = [\![c_0]\!]\,\rho$$

proviamo $P(\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1 \rightarrow c_0) \stackrel{\text{def}}{=} \forall \rho.\ [\![\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1]\!]\,\rho = [\![c_0]\!]\,\rho$

$$\begin{aligned}
[\![\textbf{if } t \textbf{ then } t_0 \textbf{ else } t_1]\!]\,\rho &= \mathit{Cond}([\![t]\!]\,\rho, [\![t_0]\!]\,\rho, [\![t_1]\!]\,\rho) && (\text{per def. di } [\![\cdot]\!]) \\
&= \mathit{Cond}(\lfloor 0 \rfloor, [\![t_0]\!]\,\rho, [\![t_1]\!]\,\rho) && (\text{per ip. induttiva}) \\
&= [\![t_0]\!]\,\rho && (\text{per def. di } \mathit{Cond}) \\
&= [\![c_0]\!]\,\rho && (\text{per ip. induttiva})
\end{aligned}$$

if false) analogo (omesso)

TH. $t \to c \Rightarrow \forall \rho.\ [\![t]\!]\,\rho = [\![c]\!]\,\rho$ (continua)

$$\frac{t \to (t_0, t_1) \qquad t_0 \to c_0}{\mathbf{fst}(t) \to c_0}$$

assumiamo

$$P(t \to (t_0,t_1)) \stackrel{\text{def}}{=} \forall \rho.\ [\![t]\!]\,\rho = [\![(t_0,t_1)]\!]\,\rho$$

$$P(t_0 \to c_0) \stackrel{\text{def}}{=} \forall \rho.\ [\![t_0]\!]\,\rho = [\![c_0]\!]\,\rho$$

proviamo $P(\mathbf{fst}(t) \to c_0) \stackrel{\text{def}}{=} \forall \rho.\ [\![\mathbf{fst}(t)]\!]\,\rho = [\![c_0]\!]\,\rho$

$$\begin{aligned}
[\![\mathbf{fst}(t)]\!]\,\rho &= \pi_1^*([\![t]\!]\,\rho) && (\text{per def. di } [\![\cdot]\!]) \\
&= \pi_1^*([\![(t_0,t_1)]\!]\,\rho) && (\text{per ip. induttiva}) \\
&= \pi_1^*(\lfloor([\![t_0]\!]\,\rho, [\![t_1]\!]\,\rho)\rfloor) && (\text{per def. di } [\![\cdot]\!]) \\
&= \pi_1([\![t_0]\!]\,\rho, [\![t_1]\!]\,\rho) && (\text{per def. di lifting}) \\
&= [\![t_0]\!]\,\rho && (\text{per def. di } \pi_1) \\
&= [\![c_0]\!]\,\rho && (\text{per ip. induttiva})
\end{aligned}$$

snd) analogo (omesso)

TH. $t \to c \Rightarrow \forall \rho.\ [\![t]\!]\rho = [\![c]\!]\rho$ (continua)

$$\frac{t_1 \to \lambda x.\ t_1' \qquad t_1'[^{t_0}/_x] \to c}{(t_1\ t_0) \to c}$$

assumiamo

$$P(t_1 \to \lambda x.\ t_1') \stackrel{\text{def}}{=} \forall \rho.\ [\![t_1]\!]\rho = [\![\lambda x.\ t_1']\!]\rho$$

$$P(t_1'[^{t_0}/_x] \to c) \stackrel{\text{def}}{=} \forall \rho.\ [\![t_1'[^{t_0}/_x]]\!]\rho = [\![c]\!]\rho$$

proviamo $P((t_1\ t_0) \to c) \stackrel{\text{def}}{=} \forall \rho.\ [\![(t_1\ t_0)]\!]\rho = [\![c]\!]\rho$

$$\begin{aligned}
[\![(t_1\ t_0)]\!]\rho &= \mathbf{let}\ \varphi \Leftarrow [\![t_1]\!]\rho.\ \varphi([\![t_0]\!]\rho) && (\text{per def. di } [\![\cdot]\!]) \\
&= \mathbf{let}\ \varphi \Leftarrow [\![\lambda x.\ t_1']\!]\rho.\ \varphi([\![t_0]\!]\rho) && (\text{per ip. induttiva}) \\
&= \mathbf{let}\ \varphi \Leftarrow \lfloor \lambda d.\ [\![t_1']\!]\rho[^d/_x] \rfloor.\ \varphi([\![t_0]\!]\rho) && (\text{per def. di } [\![\cdot]\!]) \\
&= (\lambda d.\ [\![t_1']\!]\rho[^d/_x])\ ([\![t_0]\!]\rho) && (\text{per de-lifting}) \\
&= [\![t_1']\!]\rho[^{[\![t_0]\!]\rho}/_x] && (\text{per applicazione}) \\
&= [\![t_1'[^{t_0}/_x]]\!]\rho && (\text{per lem. di sostituzione}) \\
&= [\![c]\!]\rho && (\text{per ip. induttiva})
\end{aligned}$$

TH. $t \rightarrow c \Rightarrow \forall \rho.\ [\![t]\!]\,\rho = [\![c]\!]\,\rho$ (continua)

$$\frac{t[{}^{\mathbf{rec}\ x.\ t}/_{x}] \rightarrow c}{\mathbf{rec}\ x.\ t \rightarrow c}$$

assumiamo

$$P(t[{}^{\mathbf{rec}\ x.\ t}/_{x}] \rightarrow c) \stackrel{\text{def}}{=} \forall \rho.\ [\![t[{}^{\mathbf{rec}\ x.\ t}/_{x}]]\!]\,\rho = [\![c]\!]\,\rho$$

proviamo $P(\mathbf{rec}\ x.\ t \rightarrow c) \stackrel{\text{def}}{=} \forall \rho.\ [\![\mathbf{rec}\ x.\ t]\!]\,\rho = [\![c]\!]\,\rho$

$$\begin{aligned}
[\![\mathbf{rec}\ x.\ t]\!]\,\rho &= [\![t]\!]\,\rho[{}^{[\![\mathbf{rec}\ x.\ t]\!]\rho}/_{x}] && (\text{per def.}) \\
&= [\![t[{}^{\mathbf{rec}\ x.\ t}/_{x}]]\!]\,\rho && (\text{per lemma di sostituzione}) \\
&= [\![c]\!]\,\rho && (\text{per ipotesi induttiva})
\end{aligned}$$

HOFL convergenza
Operazionale vs Denotazionale

Convergenza operazionale

$t : \tau$ chiuso

$$t \downarrow \quad \Leftrightarrow \quad \exists c \in C_\tau .\, t \longrightarrow c$$

$$t \uparrow \quad \Leftrightarrow \quad \neg\ t \downarrow$$

Esempi

$$\mathbf{rec}\ x.\ x\ \uparrow$$

$$\lambda y.\ \mathbf{rec}\ x.\ x\ \downarrow$$

$$(\lambda y.\ \mathbf{rec}\ x.\ x)\ 0\ \uparrow$$

$$\mathbf{if}\ 0\ \mathbf{then}\ 1\ \mathbf{else}\ \mathbf{rec}\ x.\ x\ \downarrow$$

Convergenza denotazionale

$t : \tau$ chiuso

$$t \Downarrow \quad \Leftrightarrow \quad \forall \rho \in Env, \exists v \in V_\tau.\ [\![t]\!]\rho = \lfloor v \rfloor$$

$$t \Uparrow \quad \Leftrightarrow \quad \neg\ t \Downarrow$$

Examples

$$[\![\mathbf{rec}\ x.\ x]\!]\rho \ \Uparrow$$

$$[\![\lambda y.\ \mathbf{rec}\ x.\ x]\!]\rho \ \Downarrow$$

$$[\![(\lambda y.\ \mathbf{rec}\ x.\ x)\ 0]\!]\rho \ \Uparrow$$

$$[\![\mathbf{if}\ 0\ \mathbf{then}\ 1\ \mathbf{else}\ \mathbf{rec}\ x.\ x]\!]\rho \ \Downarrow$$

Consistenza sulla convergenza

TH. $t : \tau$ chiuso $\quad t \downarrow \quad \Rightarrow \quad t \Downarrow$

proof. $t \downarrow \quad \Rightarrow \quad t \to c$ per def (per qualche c)

$\Rightarrow \quad \forall \rho .\ [\![t]\!]\,\rho = [\![c]\!]\,\rho$ per correttezza

le forme canoniche non sono bottom

$\Rightarrow \quad \forall \rho .\ [\![t]\!]\,\rho \neq \bot$ $\qquad [\![c]\!]\,\rho \neq \bot$

$\Rightarrow \quad t \Downarrow$ per def

TH. $t : \tau$ chiuso $\quad t \Downarrow \quad \Rightarrow \quad t \downarrow$

la prova non fa parte del programma del corso
(l'induzione strutturale non funzionerebbe)

HOFL equivalenza
Operazionale vs Denotazionale

HOFL equivalenze

$t_0, t_1 : \tau$ chiusi

$t_0 \equiv_{\mathrm{op}} t_1$ sse $\forall c.\ t_0 \to c \Leftrightarrow t_1 \to c$

$t_0 \equiv_{\mathrm{den}} t_1$ sse $\forall \rho.\ [\![t_0]\!] \rho = [\![t_1]\!] \rho$

Op e' piu' concreta

TH. $\equiv_{\mathrm{op}} \subseteq \equiv_{\mathrm{den}}$

prova. prendiamo $t_0, t_1 : \tau$ chiusi, t.c. $t_0 \equiv_{\mathrm{op}} t_1$

o $\exists c.\ t_0 \to c \ \wedge \ t_1 \to c$ oppure $t_0 \uparrow \wedge\, t_1 \uparrow$

se $\exists c.\ t_0 \to c \ \wedge \ t_1 \to c$

per la correttezza $\forall \rho.\ [\![t_0]\!]\rho = [\![c]\!]\rho = [\![t_1]\!]\rho$ quindi $t_0 \equiv_{\mathrm{den}} t_1$

se $t_0 \uparrow \wedge\, t_1 \uparrow$

per il risultato di accordo sulla convergenza $t_0 \Uparrow \wedge\, t_1 \Uparrow$

ovvero $\forall \rho.\ [\![t_0]\!]\rho = \bot_{D_\tau} = [\![t_1]\!]\rho$ quindi $t_0 \equiv_{\mathrm{den}} t_1$

Den e' strettamente piu' astratta

TH. $\equiv_{\mathrm{den}} \not\subseteq \equiv_{\mathrm{op}}$

prova.

riconsideriamo il precedente controesempio

$x : int \qquad c_0 = \lambda x.\, x + 0 \qquad c_1 = \lambda x.\, x$

Consistenza su int

TH. $t : int$ chiuso $\qquad t \to n \quad \Leftrightarrow \quad \forall \rho . [\![t]\!] \rho = \lfloor n \rfloor$

prova.

$\Rightarrow)$ se $t \to n$ allora $[\![t]\!] \rho = [\![n]\!] \rho = \lfloor n \rfloor$

$\Leftarrow)$ se $[\![t]\!] \rho = \lfloor n \rfloor$ significa che $t \Downarrow$

per il risultato di accordo sulla convergenza $t \downarrow$

quindi $t \to m$ per qualche m

per la correttezza $[\![t]\!] \rho = [\![m]\!] \rho = \lfloor m \rfloor$

e deve essere $m = n$

Equivalenza su int

TH. $t_0, t_1 : int$ $\quad t_0 \equiv_{\text{op}} t_1 \;\Leftrightarrow\; t_0 \equiv_{\text{den}} t_1$

prova. sappiamo $t_0 \equiv_{\text{op}} t_1 \;\Rightarrow\; t_0 \equiv_{\text{den}} t_1$

proviamo $t_0 \equiv_{\text{den}} t_1 \;\Rightarrow\; t_0 \equiv_{\text{op}} t_1$

assumiamo

$t_0 \equiv_{\text{den}} t_1$ quindi, o $\forall \rho.\ [\![t_0]\!]\rho = \bot_{\mathbb{Z}_\bot} = [\![t_1]\!]\rho$

o $\forall \rho.\ [\![t_0]\!]\rho = \lfloor n \rfloor = [\![t_1]\!]\rho$ per qualche n

se $\forall \rho.\ [\![t_0]\!]\rho = \bot_{\mathbb{Z}_\bot} = [\![t_1]\!]\rho$ allora $t_0 \Uparrow, t_1 \Uparrow$

per il risultato di accordo sulla convergenza

$t_0 \uparrow, t_1 \uparrow$ quindi $t_0 \equiv_{\text{op}} t_1$

se $\forall \rho.\ [\![t_0]\!]\rho = \lfloor n \rfloor = [\![t_1]\!]\rho$ allora $t_0 \to n\ ,\ t_1 \to n$

percio' $t_0 \equiv_{\text{op}} t_1$

HOFL
Semantica Unlifted

Domini Unlifted

domini lifted

$$D_\tau \triangleq (V_\tau)_\perp$$

$$V_{int} \triangleq \mathbb{Z}$$

$$V_{\tau_1 * \tau_2} \triangleq D_{\tau_1} \times D_{\tau_2} = (V_{\tau_1})_\perp \times (V_{\tau_2})_\perp$$

$$V_{\tau_1 \to \tau_2} \triangleq [D_{\tau_1} \to D_{\tau_2}] = [(V_{\tau_1})_\perp \to (V_{\tau_2})_\perp]$$

domini unlifted

$$U_{int} \triangleq \mathbb{Z}_\perp$$

$$U_{\tau_1 * \tau_2} \triangleq U_{\tau_1} \times U_{\tau_2}$$

$$U_{\tau_1 \to \tau_2} \triangleq [U_{\tau_1} \to U_{\tau_2}]$$

Semantica Unlifted

come prima

$$(\!|n|\!)\rho \triangleq \lfloor n \rfloor$$

$$(\!|x|\!)\rho \triangleq \rho(x)$$

$$(\!|t_1 \ \mathsf{op} \ t_2|\!)\rho \triangleq (\!|t_1|\!)\rho \ \underline{\mathsf{op}}_{\perp} \ (\!|t_2|\!)\rho$$

$$(\!|\mathbf{if} \ t \ \mathbf{then} \ t_1 \ \mathbf{else} \ t_2|\!)\rho \triangleq \mathsf{Cond}_{\tau}(\ (\!|t|\!)\rho \ , \ (\!|t_1|\!)\rho \ , \ (\!|t_2|\!)\rho \ )$$

$$(\!|\mathbf{rec} \ x. \ t|\!)\rho \triangleq \mathit{fix} \ \lambda d. \ (\!|t|\!)\rho[^d/_x]$$

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senza lifting

$$(\!|(t_1 \ , \ t_2)|\!)\rho \triangleq (\ (\!|t_1|\!)\rho \ , \ (\!|t_2|\!)\rho \ )$$

$$(\!|\mathbf{fst}(\ t \ )|\!)\rho \triangleq \pi_1 \ (\ (\!|t|\!)\rho \ )$$
$$(\!|\mathbf{snd}(\ t \ )|\!)\rho \triangleq \pi_2 \ (\ (\!|t|\!)\rho \ )$$

$$(\!|\lambda x. \ t|\!)\rho \triangleq \lambda d. \ (\!|t|\!)\rho[^d/_x]$$

$$(\!| \ t \ t_0 \ |\!)\rho \triangleq (\ (\!|t|\!)\rho \ ) \ (\ (\!|t_0|\!)\rho \ )$$

Inconsistenza sulla convergenza

$t_1 \triangleq \mathbf{rec}\ x.\ x \ : int \to int$
$x : int \to int$

$t_2 \triangleq \lambda y.\ \mathbf{rec}\ z.\ z \ : int \to int$
$y, z : int$

$$D_{int \to int} = [\mathbb{Z}_\perp \to \mathbb{Z}_\perp]_\perp$$

$[\![t_1]\!]\rho = \perp_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]_\perp}$

$[\![t_2]\!]\rho = \lfloor \perp_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]} \rfloor$

$t_1 \Uparrow$

$t_2 \Downarrow$

$t_1 \uparrow$

$t_2 \downarrow \qquad t_2 \to t_2$

$$U_{int \to int} = [\mathbb{Z}_\perp \to \mathbb{Z}_\perp]$$

$(\!|t_1|\!)\rho = \perp_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]}$

$(\!|t_2|\!)\rho = \perp_{[\mathbb{Z}_\perp \to \mathbb{Z}_\perp]} = \lambda d.\ \perp_{\mathbb{Z}_\perp}$

$t_1 \Uparrow_{\text{unlifted}}$

$t_2 \Uparrow_{\text{unlifted}}$

$t_2 \downarrow \not\Rightarrow t_2 \Downarrow_{\text{unlifted}}$