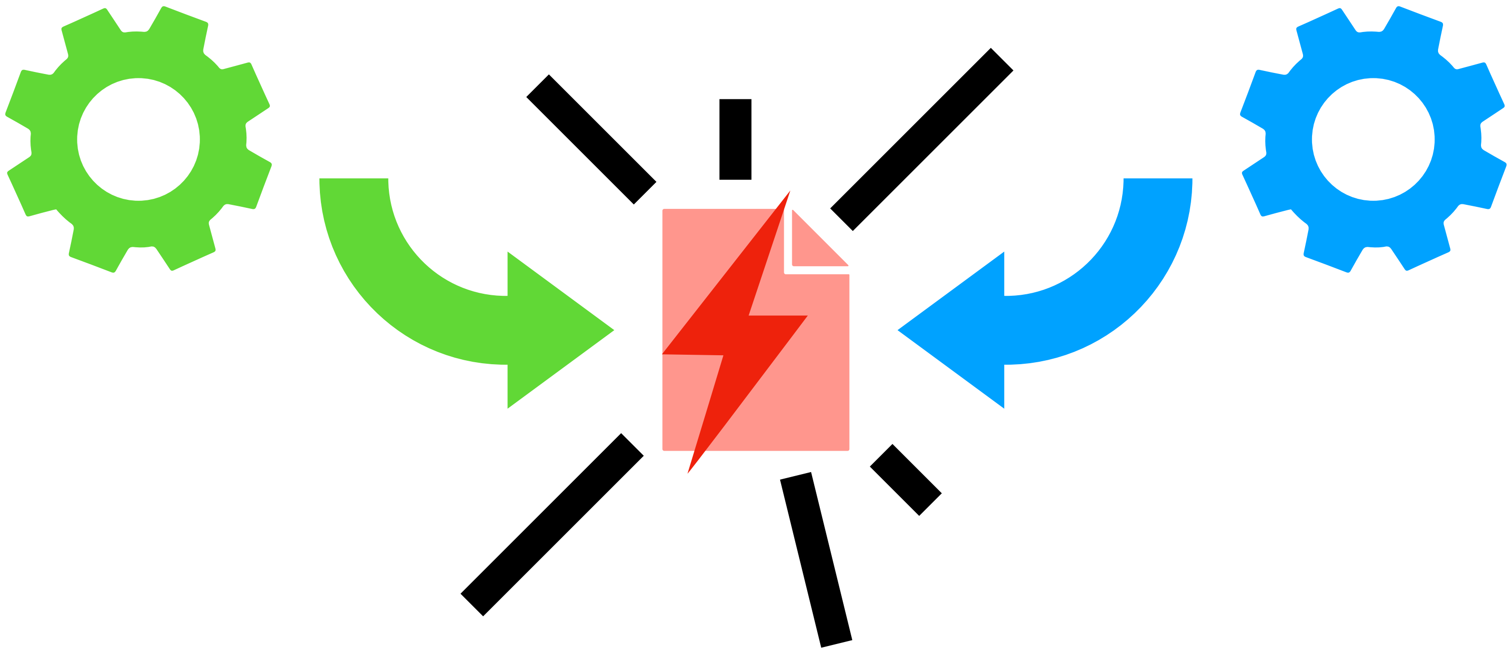
Linguaggi di Programmazione



Roberta Gori

Operatore delle conseguenze immediate- 5.3

Riassunto

Teorema di Kleene

$$p \triangleq f(p)$$

una definizione ricorsiva

esiste un tale p ?

esiste un p minimo?

se $f : D \to D$ continua D $\mathrm{OPC}_\perp$

allora $p \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp_D)$

scritto anche $\mathit{fix}(f)$

Il nostro problema

$$\mathcal{C}[\![\cdot]\!] : \mathsf{Com} \to \mathbb{M} \to \mathbb{M} \cup \{\bot\}$$

$$\mathcal{C}[\![\mathbf{while}\ b\ \mathbf{do}\ c]\!]\sigma \quad \triangleq \quad \begin{cases} \sigma & \text{if } \neg\mathcal{B}[\![b]\!]\sigma \\ \mathcal{C}[\![\mathbf{while}\ b\ \mathbf{do}\ c]\!](\mathcal{C}[\![c]\!]\sigma) & \text{otherwise} \end{cases}$$

p f(p)

Funzione 91 di McCarthy

$f = M\ f$

$\bot = \emptyset$

$\mathrm{OPC}_\bot$

$(\mathbf{Pf}(\mathbb{N},\mathbb{N}),\sqsubseteq)$

$$M\ f\ n = \begin{cases} n-10 & \text{se } n > 100 \\ f(f(n+11)) & \text{se } n \leq 100 \text{ e } f(n+11)\downarrow \\ \bot & \text{altrimenti} \end{cases}$$

M monotona? $f \sqsubseteq g \stackrel{?}{\Rightarrow} M(f) \sqsubseteq M(g)$

M continua? $M\left(\bigsqcup_i f_i\right) \stackrel{?}{\sqsubseteq} \bigsqcup_i M(f_i)$

Funzione 91 di McCarthy

$$M\ f\ n = \begin{cases} n - 10 & \text{se } n > 100 \\ f(f(n+11)) & \text{se } n \leq 100 \text{ e } f(n+11)\downarrow \\ \perp & \text{altrimenti} \end{cases}$$

M monotona?

assumiamo $f \sqsubseteq g$ e proviamo $M(f) \sqsubseteq M(g)$

prendiamo $(n, m) \in M(f)$

proviamo $(n, m) \in M(g)$

se $n > 100$

$m = n - 10$ e $(n, n-10) \in M(g)$

se $n \leq 100$

$m = f(f(n+11))$

quindi $\exists k.\ \{(n+11, k), (k, m)\} \in f \subseteq g$

e $(n, m) \in M(g)$

Funzione 91 di McCarthy

$$M\ f\ n = \begin{cases} n-10 & \text{se } n > 100 \\ f(f(n+11)) & \text{se } n \leq 100 \text{ e } f(n+11)\downarrow \\ \perp & \text{altrimenti} \end{cases}$$

M continua?

$$M\left(\bigsqcup_i f_i\right) \overset{?}{\sqsubseteq} \bigsqcup_i M(f_i)$$

consideriamo $f \triangleq \bigsqcup_i f_i$

prendiamo $(n,m) \in M(f)$

proviamo $(n,m) \in \bigsqcup_i M(f_i)$

se $n > 100$ $\quad m = n - 10 \qquad (n, n-10) \in M(f_1) \subseteq \bigsqcup_i M(f_i)$

se $n \leq 100$ $\quad m = f(f(n+11))$

quindi $\exists k.\ \{(n+11,k),(k,m)\} \in f$

quindi $\exists j_1, j_2.\ (n+11,k) \in f_{j_1}, (k,m) \in f_{j_2}$

prendiamo $j \triangleq \max\{j_1, j_2\} \qquad \{(n+11,k),(k,m)\} \in f_j$

e $(n,m) \in M(f_j) \subseteq \bigsqcup_i M(f_i)$

Funzione 91 di McCarthy

$f = M\ f$ $\quad(\mathbf{Pf}(\mathbb{N},\mathbb{N}), \sqsubseteq)$ $\quad\perp = \varnothing$ $\quad\mathrm{OPC}_\perp$

$$M\ f\ n = \begin{cases} n-10 & \text{se } n > 100 \\ f(f(n+11)) & \text{se } n \leq 100 \text{ e } f(n+11)\downarrow \\ \perp & \text{altrimenti} \end{cases}$$

$\{(n+11,k),(k,m)\} \in$

$$M^0(\varnothing) = \varnothing$$
$$M^1(\varnothing) = \{(n, n-10) \mid n > 100\} = \{(101,91),(102,92),....,(111,101),..\}$$
$$M^2(\varnothing) = \{(100,91),(101,91),(102,92),....,(110,100),...\}$$
$$M^3(\varnothing) = \{(99,91),(100,91),(101,91),(102,92),...\}$$
$$\cdots$$
$$M^{11}(\varnothing) = \{(91,91),(92,91),....,(101,91),(102,92),...\}$$
$$\cdots$$
$$M^{22}(\varnothing) = \{(80,91),(81,91),....,(101,91),(102,92),...\}$$
$$\cdots$$
$$M^{102}(\varnothing) = \{(0,91),(1,91),....,(101,91),(102,92),...\}$$

raggiunto punto fisso! funzione totale

Visione alternativa

$f = M\ f$ $\quad (\mathbf{Pf}(\mathbb{N},\mathbb{N}), \sqsubseteq)$ $\quad \bot = \varnothing$ $\quad \mathrm{OPC}_\bot$

$$M\ f\ n = \begin{cases} n - 10 & \text{se } n > 100 \\ f(f(n+11)) & \text{se } n \leq 100 \text{ e } f(n+11)\downarrow \\ \bot & \text{altrimenti} \end{cases}$$

formula: $(n,m) \in f$

o solo (n,m) per brevità

$$R_M \triangleq \left\{ \frac{}{(n, n-10)}\ n > 100 \quad , \quad \frac{(n+11, k) \quad (k, m)}{(n, m)}\ n \leq 100 \right\}$$

$$f(n) = m \qquad \Leftrightarrow \qquad (n,m) \in I_{R_M}$$

(insieme di teoremi in R_M)

Operatore delle Conseguenze Immediate (OCI)

Operatore delle Conseguenze immediate

F un insieme di formule

R un sistema logico
(con alcune restrizioni)

$(\wp(F), \subseteq) \quad \mathrm{CPO}_\perp$

$\widehat{R} : \wp(F) \to \wp(F)$

Operatore Conseguenze Immediate

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \dots \ x_n}{y} \in R.\ \{x_1, \dots, x_n\} \subseteq S \right\}$$

$\widehat{R}$ applica le regole ai fatti in S in tutti i modi possibili

$\widehat{R}(S)$: tutte le conclusioni che possiamo trarre in un passo dalle ipotesi in S

dimostriamo che il minimo punto fisso di $\widehat{R}$ e' I_R
(insieme dei teoremi di R)

Esempio

Stringhe con parentesi bilanciate $F = \{ s \in \mathcal{L} \mid s \in \{(,)\}^* \}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} \, , \, \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} \, , \, \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \emptyset$$

$$\widehat{R}(S) = \{\ \epsilon \ ?\!\in \mathcal{L}\ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \ldots \ x_n}{y} \in R. \ \{x_1, \ldots, x_n\} \subseteq S \right\}$$

Esempio

Stringhe con parentesi bilanciate $F = \{ s \in \mathcal{L} \mid s \in \{(,)\}^* \}$

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$$S = \{ \ \epsilon \in \mathcal{L} \ \}$$

$$\widehat{R}(S) = \{ ? \ \epsilon \in \mathcal{L} \ , \ () \in \mathcal{L} \ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \ldots \ x_n}{y} \in R. \ \{x_1, \ldots, x_n\} \subseteq S \right\}$$

Esempio

Stringhe con parentesi bilanciate $F = \{ s \in \mathcal{L} \mid s \in \{(,)\}^* \}$

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$$S = \{\ () \in \mathcal{L}\ \}$$

$$\widehat{R}(S) = \{ ?\ \epsilon \in \mathcal{L}\ ,\ (()) \in \mathcal{L}\ ,\ ()() \in \mathcal{L}\ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \ldots\ x_n}{y} \in R.\ \{x_1, \ldots, x_n\} \subseteq S \right\}$$

Esempio

Stringhe con parentesi bilanciate $F = \{s \in \mathcal{L} \mid s \in \{(,)\}^*\}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} \,,\, \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} \,,\, \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{\ )(\in \mathcal{L}\ \}$$

$$\widehat{R}(S) = \{?\epsilon \in \mathcal{L}\ ,\ ()() \in \mathcal{L}\ ,\ )()(\in \mathcal{L}\ \}$$

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Esempio

Stringhe con parentesi bilanciate $\quad F = \{s \in \mathcal{L} \mid s \in \{(,)\}^*\}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} \,,\ \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} \,,\ \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$S = \{\ )(\in \mathcal{L} \,,\ \ () \in \mathcal{L}\ \}$$

$$\widehat{R}(S) = ?$$

$$\widehat{R}(S) = \{\epsilon \in \mathcal{L} \,,\ ()() \in \mathcal{L} \,,\ (()) \in \mathcal{L} \,,\ )()(\in \mathcal{L} \,,\ )(() \in \mathcal{L} \,,\ ())(\in \mathcal{L}\ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \dots \ x_n}{y} \in R.\ \{x_1, \dots, x_n\} \subseteq S \right\}$$

Esempio

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$ $\qquad F = \{(n, m) \mid n, m \in \mathbb{N}\}$

$$R = \left\{ \ \frac{}{(0,0)} \ , \ \frac{}{(1,1)} \ , \ \frac{(n,h) \quad (n+1,k)}{(n+2, h+k)} \ \right\}$$

$$S = \emptyset$$

$$\widehat{R}(S) = \{ (0,0) \ , \ (1,1) \ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \dots \ x_n}{y} \in R. \ \{x_1, \dots, x_n\} \subseteq S \right\}$$

Esercizio

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$ $\qquad$ $F = \{(n, m) \mid n, m \in \mathbb{N}\}$

$$R = \left\{ \ \frac{}{(0,0)} \ , \ \frac{}{(1,1)} \ , \ \frac{(n,h) \quad (n+1,k)}{(n+2, h+k)} \ \right\}$$

$$S = \{ \ (2,1) \ \}$$

$$\widehat{R}(S) = \{ \ (0,0) \ , \ (1,1) \ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \dots \ x_n}{y} \in R. \ \{x_1, \dots, x_n\} \subseteq S \right\}$$

Esercizio

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$F = \{(n, m) \mid n, m \in \mathbb{N}\}$

$$R = \left\{ \ \frac{}{(0,0)} \ , \ \frac{}{(1,1)} \ , \ \frac{(n,h) \quad (n+1,k)}{(n+2, h+k)} \ \right\}$$

$$S = \{ \ (5,5) \ , \ (6,8) \ \}$$

$$\widehat{R}(S) = \{?(0,0) \ , \ (1,1) \ , \ (7,13) \ \}$$

$$\widehat{R}(S) \triangleq \left\{ y \mid \exists \frac{x_1 \ \dots \ x_n}{y} \in R. \ \{x_1, \dots, x_n\} \subseteq S \right\}$$

OCI e' monotono

TH. $\widehat{R}$ e' monotono

prova. Prendiamo $S_1 \subseteq S_2$

vogliamo provare $\widehat{R}(S_1) \subseteq \widehat{R}(S_2)$

prendiamo $y \in \widehat{R}(S_1)$ vogliamo provare $y \in \widehat{R}(S_2)$

$$\Downarrow$$

$$\exists \frac{x_1 \ \ldots \ x_n}{y} \in R \text{ con } \{x_1, \ldots, x_n\} \subseteq S_1$$

$$\Downarrow \quad S_1 \subseteq S_2$$

$$\{x_1, \ldots, x_n\} \subseteq S_2$$

$$\Downarrow$$

$$y \in \widehat{R}(S_2)$$

OCI e' continuo

TH. $\widehat{R}$ is continuo (sotto alcune ipotesi)

prova. Prendiamo $\{S_i\}_{i\in\mathbb{N}}$ una catena in $\wp(F)$

Vogliamo provare $$\bigcup_{i\in\mathbb{N}} \widehat{R}(S_i) = \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$$

$$\bigcup_{i\in\mathbb{N}} \widehat{R}(S_i) \subseteq \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$$ perche' $\widehat{R}$ e' monotona

$$\bigcup_{i\in\mathbb{N}} \widehat{R}(S_i) \supseteq \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$$?

OCI e' continuo (con.)

$$\bigcup_{i\in\mathbb{N}} \widehat{R}(S_i) \supseteq \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$$

Prendiamo

$y \in \widehat{R}\left(\bigcup_{i\in\mathbb{N}} S_i\right)$ vogliamo provare $y \in \bigcup_{i\in\mathbb{N}} \widehat{R}(S_i)$

$\Downarrow$

$\exists \dfrac{x_1 \ \dots \ x_n}{y} \in R$ con $\{x_1, \dots, x_n\} \subseteq \bigcup_{i\in\mathbb{N}} S_i$

percio' $\forall j \in [1, n].\ \exists k_j \in \mathbb{N}.\ x_j \in S_{k_j}$ prendiamo $k = \max\{k_1, \dots, k_n\}$

chiaramente $\{x_1, \dots, x_n\} \subseteq S_k$

quindi $y \in \widehat{R}(S_k) \subseteq \bigcup_{i\in\mathbb{N}} \widehat{R}(S_i)$

possibile sse ogni regola ha un numero finito di premesse

Teoremi dimostrabili

I_R l'insieme di teoremi dimostrabili in R

I_R^n teoremi dimostrabili con una derivazione di altezza al max n

$$I_R = \bigcup_{n \in \mathbb{N}} I_R^n$$

dove

$$I_R^0 = \emptyset$$

$$I_R^{n+1} = I_R^n \cup \widehat{R}(I_R^n)$$

dimostrabili usando un passo in piu'

teoremi dimostrabili con una derivazione di altezza al max n

Teoremi di altezza n

TH. Sia $P(n) \triangleq I_R^n = \widehat{R}^n(\emptyset)$ $\forall n \in \mathbb{N}.\ P(n)$

prova. per induzione matematica

$P(0) \triangleq I_R^0 = \widehat{R}^0(\emptyset)$ $I_R^0 = \emptyset = \widehat{R}^0(\emptyset)$

$\forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)$

prendiamo un generico n

assumiamo $P(n) \triangleq I_R^n = \widehat{R}^n(\emptyset)$

vogliamo provare $P(n+1) \triangleq I_R^{n+1} = \widehat{R}^{n+1}(\emptyset)$

$$
\begin{aligned}
I_R^{n+1} &= I_R^n \cup \widehat{R}(I_R^n) && \text{per def} \\
&= \widehat{R}^n(\emptyset) \cup \widehat{R}(\widehat{R}^n(\emptyset)) && \text{per ipotesi induttiva} \\
&= \widehat{R}^n(\emptyset) \cup \widehat{R}^{n+1}(\emptyset) && \text{per def} \\
&= \widehat{R}^{n+1}(\emptyset) && \text{perche'} \quad \widehat{R}^n(\emptyset) \subseteq \widehat{R}^{n+1}(\emptyset)
\end{aligned}
$$

Punto fisso di OCI

TH. $\mathit{fix}(\widehat{R}) = I_R$ (sotto alcune ipotesi)

ogni regola deve avere un numero finito di premesse

prova.

per il th. di punto fisso di Kleene sappiamo che $\mathit{fix}(\widehat{R})$ esiste

$$\mathit{fix}(\widehat{R}) = \bigcup_{n \in \mathbb{N}} \widehat{R}^n(\emptyset)$$ per def.

$$= \bigcup_{n \in \mathbb{N}} I_R^n$$ per il risultato precedente

$$= I_R$$ per def

Esempio

Stringhe con parentesi bilanciate $\quad F = \{s \in \mathcal{L} \mid s \in \{(,)\}^*\}$

$$R = \left\{ \frac{}{\epsilon \in \mathcal{L}} \, , \; \frac{s \in \mathcal{L}}{(s) \in \mathcal{L}} \, , \; \frac{s_1 \in \mathcal{L} \quad s_2 \in \mathcal{L}}{s_1 s_2 \in \mathcal{L}} \right\}$$

$$\widehat{R}^0(\emptyset) = \emptyset$$

$$\widehat{R}^1(\emptyset) = \{ \; \epsilon \in \mathcal{L} \; \}$$

$$\widehat{R}^2(\emptyset) = \{ \; \epsilon \in \mathcal{L} \; , \; () \in \mathcal{L} \; \}$$

$$\widehat{R}^3(\emptyset) = \{ \; \epsilon \in \mathcal{L} \; , \; () \in \mathcal{L} \; , \; (()) \in \mathcal{L} \; , \; ()() \in \mathcal{L} \; \}$$

$$\begin{aligned}\widehat{R}^4(\emptyset) = \{ \; & \epsilon \in \mathcal{L} \; , \; () \in \mathcal{L} \; , \; (()) \in \mathcal{L} \; , \; ()() \in \mathcal{L} \; , \; ((())) \in \mathcal{L} \; , \\ & (()()) \in \mathcal{L} \; , \; ()(()) \in \mathcal{L} \; , \; (())() \in \mathcal{L} \; , \; ()()() \in \mathcal{L} \; , \\ & (())()() \in \mathcal{L} \; , \; ()()(()) \in \mathcal{L} \; , \; (())(()) \in \mathcal{L} \; , \; ()()()() \in \mathcal{L} \; \}\end{aligned}$$

Example

$\mathbf{Pf}(\mathbb{N},\mathbb{N})$ $\qquad F = \{(n,m) \mid n,m \in \mathbb{N}\}$

$$R = \left\{ \ \frac{}{(0,0)}\ ,\ \frac{}{(1,1)}\ ,\ \frac{(n,h) \quad (n+1,k)}{(n+2,h+k)} \ \right\}$$

$$\widehat{R}^0(\emptyset) = \emptyset$$

$$\widehat{R}^1(\emptyset) = \{\ (0,0)\ ,\ (1,1)\ \}$$

$$\widehat{R}^2(\emptyset) = \{\ (0,0)\ ,\ (1,1)\ ,\ (2,1)\ \}$$

$$\widehat{R}^3(\emptyset) = \{\ (0,0)\ ,\ (1,1)\ ,\ (2,1)\ ,\ (3,2)\ \}$$

$$\widehat{R}^4(\emptyset) = \{\ (0,0)\ ,\ (1,1)\ ,\ (2,1)\ ,\ (3,2)\ ,\ (4,3)\ \}$$

$$\widehat{R}^5(\emptyset) = \{\ (0,0)\ ,\ (1,1)\ ,\ (2,1)\ ,\ (3,2)\ ,\ (4,3)\ ,\ (5,5)\ \}$$

$$\widehat{R}^6(\emptyset) = \{\ (0,0)\ ,\ (1,1)\ ,\ (2,1)\ ,\ (3,2)\ ,\ (4,3)\ ,\ (5,5)\ ,\ (6,8)\ \}$$