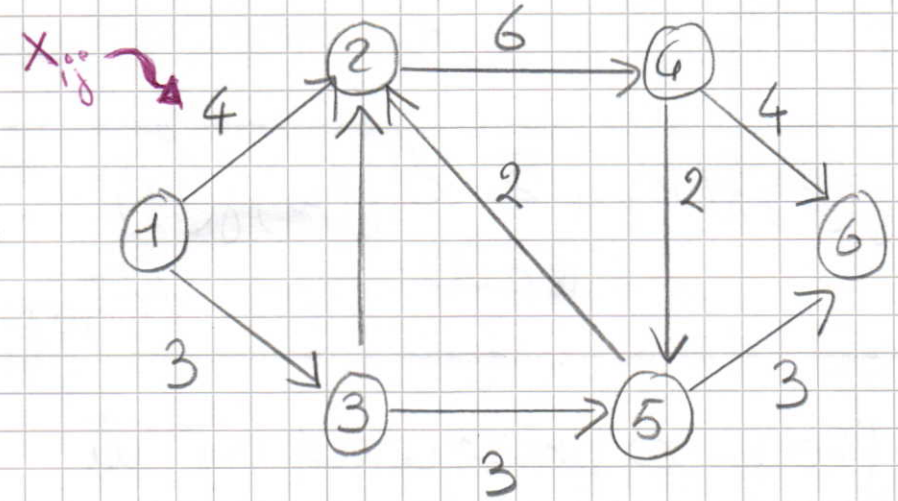
Flow decomposition

(Ahuja - Magnanti - Orlin : Chapter 3 (3.5))

Observation : any flow x can be defined in terms of flows on arcs (as previously formulated) or in terms of flows on paths and cycles

example



$-7 <$ flows on arcs $> x = (x_{ij})$

or

4 units along the path $(1, 2, 4, 6) = P_1$

3 units along the path $(1, 3, 5, 6) = P_2$

2 units along the cycle $(2, 4, 5) = W$

$x <$ flows on paths and cycle $>$

Let $\mathcal{P}$: set of all paths

$f(P)$: flow on $P \in \mathcal{P}$

$\mathcal{W}$: set of all cycles

$f(w)$: flow on $w \in \mathcal{W}$

$$\delta_{ij}(P) = \begin{cases} 1 & \text{if } (i,j) \in P \\ 0 & \text{otherwise} \end{cases}$$

$\delta_{ij}(w)$

Then : any flow representation in terms of path and cycle flows determines arc flows uniquely :

$$x_{ij} = \sum_{P \in \mathcal{P}} \delta_{ij}(P) f(P) + \sum_{w \in \mathcal{W}} \delta_{ij}(w) f(w) \qquad \forall (i,j) \in A$$

Viceversa : any flow representation in terms of arc flows "decomposes" into path and cycle flow (not uniquely)

⇓

Flow decomposition Theorem : each path and cycle flow has a unique representation in terms of (nonnegative) arc flows. Conversely, each arc

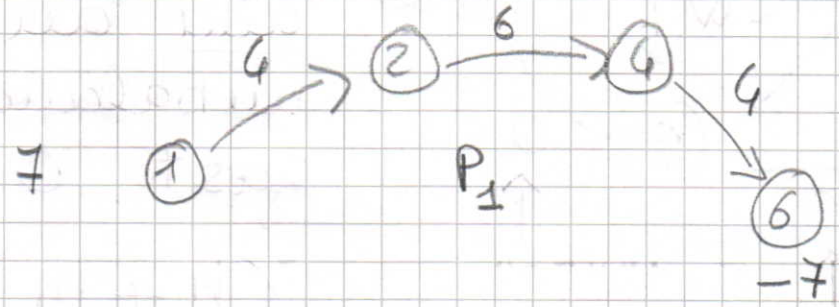
flow x can be represented as path and cycle flow (not uniquely) s.t:

a) each directed path with a positive flow connects a source to a destination

b) at most $n+m$ paths and cycles have a positive flow; out of these, at most m cycles have positive flow

Proof (intuition) <example cont.>

select a source node (1):

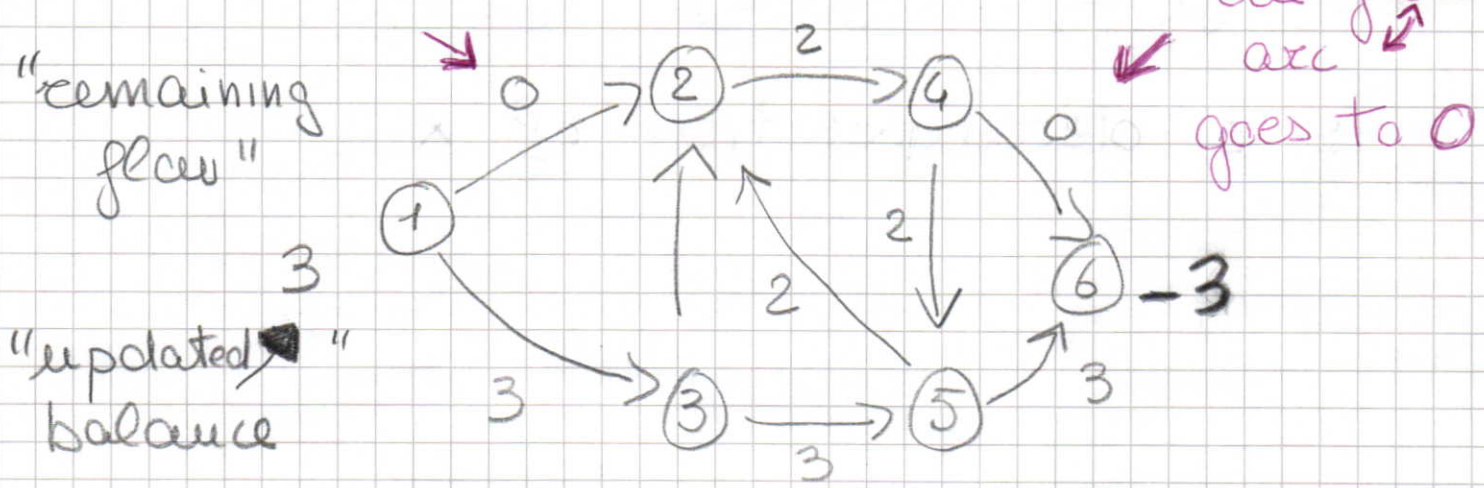


$\min\{c(P_1), b(1), -b(7)\}$

$= 4$

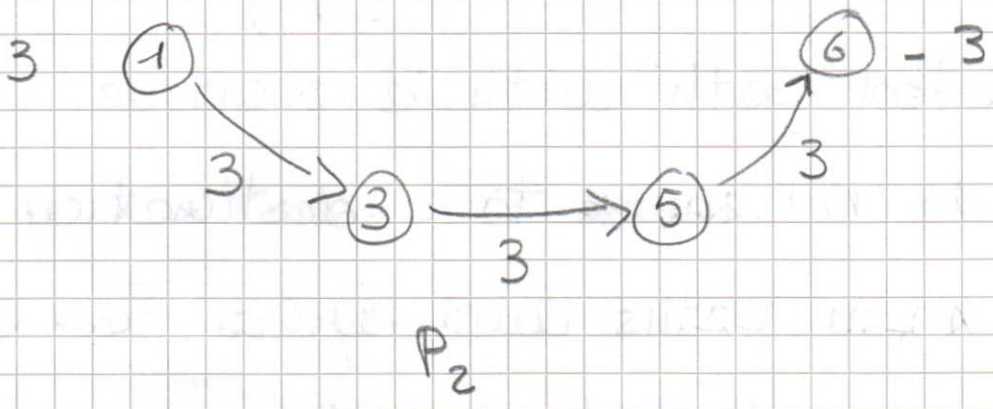
- subtract 4 units of flow along P_1:

$f(P_1) = 4$



at least one flow arc goes to 0

select a source node (1):



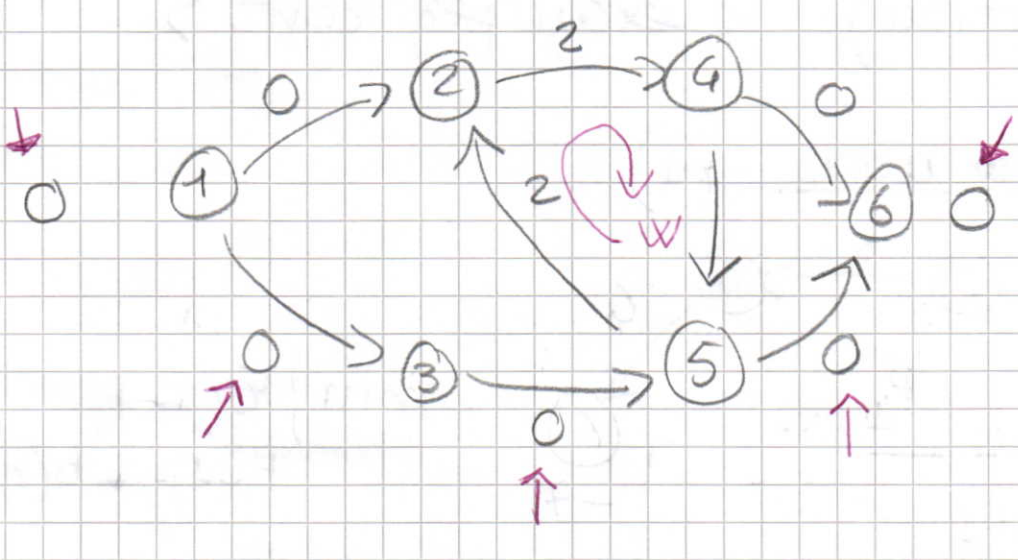
P_2

$\min\{r(P_2), b(1), -b(6)\}$

$= 3$

subtract 3 units of flow along P_2:

$f(P_2) = 3$



at least one flow arc goes to 0 and an imbalance goes to 0

$(n+m)$

remaining cycle w : $f(w) = 2$ at most m

$\Rightarrow f(P_1)$ $f(P_2)$ $f(w)$

is a decomposition of x

A <u>circulation</u> is a flow s.t.

$b(i) = 0 \quad \forall i \in N$

<u>Corollary</u> : a circulation x can be represented in terms of cycle flow along $\leq m$ directed cycles.

<u>Important consequences of the flow decomposition Theorem</u>

Let x be a flow and $G(x)$ its residual network.

<u>Def</u> : an <u>augmenting cycle</u> w w.r.t. x is a directed cycle in $G(x)$; <u>its cost</u>

$c(w) = \sum_{(i,j) \in w} c_{ij} = \sum_{(i,j)} c_{ij} \delta_{ij}(w)$ is the

costs in $G(x)$

mistake in AHO

change of the cost of x if we push 1 unit of flow along w.

Consider the minimum cost flow problem on $G=(N,A)$, and let x and x° be any two feasible solutions: how can we compare cx and cx°?

Indeed, the flow decomposition Theorem can be extended so as to estabish a relationship between x and x°:

Flow decomposition theorem (ext 1): given a feasible solution x° to the minimum cost flow problem, any other feasible solution x can be obtained from x° by sending flow along $\leq m$ augmenting cycles w.r.t. x°:

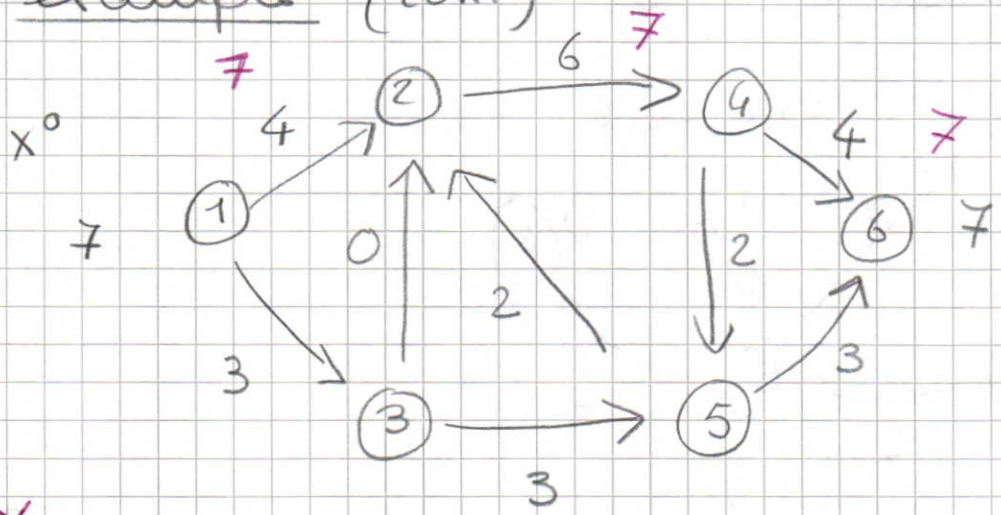
$$x_{ij} = x^\circ_{ij} + \delta_{ij}(w_1)\,\beta(w_1) + \dots + \delta_{ij}(w_r)\,\beta(w_r) \qquad \forall (i,j)\in A$$

$(r \leq m)$

now: $\delta_{ij}(w_k) = \begin{cases} +1 & \text{if } (i,j) \text{ is forward} \\ -1 & \text{if } (i,j) \text{ is backward} \\ 0 & \text{if } (i,j) \notin w_k \end{cases}$

$k = 1, \dots r$

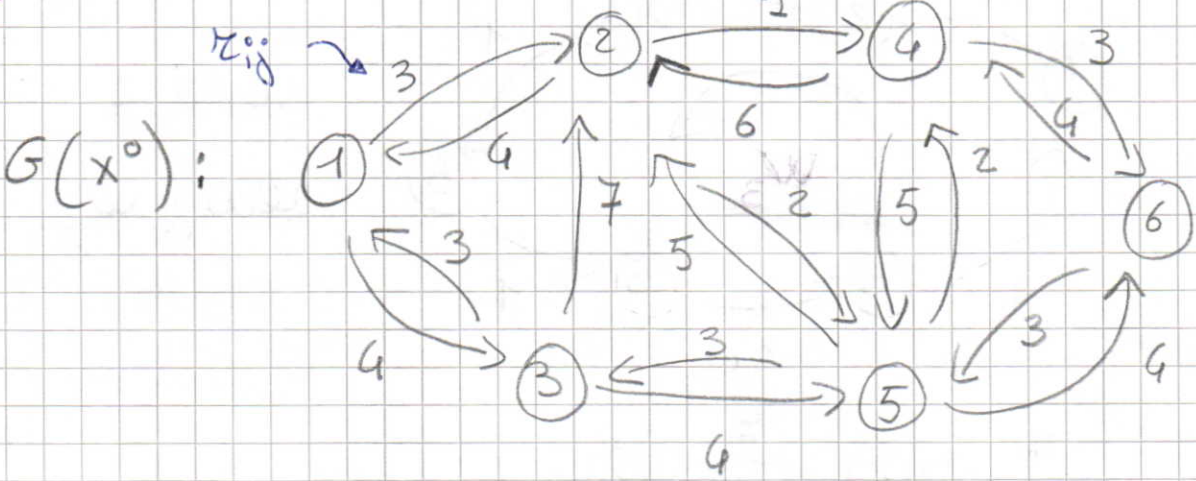
example (cont)



$u_{ij} = 7$

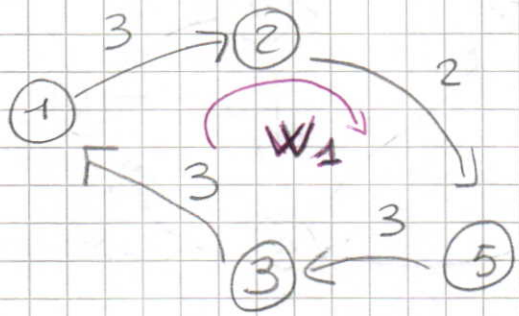
$\forall (i,j) \in A$

X



In fact:

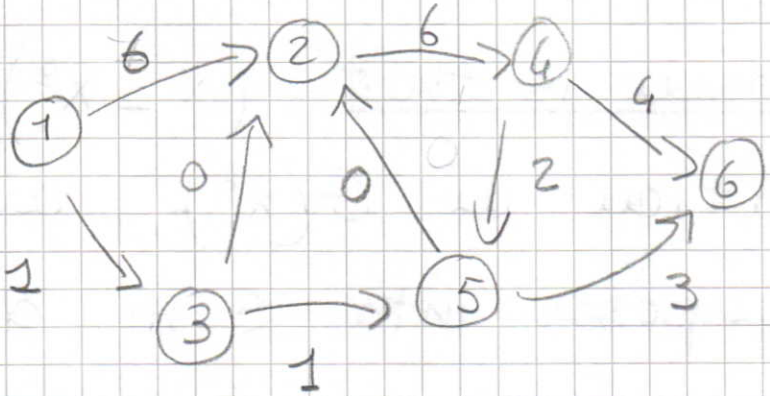
start from x°



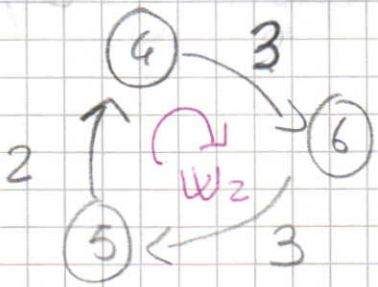
$\delta_{25} = -1$

send 2 along w_1

< intermediate flow >

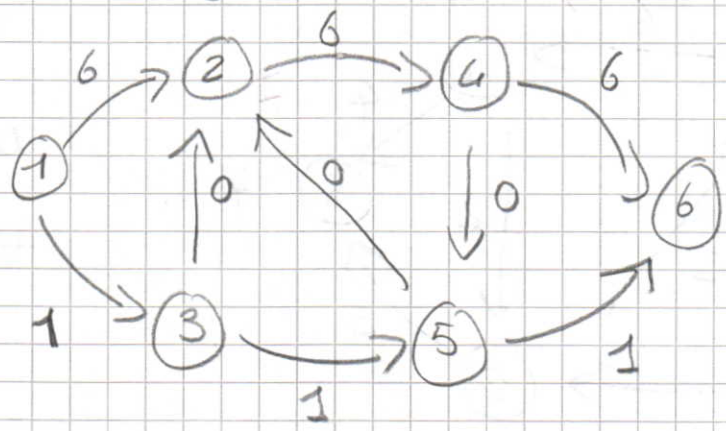


then

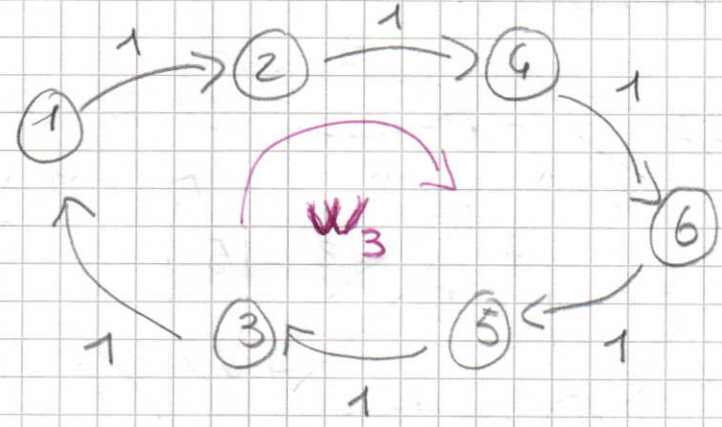


send 2 along w_2

< intermediate flow >

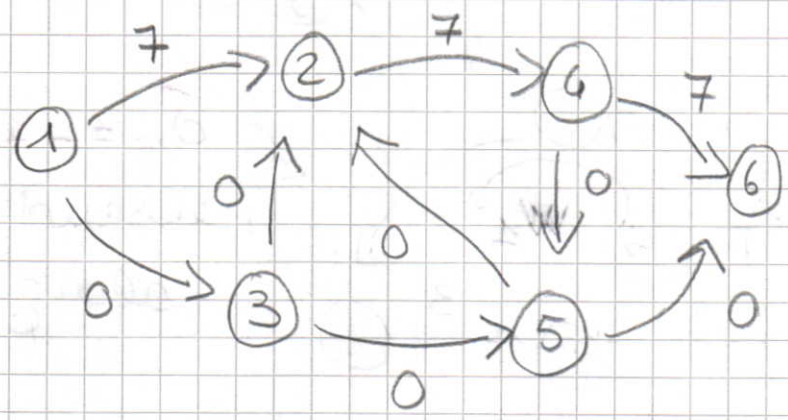


and finally



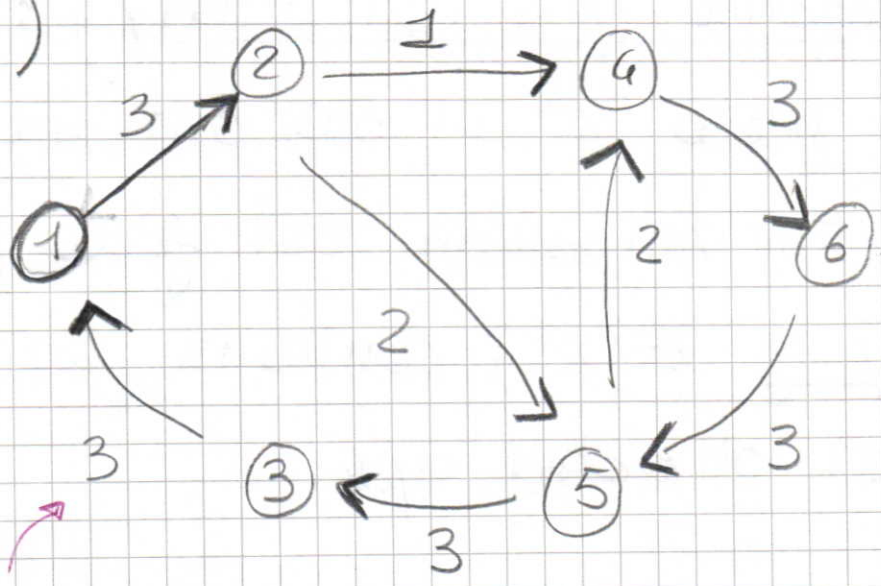
send 1 along w_3

x !!



<u>The reason is that</u> $(x - x^\circ)$ is a circulation in $G(x^\circ)$, and therefore it decomposes into $\leq m$ directed cycles ($\equiv$ augmenting cycles) in $G(x^\circ)$

$(x - x^0)$



circulation in $G(x^0)$

$x_{13} - x^0_{13} = -3$

(reverse arc in $G(x^0)$)

Therefore:

$$\sum_{(i,j)\in A} c_{ij} x_{ij} = \sum_{(i,j)\in A} c_{ij} x^0_{ij} + \sum_{(i,j)\in A} c_{ij} \delta_{ij}(w_1) f(w_1) +$$

$$+ \dots + \sum_{(i,j)\in A} c_{ij} \delta_{ij}(w_k) f(w_k) =$$

$$= \sum_{(i,j)\in A} c_{ij} x^0_{ij} + c(w_1) f(w_1) + \dots + c(w_k) f(w_k)$$

We can thus derive the following optimality conditions for the minimum cost flow problem:

Theorem (Negative Cycle Optimality):
a feasible flow x^* is an optimal solution for the minimum cost flow problem if and only if $G(x^*)$ contains no negative cost directed cycle.

Algorithms for minimum cost flow

(Ahuja - Magnanti - Orlin: Chapter 9 (9.1, 9.3 ("Negative Cycle Optimality Conditions" and "Reduced cost optimality conditions"), 9.6, 9.7)

Notation: $z(x) = \sum_{(i,j)\in A} c_{ij} x_{ij}$ cost of flow x

Assumptions:
- integral data
- $\sum_{i\in N} b(i) = 0$ and feasibility
- $c_{ij} \geq 0 \quad \forall (i,j) \in A$

Optimality Conditions

(1) Negative Cycle Optimality Conditions

< already introduced >

Proof : to be studied

(2) Reduced Cost Optimality Conditions

Let us associate $\pi(i) \in \mathbb{R}$ with each node i

potential of i

Given a potential vector π, set:

$$c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j)$$ reduced cost of (i,j)

N.B.: reduced costs are defined also for the arcs in a residual network (using the residual cost instead of c_{ij})

↑ mistake in AMO

We already proved:

Property:

(a) for any directed path P from k to ℓ, $\sum_{(i,j)\in P} c_{ij}^{\pi} = \sum_{(i,j)\in P} c_{ij} - \pi(k) + \pi(\ell)$

(b) for any directed cycle W, $\sum_{(i,j)\in W} c_{ij}^{\pi} = \sum_{(i,j)\in W} c_{ij}$.

Theorem (Reduced cost Optimality conditions): a feasible flow x^* is an optimal solution to the minimum cost flow problem if and only if there exists a set of node potentials π s.t.

$$c_{ij}^{\pi} \geq 0 \quad \forall (i,j) \in G(x^*).$$

Proof

$\Leftarrow$: assume x^* s.t. $c_{ij}^{\pi} \geq 0 \quad \forall (i,j) \in G(x^*)$.

Therefore $\sum_{(i,j)\in W} c_{ij}^{\pi} \geq 0 \quad \forall W$ directed cycle in $G(x^*)$

costs in $G(x^*)$ ("augmenting cycle")

From Property (b), $\sum_{(i,j)\in W} c_{ij} \geq 0 \quad \forall W$ cycle in $G(x^*)$

Therefore, $G(x^*)$ contains no negative cost directed cycle. From the "Negative Cycle Optimality Conditions" x^* is an optimal flow!

$\Rightarrow$: Let x^* optimal. Then $G(x^*)$ contains no negative cost directed cycle.

Compute a shortest path tree of root 1 in $G(x^*)$ (well-defined due to), and $d(i)$ denote the shortest path label of node i.

From Bellman's optimality conditions

$$d(j) \leq d(i) + c_{ij} \quad \forall (i,j) \in G(x^*)$$

(c_{ij}: cost in $G(x^*)$)

$$\Updownarrow$$

$$c_{ij} - (\underbrace{-d(i)}_{\pi(i)} + \underbrace{(-d(j))}_{\pi(j)}) \geq 0$$

So $c_{ij}^{\pi} \geq 0 \quad \forall (i,j) \in G(x^*)$ if we set $\pi(i) = -d(i)$

In other words: x^* satisfies the reduced

cost optimality conditions. □

Economic interpretation of reduced cost:

if. c_{ij} : cost of transporting 1 unit of commodity from i to j

. $u(i) = -\pi(i)$ cost of obtaining 1 unit of commodity at i

$$c_{ij} - \pi(i) + \pi(j) \geq 0$$

$$\Updownarrow$$

$$c_{ij} + u(i) - u(j) \geq 0$$

$$\Updownarrow$$

$$u(j) \leq c_{ij} + u(i)$$

The cost of obtaining 1 unit of commodity at node j must be no more than the cost of obtaining the unit at i plus the cost of sending it from i to j.

Cycle-canceling algorithm

(Ahuja-Magnanti-Orlin: 9.6 (until page 319, "Augmenting flow..." excluded)

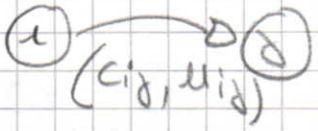
Given a feasible flow x (e.g. via a maximum flow algorithm), at each iteration find a negative augmenting cycle W in $G(x)$, and push $\delta = \min\{x_{ij} : (i,j) \in W\}$ along W, until no negative augmenting cycle exists (minimum cost flow)

- See Figure 9.7, page 317

How to find a negative cycle in $G(x)$?
via Bellman-Ford's shortest path algorithm (← course RO): $O(m \cdot n)$

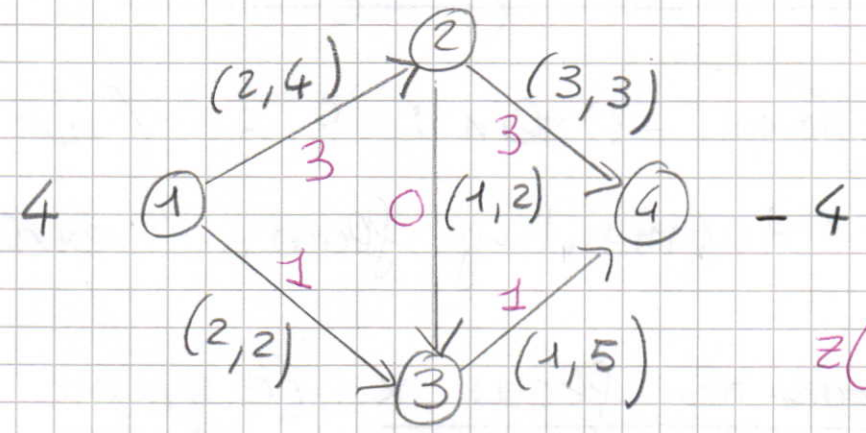
Obs: a feasible flow at each iteration

Example



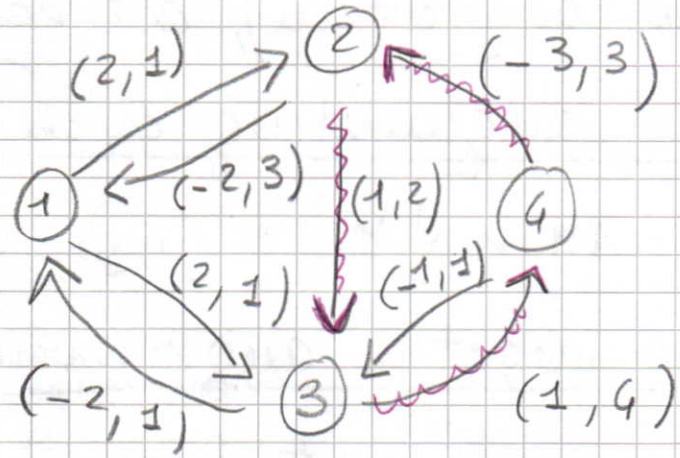
1)

$\{x_{ij}\}$

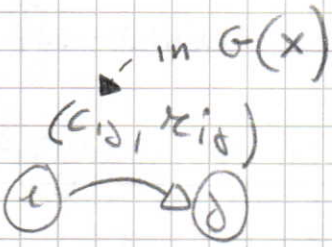


$z(x) = cx = 18$

$G(x)$



in $G(x)$



A negative augmenting cycle:

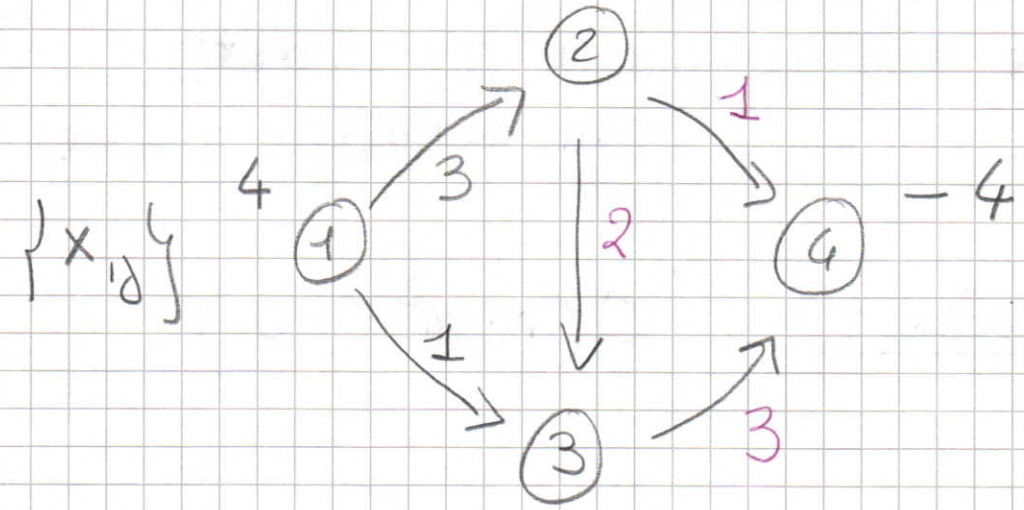
$W = (2, 3, 4)$

$c(W) = -3 + 1 + 1 = -1$

$\delta = \min\{3, 2, 4\} = 2$

< Augment δ along W >

2)



$$z(x) = cx = 18 + (-1)\cdot(2) = 16$$
$$\qquad\qquad\quad c(w) \quad \delta$$

⋮

As a by-product:

<u>Theorem (integrality)</u>: if arc capacities and supplies/demands are integer, then there exists an integer minimum cost flow.

<u>Proof</u>

- maximum flow algorithm finds an integer initial flow x;
- at each iteration $x_{ij} \in \mathbb{Z}^+$, so δ is integer.

□

Let C be the maximum arc cost. (in absolute value)

Then:

Time complexity : $O(n \cdot m^2 C \cdot U)$

Proof

1) • mCU : upper bound on the initial flow cost

$(c_{ij} \leq C \;,\; x_{ij} \leq U \quad \forall (i,j))$

• $-mCU$: lower bound on the optimal flow cost

$(c_{ij} \geq -C \;,\; x_{ij} \leq U \quad \forall (i,j))$

due to integrality, at each iteration $z(x)$ decreases by an integer ≥ 1

$\Rightarrow$ $O(mCU)$ iterations

2) cost per iteration : $O(m \cdot n)$

1) & 2) $\Rightarrow$ $O(n \cdot m^2 \cdot C \cdot U)$ □

Successive shortest path algorithm

(Ahuja-Magnanti-Orlin: 9.7 (until page 323, "The successive shortest path ... " excluded))

In contrast to cycle-canceling algorithm, the successive shortest path algorithm maintains the reduced cost optimiality, and strives to attain feasibility >

In fact, it maintains "pseudoflows":

Def: a pseudoflow $x: A \to \mathbb{R}^+ \cup \{0\}$ satisfies the capacity constraints; it may not satisfy the flow conservation constraints.

NB: a flow is a special pseudoflow

Then

$$e(i) = b(i) + \sum_{(j,i) \in BS(i)} x_{ji} - \sum_{(i,j) \in FS(i)} x_{ij} \quad \forall i \in N$$

imbalance of i

If:

$e(i) > 0$ — excess node (must send flow)

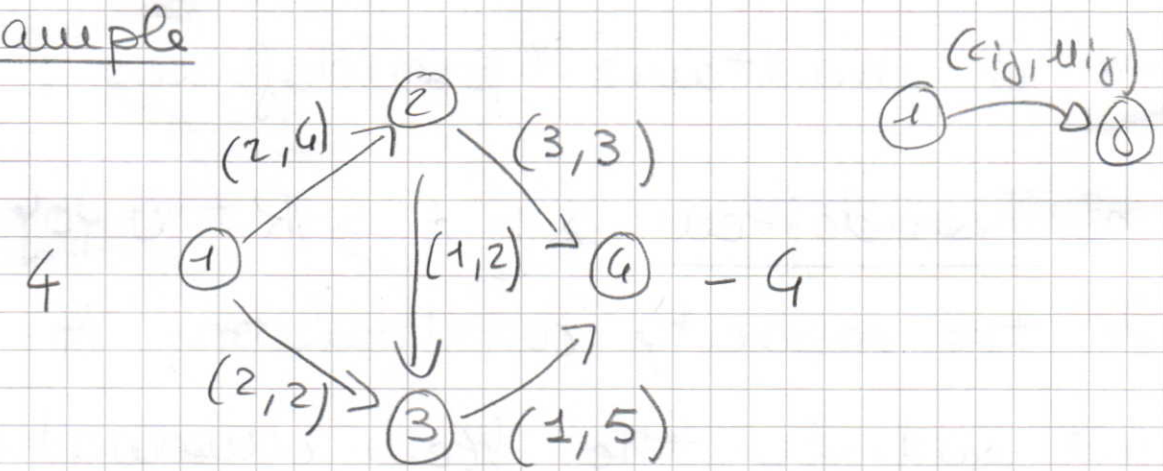
$e(i) < 0$ — deficit node (must receive flow)

$e(i) = 0$ — balanced

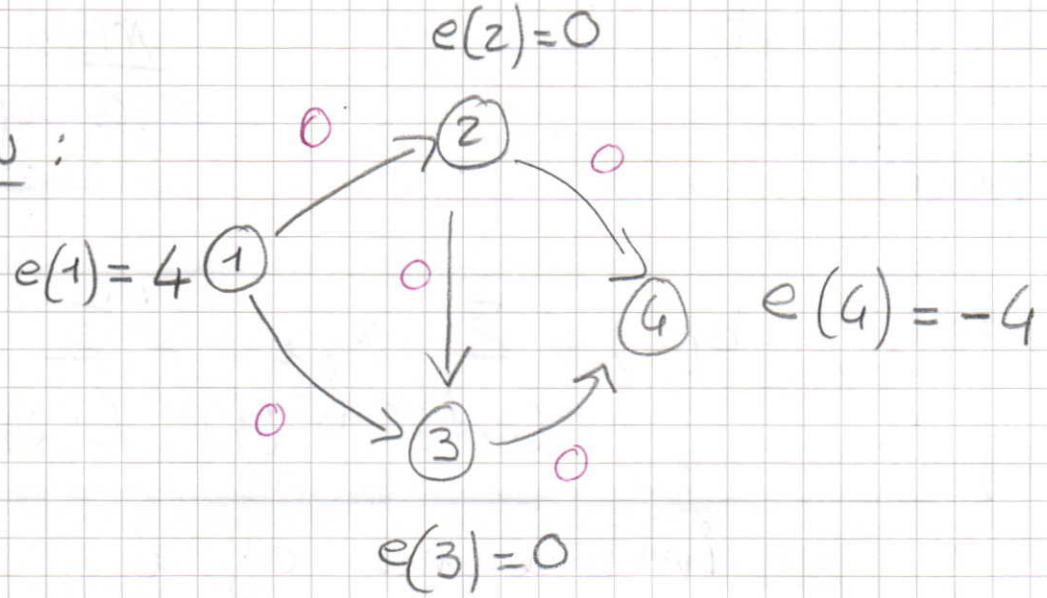
If E set of excess nodes and D set of deficit nodes, then:

$$\sum_{i \in N} e(i) = \sum_{i \in N} b(i) = 0 \Rightarrow \sum_{i \in E} e(i) = -\sum_{i \in D} e(i)$$

example



a pseudoflow:



$E = \{1\}$

$D = \{4\}$

Define the residual network $G(x)$ as before.

Lemma : Suppose a pseudoflow x satisfies the reduced cost optimality w.r.t. some potentials $\{\pi(i)\}$. Let $d(i)$ the shortest path distance from some node s to i in $G(x)$ w.r.t. c_{ij}^{π}. Then:

(a) x also satisfies the reduced cost optimality w.r.t. $\{\pi'(i)\}$, with $\pi'(i) = \pi(i) - d(i) \quad \forall i \in N$ (i.e. $c_{ij}^{\pi'} \geq 0 \quad \forall (i,j)$)

(b) $c_{ij}^{\pi'} = 0$ if (i,j) belongs to a shortest path from s to some other node, (w.r.t. c_{ij}^{π})

Proof :

(a) $\cdot\ c_{ij}^{\pi} \geq 0 \quad \forall (i,j) \in G(x)$ (by hyp.)

$\Rightarrow \cdot\ d(j) \leq d(i) + c_{ij}^{\pi} \quad \forall (i,j) \in G(x)$

$(d(s) = 0)$ shortest path optimality conditions

Remember that:

$c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j)$; by substituting in $\Rightarrow$ we get:

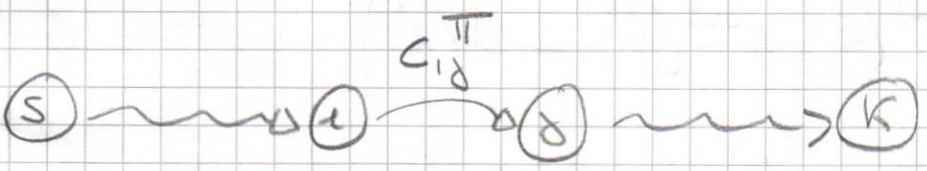
$$d(j) \leq d(i) + c_{ij} - \pi(i) + \pi(j) \quad \forall (i,j) \in G(x)$$

$$c_{ij} - (\underbrace{\pi(i) - d(i)}_{\pi'(i)}) + (\underbrace{\pi(j) - d(j)}_{\pi'(j)}) \geq 0 \quad " \; "$$

i.e.

i.e. $c_{ij}^{\pi'} \geq 0 \quad \forall (i,j) \in G(x)$.

(b) Consider a shortest path from s to some node k:



it is $d(j) = d(i) + c_{ij}^{\pi} \quad \forall (i,j) \in P$

Since $c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j)$, we get

$$c_{ij} - (\underbrace{\pi(i) - d(i)}_{\pi'(i)}) + (\underbrace{\pi(j) - d(j)}_{\pi'(j)}) = 0$$

$$c_{ij}^{\pi'} = 0$$

□

Lemma: Suppose a pseudoflow x satisfies the reduced cost optimality conditions. If x' is obtained from x by sending flow along a shortest path (w.r.t. c_{ij}^{π}) from some node s to some node k, then x' also satisfies the reduced cost optimality conditions.

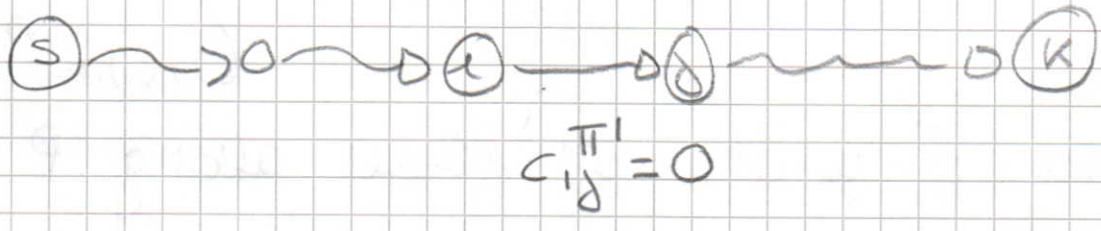
Proof: Let $\{\pi(i)\}$ and $\{\pi'(i)\}$ as in the previous Lemma.

x: $c_{ij}^{\pi'} \geq 0 \quad \forall (i,j) \in G(x)$

$c_{ij}^{\pi'} = 0 \quad \forall (i,j)$ in the shortest path tree rooted at s

} previous Lemma

Send δ to obtain x':



The augmentation may add (j,i) to $G(x')$: $c_{ji}^{\pi'} = 0$, so also this arc satisfies the reduced cost optimality! (w.r.t. $\{\pi'(i)\}$) □

Successive shortest path

$x := 0$, $\pi := 0$, (NB: $c^{\pi}_{i,\gamma} \geq 0 \ \forall (i,\gamma)$)

$e(i) := b(i) \ \forall i \in N$;

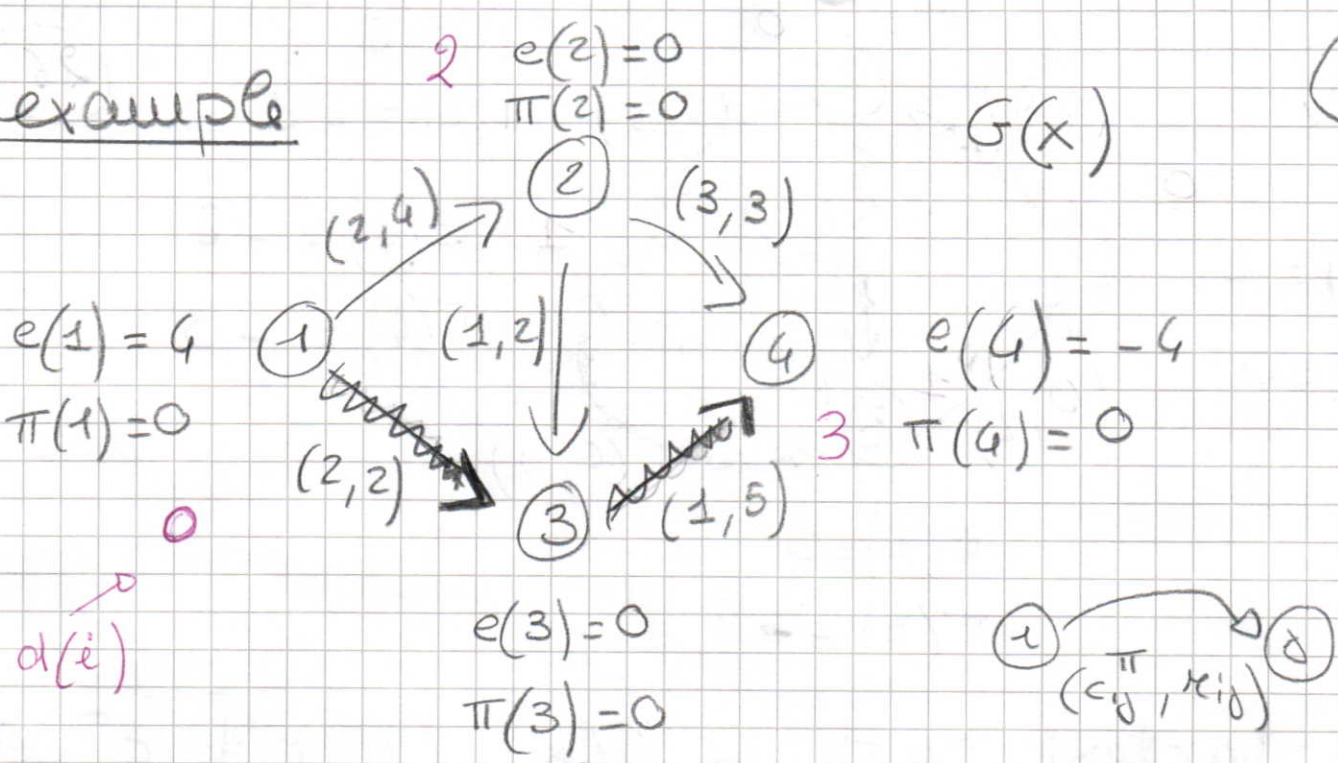
$E = \{ i : e(i) > 0 \}$; $D = \{ i : e(i) < 0 \}$;

<u>while</u> $E \neq \emptyset$ <u>do</u>

- select $s \in E$ and $k \in D$;
- find the shortest path tree of root s in $G(x)$ w.r.t. $\{ c^{\pi}_{i,\gamma} \}$; ← NB: $c^{\pi}_{i,\gamma} \geq 0$
- let $d(i)$ be the shortest distance of i, $\forall \in N$, and P the shortest path from s to k;
- $\pi(i) := \pi(i) - d(i)$, $\forall i \in N$; ← $\pi'(i)$ in the Lemmas
- $\delta = \min \{ e(s), -e(k), \min \{ r_{i,\gamma} : (i,\gamma) \in P \} \}$;
- send δ units of flow along P;
- update x, $G(x)$, E, D and the reduced costs (now $\{ c^{\pi'}_{i,\gamma} \}$)

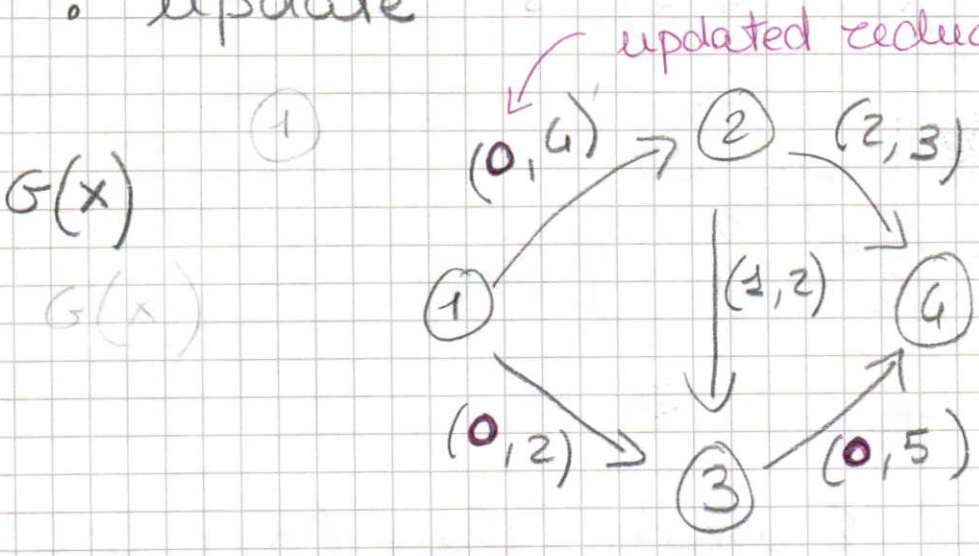
<u>end</u>

example



1). shortest path in $G(x)$ (w.r.t. $c^{\pi}_{ij} \geq 0$) from $1 \in E$ to $4 \in D$: $(1, 3, 4)$

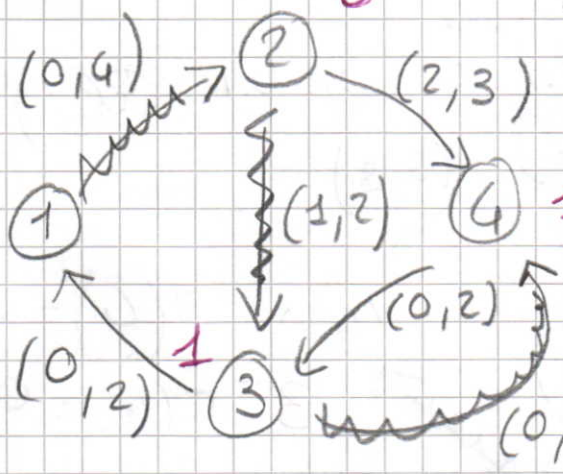
- $\pi(i) := \pi(i) - d(i) \quad \forall i \quad (0, -2, -2, -3)$
- $\delta = \min\{4, 4, \min\{2, 5\}\} = 2$
- update



+

sending of 2 units along $(1, 3, 4)$

$G(x)$



$e(1) = 2$

$\pi(1) = 0$

$e(2) = 0$

$\pi(2) = -2$

$e(4) = -2$

$\pi(4) = -3$

$e(3) = 0$

$\pi(3) = -2$

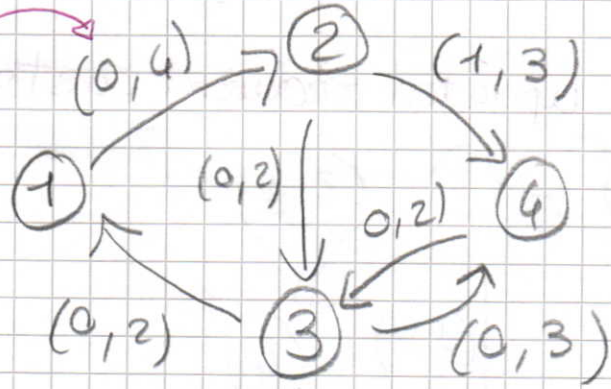
2) shortest path in $G(x)$ ($w, r, t, c_{ij}^{\pi} \geq 0$) from $1 \in E$ to $4 \in D$: $(1\ 2\ 3\ 4)$

- $\pi(i) := \pi(i) - d(i) \quad \forall i$

  $(0, -2, -3, -4)$

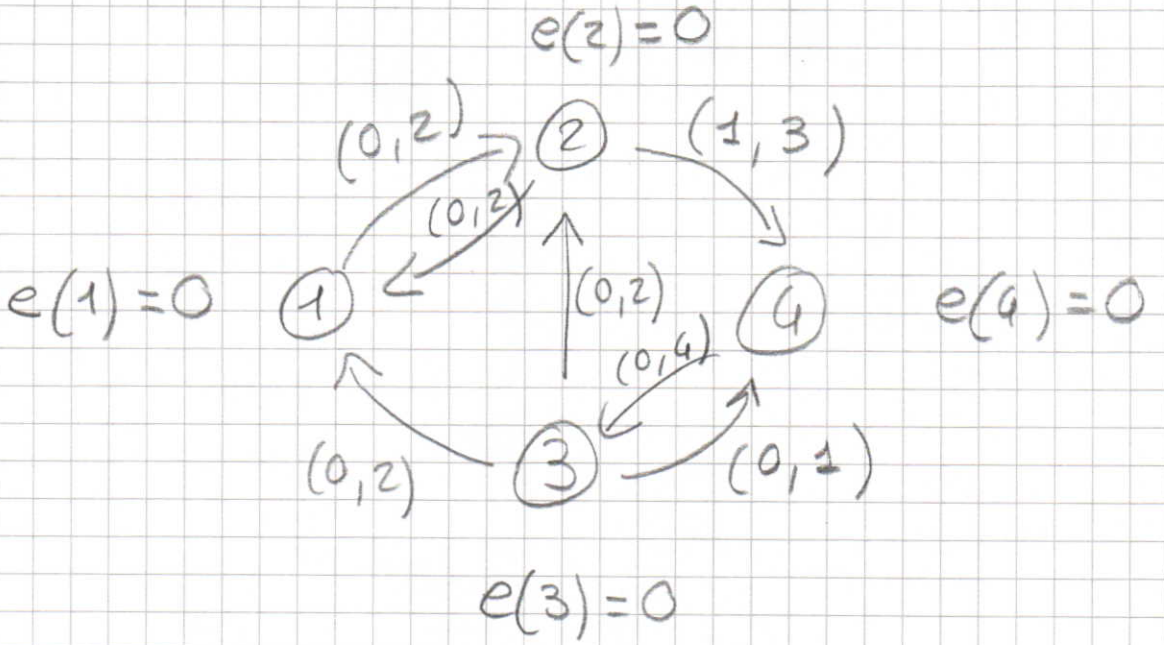
- $\delta = \min\{2, 2, \min\{4, 2, 3\}\} = 2$

updated reduced costs

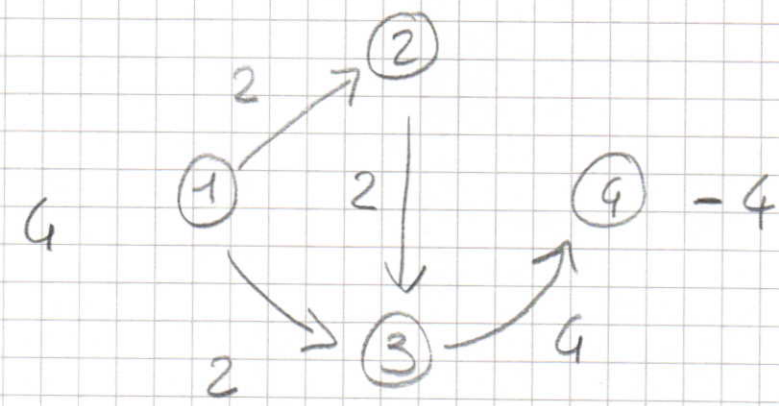


+

sending of 2 units along $(1\ 2\ 3\ 4)$



STOP : The pseudoflow is a feasible (and optimal) flow



Time Complexity : $O(nU \cdot S(n,m))$

Proof

- $\leq nU$ iterations
  (the excess of some nodes strictly decreases at each iteration)
- $S(n,m)$ time to compute a shortest path tree
  (e.g. $O(n^2)$ ← Dijkstra)