Flow decomposition

(Ahuja - Magnanti - Orlin: Chapter 3 (3.5))

Observation: any flow $x$ can be defined in terms of flows on arcs (as previously formulated) or in terms of flows on paths and cycles.

Example

4 units along the path $(1, 2, 4, 6) = P_1$

3 units along the path $(1, 3, 5, 6) = P_2$

2 units along the cycle $(2, 4, 5) = C$

$X = (x_{ij})$
Let \( \mathcal{P} \) : set of all paths
\[ f(P) : \text{flow on } P \in \mathcal{P} \]
\( W \) : set of all cycles
\[ f(w) : \text{flow on } w \in W \]
\[ \delta_{i,j}^0(P) = \begin{cases} 1 & \text{if } (i,j) \in E(P) \\ 0 & \text{otherwise} \end{cases} \]

Then: any flow representation \( \mathbf{x} \) in terms of path and cycle flows determines arc flows uniquely:

\[
\mathbf{x}_{i,j} = \sum_{P \in \mathcal{P}} \delta_{i,j}(P) f(P) + \sum_{w \in W} \delta_{i,j}^0(w) f(w) \quad \forall (i,j) \in \mathcal{A}
\]

Vice versa: any flow representation \( \mathbf{x} \) in terms of arc flows "decomposes" into path and cycle flow (not uniquely).

\[ \Rightarrow \text{ Flow decomposition Theorem: each path and cycle flow has a unique representation in terms of (nonnegative) arc flows. Conversely, each arc} \]
flow \( x \) can be represented as path and cycle flow (not uniquely) s.t.

a) each directed path with a positive flow connects a source to a destination

b) at most \( n + m \) paths and cycles have a positive flow; out of these, at most \( m \) cycles have positive flow

Proof (intuition) (example cont.):

select a source node \((1)\):

![Diagram of a network showing flow](image)

\[ \min \{ \bar{e}(P_1), b(1), -b(4) \} = 4 \]

substract 4 units of flow along \( P_1 \):

\[ \bar{f}(P_1) = 4 \]

"remaining flow" 3

"updated balance" 3

at least one flow arc goes to 0
select a source node \((1)\):

\[
3 \quad 1 \quad 3 \rightarrow 3 \rightarrow 5 \rightarrow 6 \quad -3 \quad \min \{x(P_2), x(P_1), -b(1), -b(6)\} = 3
\]

subtract 3 units of flow along \(P_2\):

\[
\ell(P_2) = 3
\]

remaining cycle \(w\):

\[
g(w) = 2
\]

\[
\Rightarrow g(P_1) \quad g(P_2) \quad g(w)
\]

is a decomposition of \(x\) at most \(m\) flows are 0 and all imbalance goes to 0.
A circulation is a flow s.t. 
\[ b(i) = 0 \quad \forall i \in \mathbb{N} \]

**Corollary:** a circulation \( x \) can be represented in terms of cycle flow along \( \leq m \) directed cycles.

**Important consequences of the flow decomposition Theorem**

Let \( x \) be a flow and \( G(x) \) its residual network.

**Def:** an **augmenting cycle** \( w \) w.r.t. \( x \) is a directed cycle \( w \in G(x) \); its cost \( c(w) = \sum c_{i\delta} = \sum c_{i\delta} \delta_{i\delta}(w) \) is the change of the cost of \( x \) if we push 1 unit of flow along \( w \).
Consider the minimum cost flow problem on $G = (V, A)$, and let $x$ and $x^0$ be any two feasible solutions; how can we compare $cx$ and $cx^0$?

Indeed, the flow decomposition theorem can be extended so as to establish a relationship between $x$ and $x^0$:

Flow decomposition theorem \((\text{ext+1})\): given a feasible solution $x^0$ to the minimum cost flow problem, any other feasible solution $x$ can be obtained from $x^0$ by sending flow along $\leq m$ augmenting cycles $\omega$, viz., $x^0$:

$$x_i = x^0_i + \sum_{\omega \in \Omega} \delta_{i,0} (\omega) s(\omega) + \ldots + \delta_{i,0} (\omega_k) s(\omega_k) \quad \forall (i, 0) \in A$$

\[x \leq m\]

Now: $\delta_{i,0} (\omega_k) =$

$$\begin{cases} +1 & \text{if } (i, 0) \text{ is forward} \\ -1 & \text{if } (i, 0) \text{ is backward} \\ 0 & \text{if } (i, 0) \notin \omega_k \end{cases}$$

$k = 1, \ldots, \omega_k$
example (cont)

\[ \mu_{18} = 7 \]
\[ \forall (i, j) \in A \]

\( G(x^0) \):

In fact:

start from \( x^0 \)

< intermediate flow >

them

send 2 along \( W_1 \)

send 2 along \( W_2 \)
The reason is that \((x - x^0)\) is a circulation in \(G(x^0)\), and therefore it decomposes into \(\leq m\) directed cycles (\(=\) augmenting cycles) in \(G(x^0)\).
\[(X - x^0)\]

Diagram:

1 -> 2 -> 4

2 -> 3 -> 5

3 -> 6

\[x_{13} - x^0_{13} = -3\]

(reverse arc \(\mu \in \mathcal{G}(x^0)\))

\[
\begin{align*}
\sum_{(i, j) \in A} c_{ij} x_{ij} &= \sum_{(i, j) \in A} c_{ij} x_{ij}^0 + \sum_{(i, j) \in A} c_{ij} \delta_{ij}(w_1) g(w_1) + \\
&+ \ldots + \sum_{(i, j) \in A} c_{ij} \delta_{ij}(w_{\kappa}) g(w_{\kappa}) = \\
&= \sum_{(i, j) \in A} c_{ij} x_{ij}^0 + c(w_1) g(w_1) + \ldots + c(w_{\kappa}) g(w_{\kappa})
\end{align*}
\]

We can thus derive the following optimality conditions for the minimum cost flow problem:
Theorem (Negative Cycle Optimality): a feasible flow $x^*$ is an optimal solution for the minimum cost flow problem if and only if $G(x^*)$ contains no negative cost directed cycle.

Algorithms for minimum cost flow (Ahuya - Magnanti - Orlin: Chapter 9)

(9.1, 9.3 ("Negative Cycle Optimality Conditions" and "Reduced Cost Optimality Conditions"), 9.6, 9.7)

Notation: $z(x) = \sum_{(i,j) \in A} c_{ij} x_{ij}$ cost of flow

Assumptions: integral data

- $\sum_{i \in \mathbb{N}} b(i) = 0$ and feasibility
- $c_{ij} \geq 0 \quad \forall (i,j) \in A$
Optimality Conditions

1. Negative Cycle Optimality Conditions
   < already introduced >
   Proof: to be studied

2. Reduced Cost Optimality Conditions

   Let us associate \( \pi(i) \in \mathbb{R} \) with each node \( i \)
   potential of \( i \).

   Given a potential vector \( \pi \), let:
   \[
   c_{ij}^\pi = c_{ij} - \pi(i) + \pi(j)
   \]
   reduced cost of \((i, j)\)

   N.B.: reduced costs are defined also for the
   arcs in a residual network (using the
   residual cost instead of \( c_{ij} \))
We already proved:

Property:

(a) for any directed path \( P \) from \( k \) to \( e \),
\[
\sum_{(i,j) \in P} c_{ij} = \sum_{(i,j) \in P} c_{ij} - \pi(k) + \pi(e)
\]

(b) for any directed cycle \( W \),
\[
\sum_{(i,j) \in W} c_{ij} = \sum_{(i,j) \in W} c_{ij}.
\]

**Theorem (Reduced Cost Optimality Conditions):**
a feasible flow \( x^* \) is an optimal solution to the minimum cost flow problem if and only if there exists a set of node potentials \( \pi \) s.t.
\[
c_{ij} \geq 0 \quad \forall (i,j) \in G(x^*).
\]

**Proof**

\( \leftarrow \): assume \( x^* \) s.t. \( c_{ij} \geq 0 \quad \forall (i,j) \in G(x^*). \)

Therefore
\[
\sum_{(i,j) \in W} c_{ij} \geq 0 \quad \forall W \text{ directed cycle in } G(x^*).\]

Costs \( c_{ij} \) are less than or equal to \( g(x^*) \) (augmenting cycle).

From Property (b),
\[
\sum_{(i,j) \in W} c_{ij} \geq 0 \quad \forall W \text{ cycle in } G(x^*).\]
Therefore, $G(x^*)$ contains no negative cost directed cycle. From the "Negative Cycle Optimality Conditions" $x^*$ is an optimal flow.

Let $x^*$ optimal. Then $G(x^*)$ contains no negative cost directed cycle. Compute a shortest path tree of cost 1 in $G(x^*)$ (well-defined due to ), and denote the shortest path label of node $i$.

From Bellman's optimality condition:

$$d(\delta) \leq d(i) + c_{i\delta} \quad \forall (i, \delta) \in G(x^*)$$

Cost in $G(x^*)$

$$c_{i\delta} = (-d(i) + (-d(\delta))) \geq 0$$

$$\pi(i) \quad \pi(\delta)$$

So, $c_{i\delta} \geq 0 \quad \forall (i, \delta) \in G(x^*)$ if we set $\pi(i) = -d(i)$

In other words, $x^*$ satisfies the reduced
Economic interpretation of reduced cost:

If \( c_{ij} \) is the cost of transporting 1 unit of commodity from \( i \) to \( j \), then:

\[
\mu(i) = -\pi(i) \quad \text{cost of obtaining 1 unit of commodity at } i
\]

\[
c_{ij} - \pi(i) + \pi(j) \geq 0
\]

\[
c_{ij} + \mu(i) - \mu(j) \geq 0
\]

\[
\mu(j) \leq c_{ij} + \mu(i)
\]

The cost of obtaining 1 unit of commodity at node \( j \) must be no more than the cost of obtaining the unit at \( i \) plus the cost of sending it from \( i \) to \( j \).
Cycle-canceling algorithm

(Ahuja - Magnanti - Orlin: 9.6 (until page 349) "Augmenting flow..." excluded)

Given a feasible flow \( \mathbf{x} \) (e.g., via a maximum flow algorithm), at each iteration find a negative augmenting cycle \( \mathbf{w} \) in \( G(\mathbf{x}) \), and push \( \delta = \min \{ x_{ij} : (i,j) \in \mathbb{W} \} \) along \( \mathbf{w} \), until no negative augmenting cycle exists (minimum cost flow).

See Figure 9.7, page 347

How to find a negative cycle \( \mathbf{w} \) in \( G(\mathbf{x}) \)?

via Bellman-Ford's shortest path algorithm (course RO): \( O(mn) \)

Obs: a feasible flow at each iteration
Example

\[ \begin{align*}
G(x) &= G(x) \\
Z(x) &= c_x = 18
\end{align*} \]

A negative augmenting cycle:

\[ W = (2, 3, 4) \]

\[ c(W) = -3 + 1 + 1 = -1 \]

\[ \delta = \min \{ 3, 2, 4 \} = 2 \]

< Augment \( \delta \) along \( W \) >
2) \[ x \in \mathcal{A} \]

\[ z(x) = c x = 18 + (-1) \cdot (2) = 16 \]

\[ c(w) \leq \delta \]

As a by-product:

**Theorem (Integrality):** If arc capacities and supplies/demands are integer, then there exists an integer minimum cost flow.

**Proof**

- Maximum flow algorithm finds an integer initial flow \( x \);
- at each iteration \( x_{ij} \in \mathbb{Z}^+ \), so \( \delta \) is integer. \( \square \)
Let $c$ be the maximum arc cost. (in absolute value)

Then:

Time complexity: $O(n \cdot m^2 \cdot c \cdot U)$

Proof

1). $m < U$: upper bound on the initial flow cost

$$(c_{ij} \leq C, x_{ij} \leq U \quad \forall (i,j))$$

2) $m \geq U$: lower bound on the optimal flow cost

$$(c_{ij} \geq -C, x_{ij} \leq U \quad \forall (i,j))$$

due to integrality, at each iteration $z(x)$ decreases by an integer $\geq 1$.

$\Rightarrow O(m \cdot C \cdot U)$ iterations

2) cost per iteration: $O(m \cdot n)$

1) & 2) $\Rightarrow O(n \cdot m^2 \cdot c \cdot U)$
Successive shortest path algorithm

(Ahuja - Magnanti - Orlin: 9.7 (until page 323, "The successive shortest path ... " excluded))

In contrast to cycle-canceling algorithm, the successive shortest path algorithm maintains the reduced cost optimality, and strives to attain feasibility.

In fact, it maintains "pseudoflow":

**Def:** a pseudoflow \( x: A \rightarrow \mathbb{R}^+ \cup \{0\} \) satisfies the capacity constraints; it may not satisfy the flow conservation constraints. *NB:* a flow is a special pseudoflow.

Then:

\[
e(i) = b(i) + \sum x_{ij} - \sum x_{ji} \quad \forall i \in V
\]

\[
(\forall (j,i) \in BS(i), \ (i,d) \in FS(i))
\]

imbalance of \( i \)
If:

\[ e(i^*) > 0 \quad \text{excess mode (must send flow)} \]
\[ e(i^*) < 0 \quad \text{deficit mode (must receive flow)} \]
\[ e(i) = 0 \quad \text{balanced} \]

If \( E \) set of excess modes and \( D \) set of deficit modes, then:

\[ \sum_{i \in E} e(i) = \sum_{i \in N} b(i) = 0 = \sum_{i \in D} e(i) = -\sum_{i \in N} e(i) \]

**Example**

![Graph with numbered nodes and arrows representing flows](image)

\[ E = \{ 1, 4 \} \]
\[ D = \{ 3, 4 \} \]

A pseudo-flow:

![Pseudo-flow graph](image)
Lemma: Suppose a pseudoflow $x$ satisfies the reduced cost optimality w.r.t. some potentials $\{\Pi^{'}(i)\}$. Let $d(i')$ the shortest path distance from some node $s$ to $i'$ w.r.t. $G(x)$ w.r.t. $c_{ij}^{\Pi}$. Then:

(a) $x$ also satisfies the reduced cost optimality w.r.t. $\{\Pi^{'}(i)\}$, with $\sum_{(i',j)} c_{ij}^{\Pi} \geq 0$ for $i' \in V$,

$$\Pi^{'}(i) = \Pi(i) - d(i') \quad \forall i \in V$$

(b) $c_{ij}^{\Pi} = 0$ if $(i,j)$ belongs to a shortest path from $s$ to some other node,$(w.r.t. c_{ij}^{\Pi})$

Proof:

(a) $c_{ij}^{\Pi} \geq 0 \quad \forall (i,j) \in G(x)$ \ (by hyp.)

$$= 0 \cdot d(s) = d(i') + c_{ij}^{\Pi} \quad \forall (i,j) \in G(x)$$

$(d(s) = 0)$

shortest path optimality conditions
Remember that:

\[ c_{i'j'} = c_{ij} - \Pi(i') + \Pi(j') \]

By substituting in \( n = 0 \) we get:

\[ d(j) = d(i') + c_{i'j'} - \Pi(i') + \Pi(j') \quad \forall (i', j') \in E(x) \]

\[ c_{i'j'} = \left( \frac{\Pi(i') - d(i')}{\Pi'(i')} \right) + \left( \frac{\Pi(j') - d(j')}{\Pi'(j')} \right) \]

i.e., \( c_{i'j'} \geq 0 \quad \forall (i', j') \in E(x) \).

(b) Consider a shortest path from \( s \) to some node \( k \):

\[ P \quad s \rightsquigarrow i \rightsquigarrow j \rightsquigarrow k \]

It is \( d(j) = d(i') + c_{i'j'} \quad \forall (i', j') \in P \)

Since \( c_{i'j'} = c_{ij} - \Pi(i') + \Pi(j') \), we get

\[ c_{ij} - \left( \frac{\Pi(i') - d(i')}{\Pi'(i')} \right) + \left( \frac{\Pi(j') - d(j')}{\Pi'(j')} \right) = 0 \]

\[ \frac{\Pi'(i')}{\Pi'(j')} \]

\[ c_{ij} = 0 \]

\( \Box \)
Lemma: Suppose a pseudoflow $x$ satisfies the reduced cost optimality conditions. If $x'$ is obtained from $x$ by sending flow along a shortest path (w.r.t. $c_{ij}$) from some node $s$ to some node $k$, then $x'$ also satisfies the reduced cost optimality conditions.

Proof: Let $\vec{\pi}(i)$ and $\vec{\pi}'(i)$ as in the previous lemma.

\[
x: \quad c_{ij}^{\vec{\pi}} \geq 0 \quad \forall (i,j) \in G(x) \\
\quad c_{ij}^{\vec{\pi}'} = 0 \quad \forall (i,j) \text{ in the shortest path tree rooted at } s
\]

Send $\delta$ to obtain $x'$:

The augmentation may add $(g,i)$ to $G(x')$: $c_{ij}^{\vec{\pi}'} = 0$, so also this arc satisfies the reduced cost optimality.
Successive shortest path

\[ x_0 = 0, \quad \pi_0 = 0, \quad (\text{NB: } c_{l^0} \geq 0 \quad \forall (i, l^0)) \]

\[ e(i) = b(i) \quad \forall i \in N \]

\[ E = \{ i : e(i) > 0 \}, \quad D = \{ i : e(i) < 0 \} \]

while \( E \neq \emptyset \) do

- select \( s \in E \) and \( k \in D \)
- find the shortest path tree of root \( s \) \( \omega \in G(x) \) w.r.t. \( \Pi^0, \quad \sum_{i} c_{i^0}^j \) \hfill \text{(NB: } c_{i^0}^j \geq 0) \)
- let \( d(i) \) be the shortest distance of \( i, \forall i \in N \), and \( P \) the shortest path from \( s \) to \( k \)
- \( \Pi(i) = \Pi(i) - d(i), \quad \forall i \in N \)
- \( \delta = \min \{ e(s), -e(k), \min_{(i, j) \in \pi} c_{j^0}^i \} \)
- send \( \delta \) units of flow along \( P \)
- update \( x, G(x), E, D \) and the reduced costs \( (\text{now } \gamma c_{i^0}^j \) \)

end
Example:

\[ e(1) = 4 \quad \pi(1) = 0 \]

\[ e(2) = 0 \quad \pi(2) = 0 \]

\[ e(3) = 0 \quad \pi(3) = 0 \]

1. Shortest path in \( G(x) \) (w.r.t. \( c_{ij} \geq 0 \)) from \( i \in E \) to \( g \in D \): \((1, 3, 4)\)

\[ \pi(i) := \pi(i) - d(i) \quad \forall i \in (0, -2, -2, -3) \]

\[ \delta = \min \{ 4, 4, \min \{ 2, 5 \} \} = 2 \]

Update:

Updated reduced costs:

\[ G(x) \]

Sending of 2 units along \((1, 3, 4)\)
2) Shortest path in $G(x)$ \((w,z,t, c_{ij})\) from $1 \in E$ to $4 \in D$: \((1\ 2\ 3\ 4)\)

- $\pi(i) = \pi(i) - d(i) + i$
  - \((0, -2, -3, -4)\)

- $\delta = \min \{ 2, -2, \min \{ 4, 2, 3 \}\} = 2$

Updated reduced costs:
STEP: The pseudo-flow is a feasible (and optimal) flow.

Time Complexity: $O(nU + S(n,m))$

Proof:
- $\leq nU$ iterations (the excess of some nodes strictly decreases at each iteration)
- $S(n, m)$ time to compute a shortest path tree (e.g. $O(n^2)$ for Dijkstra)