

## The augmenting path algorithm

Def : given a flow  $x$ , an augmenting path is a directed path from  $s$  to  $t$  in  $G(x)$ ; its residual capacity is the minimum residual capacity of its arcs.

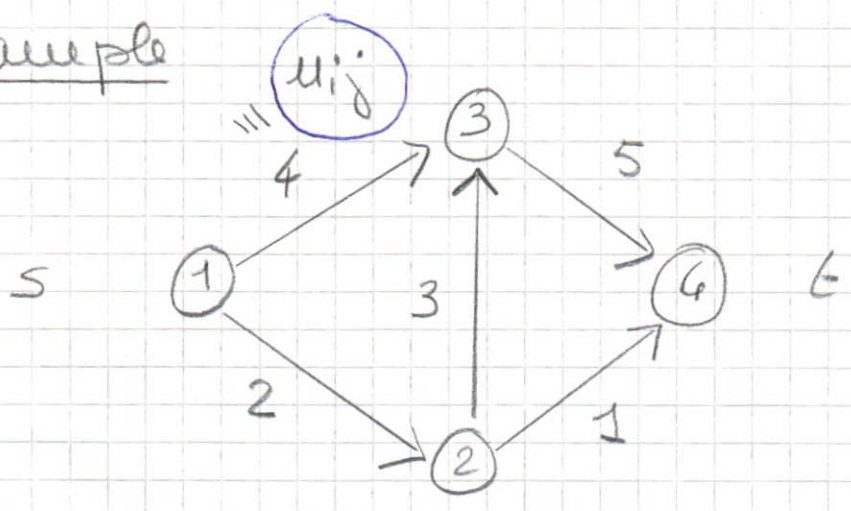
The algorithm : at each iteration finds an augmenting path and augments flow on this ( $\equiv$  residual capacity) until there is no augmenting path.

Theorem : a flow  $x^*$  is a maximum flow if and only if  $G(x^*)$  contains no augmenting path.

Proof : from the max-flow min-cut theorem (course RO)

Theorem (integrality): if the arc capacities are integer, then there exists an integer maximum flow.

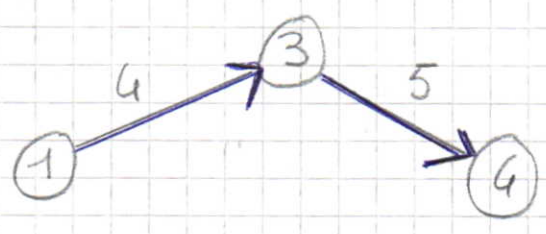
example



Initial flow  
 $x = 0$   
 $v = 0$

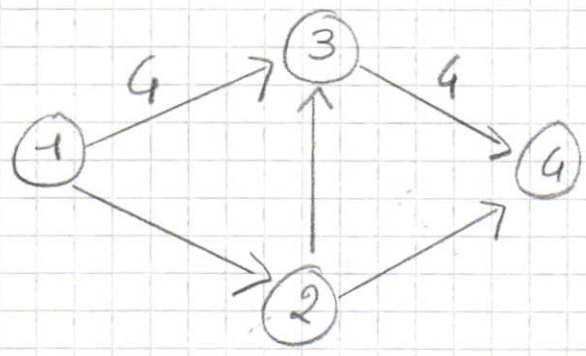
1)  $G(x) = G$

aug. path



$\delta = \min\{4, 5\} = 4$   
4 residual capacity

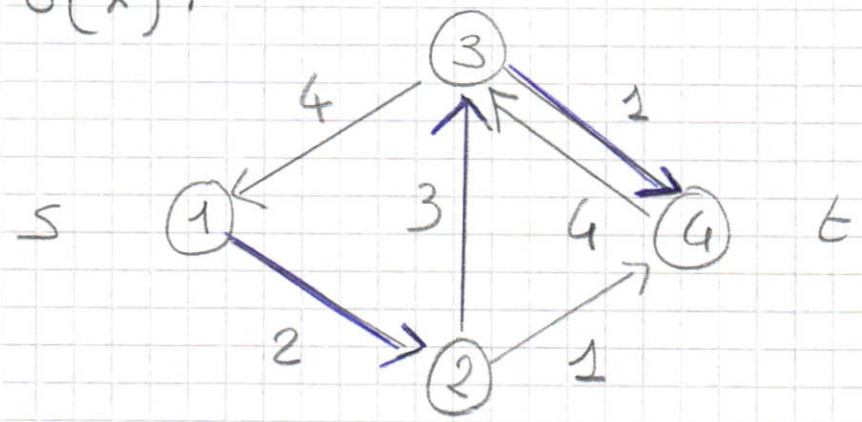
flow x



$v = 4$

NB: an integer flow at each iteration!

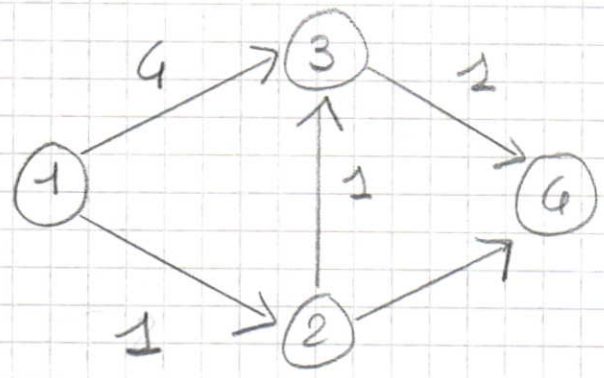
2)  $G(x)$ :



aug. path : (1, 2, 3, 4)

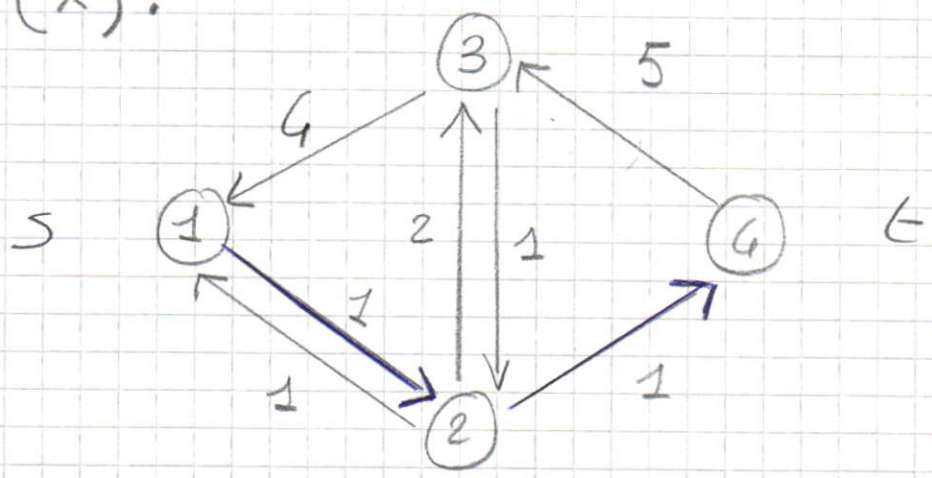
$$\delta = \min \{2, 3, 1\} = 1$$

flow x:



$$v = 4 + \delta = 5$$

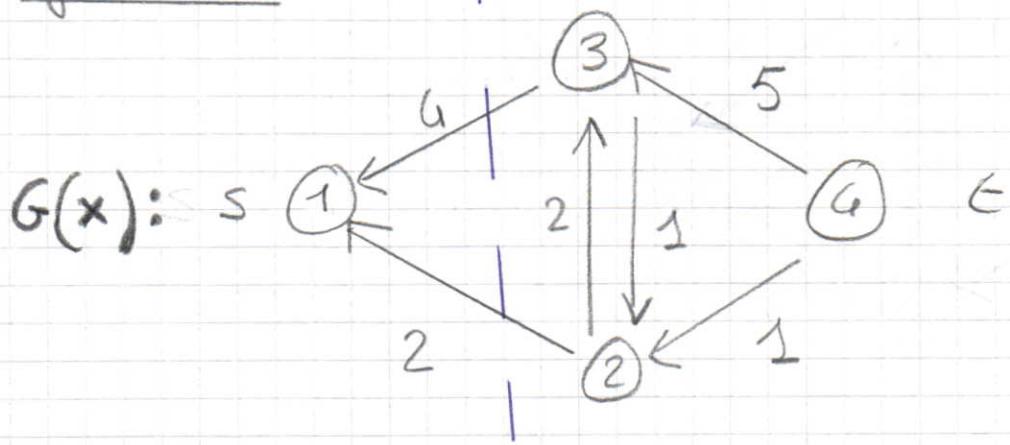
2)  $G(x)$ :



aug. path : (1, 2, 4)

$$\delta = \min \{1, 1\} = 1$$

flow x:



$v = 5 + 1 = 6$

no augmenting path:

STOP, x is an (integer) maximum flow

A minimum cut:  $S = \{1, 4\}$   $\bar{S} = \{2, 3, 4\}$

$u[S, \bar{S}] = u_{12} + u_{13} = 6 = v$

Time complexity ( $u_{ij} \in \mathbb{Z}^+$ ):  $O(n \cdot m \cdot U)$ , where U maximum arc capacity

Proof:

- capacity of any s-t cut:  $O(n \cdot U)$
- time to discover an augmenting path:  $O(m)$

□