More on Lagrangian relaxation

(Wolsey : Chapter 10)

Consider an integer optimization problem of form :

(P) $\quad z = \max cx$

$Dx \leq d$ "complicating constraints"

$x \in X$ ← include the integrality constraints

Let D $m \times n$ matrix

$d \in \mathbb{R}^m$

$x \in \mathbb{R}^n$

For any $u = (u_1, .. u_m) \geqslant 0$, consider the <u>Lagrangian relaxation</u>:

(P_u) $\quad z(u) = \max cx + u(d - Dx)$

$x \in X$

u : price or dual variables or Lagrangian multipliers associated with $Dx \leq d$

Proposition : (P_u) is a relaxation of (P) for all $u \geq 0$.

Proof:

(i) $\{x : Dx \leq d, x \in X\} \subseteq X$

the feasible region of (P_u) is at least as large

(ii) $$cx + \underbrace{u}_{\geq 0}\underbrace{(d - Dx)}_{\geq 0} \geq cx \quad \forall x \in X$$

$\forall x \in X$ the objective value in (P_u) is at least as great as in (P)

□

Therefore : $\boxed{z(u) \geq z \qquad \forall u \geq 0}$

(maximization problem)

To find the best ($\equiv$ smallest) upper bound $z(u)$ for $u \geq 0$, we solve the Lagrangian Dual Problem:

$$(LD) \qquad w_{LD} = \min\{z(u) : u \geq 0\}$$

Solving a Lagrangian relaxation may sometimes lead to solve (P):

Let $u \geq 0$:

Proposition: If:

(i) $x(u)$ is an optimal solution to (P_u)

(ii) $D\,x(u) \leq d$

(iii) if $u_i > 0$ then $(D\,x(u))_i = d_i$

(complementarity)

then $x(u)$ is optimal for (P).

Proof:

From (i):

$$w_{LD} \leq z(u) = c\,x(u) + u\,(d - D\,x(u))$$