

# Optimality, relaxations

15

## and bounds

(Wolsey: Chapter 2)

Given an optimization problem of form

$$z = \max_{x \in X} c(x)$$

how can we prove that a feasible  $x^*$  is optimal? If we find a lower bound  $\underline{z} \leq z$  and an upper bound  $\bar{z} \geq z$  such that  $\underline{z} = c(x^*) = \bar{z}$ , then  $x^*$  is optimal.

### Algorithmic consequence:

- find a decreasing sequence of u.b.:

$$\bar{z}_1 > \bar{z}_2 > \dots > \bar{z}_s$$

- find an increasing sequence of l.b.:

$$\underline{z}_1 < \underline{z}_2 < \dots < \underline{z}_t$$

• stop when

$$\bar{z}_s - \underline{z}_e \leq \varepsilon$$

for a given  $\varepsilon \geq 0$ .

So: how to compute l.b. and u.b.?

Primal bounds ( $\equiv$  l.b.)

each feasible solution  $\bar{x} \in X$  gives a l.b. (for a maximization problem);

in fact

$$c(\bar{x}) \leq \bar{z}$$

\* this is the way to compute l.b. \*

heuristics

Dual bounds ( $\equiv$  u.b.)

finding u.b. for a maximization problem (l.b. for a minimization problem) is not immediate:

is not immediate:

\* the most important approach is \*

"by relaxation"

Idea: replace a "difficult"

(17)

max (min) optimization problem by a "simplex" optimization problem s.t.:

Def: a problem (RP):

$$(RP) \quad z^R = \max_{x \in T} f(x)$$

is a *relaxation* of problem (P):

$$(P) \quad z = \max_{x \in X} c(x)$$

if:

(i)  $X \subseteq T$

(ii)  $f(x) \geq c(x) \quad \forall x \in X$

Proposition: if (RP) is a relaxation of (P), then  $z^R \geq z$  (obv!)

Proof:

Assume  $x^*$  optimal solution to (P)