Network flows

(Ahuja - Magnanti - Orlin : chapters 1, 2, 3 (essentially 3.5))

Review of basic concepts and used notation

Chapter 1 (until page 7, row + 3)

- Introduction to network flows and their application in nowaday life

  - how to move some entity (electricity, a consumer product, a person, a vehicle ...) from some points to other points in an underlying network in an "efficient" way;

  - typical of areas such as applied mathematics, computer science, engineering, management ... ;

Basic problems:

- Shortest path problem (assumed known!)
- Maximum flow problem: given arc capacities, how can we send as much flow (good) as possible from a source to a destination?
- Minimum cost flow problem: if we incur a cost per unit flow in a capacitated network, how can we send units of a good from some points to other points at a minimum cost?

NB: these are special LPs (and so, polynomially solvable); however, for efficiency reasons they are addressed directly via graph theory, and not via a LP perspective (although many concepts derive from LP theory!)

Minimum cost flow

- distribution of a product from plants to warehouses
- routing of vehicles along a street network .....

→ See Chapter 2 (from 2.1 to 2.3 : TO READ) for basic notation and definitions of graph theory

Let $G = (N, A)$ directed network

- N set of n nodes
- A set of m directed arcs
- c_{ij} cost per unit flow on (i,j), $\forall (i,j) \in A$
- u_{ij} capacity of (i,j), $\forall (i,j) \in A$
  ("maximum" amount of flow)
- $b(i) \in \mathbb{Z}$ supply / demand of node i, $\forall i \in N$:

Assumption (the opposite w.r.t. RO course):

$b(i) > 0$ supply node

$b(i) < 0$ demand node (with demand $-b(i)$)

$b(i) = 0$ transshipment node

Decision variables (flow variables):

x_{ij} flow to push along (i,j), $\forall (i,j) \in A$

Mathematical model (LP):

$$(MCF) \quad \text{Min} \sum_{(i,j) \in A} c_{ij} x_{ij}$$

Flow conservation constraints

$$\underbrace{\sum_{(i,j) \in FS(i)}}_{\text{Forward Star of } i} x_{ij} - \underbrace{\sum_{(j,i) \in BS(i)}}_{\text{Backward Star of } i} x_{ji} = b(i) \quad \forall i \in N$$

$$0 \le x_{ij} \le u_{ij} \quad \forall (i,j) \in A$$

l_{ij} in A.M.O.

Capacity constraints

Necessary condition for a feasible solution : $\sum_{i \in N} b(i) = 0$

Compact form :

$$\text{Min } c \cdot x$$
$$N x = b$$
$$0 \leq x \leq u$$

N $(n \times m)$ is the node-arc incidence matrix of G :

$$N : \begin{array}{c c} & (i,j) \\ \begin{array}{c} i \\ \\ j \end{array} & \begin{bmatrix} +1 \\ \\ -1 \end{bmatrix} \end{array}$$

0 elsewhere

$c \in \mathbb{R}^m$ cost vector

$x \in \mathbb{R}^m$ variable vector

$b \in \mathbb{R}^n$ balance vector

$u \in \mathbb{R}^m$ capacity vector

<u>Integrality assumption</u> : c, u and b are integer-valued

Special cases

1. **Shortest path from s to t in G**

< send 1 unit of flow from s to t at a minimum cost >

$$s \xrightarrow{1} \circ \xrightarrow{1} \circ \xrightarrow{1} t$$

Special case of model (MCF) with

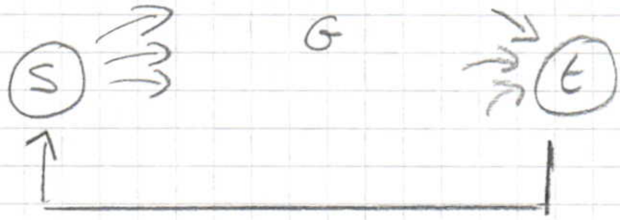
$$\begin{cases} b(s) = 1 \\ b(t) = -1 \\ b(i) = 0 \qquad \forall i \neq s, t \end{cases}$$

<u>no arc capacity</u>

2. **Maximum flow from s to t in G**

<u>Complementary</u> : no costs but arc capacities (u_{ij})

Special case of model (MCF) with an extra arc from t to s:



$c_{ij} = 0 \quad \forall (i,j) \in A$

$c_{ts} = -1$

and $b(i) = 0 \quad \forall i \in N$

<u>then</u> : the minimum cost flow solution maximizes the flow along (t,s), i.e. the flow amount sent from s to t along G !

Chapter 2

(Section 2.4 (only "Working with Reduced Costs" and "Working with Residual Networks")) until page 45 (row 12)

Many network flow algorithms work with "reduced costs" c_{ij}^{π} instead of the actual costs c_{ij}:

given a number $\pi(i)$ associated with $i \in N$ (potential of i), the reduced cost of (i,j) is

$$\boxed{c_{ij}^{\pi} = c_{ij} - \pi(i) + \pi(j)}$$

Let $z(0)$ be the objective function w.r.t. $\{c_{ij}\}$ and $z(\pi)$ " " " " $\{c_{ij}^{\pi}\}$.

Then:

Property: $z(\pi) = z(0) - \sum_{i \in N} \pi(i) \cdot b(i) =$

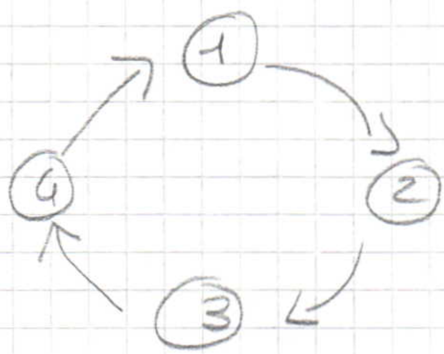
$= z(0) - \pi b.$

Therefore, since πb is a constant, the minimum cost flow problems with costs $\{c_{ij}\}$ and $\{c_{ij}^{\pi}\}$ have the same optimal solutions. □

Therefore, we can use either $\{c_{ij}\}$ or $\{c_{ij}^{\pi}\}$

In particular, what is the effect of working with reduced costs on cycles and paths (important structures at algorithmic level)?

W directed cycle :



$$\boxed{\sum_{(i,j)\in W} c_{ij}^{\pi}} = c_{12} - \cancel{\pi(1)} + \cancel{\pi(2)} + c_{23} - \cancel{\pi(2)} + \cancel{\pi(3)} +$$

$$+ c_{34} - \cancel{\pi(3)} + \cancel{\pi(4)} + c_{41} - \cancel{\pi(4)} + \cancel{\pi(1)} =$$

$$\boxed{\sum_{(i,j)\in W} c_{ij}}$$

The cost is the same!

<u>P directed path from s to t</u> : $s \rightarrow 1 \rightarrow 2 \rightarrow t$

$$\boxed{\sum_{(i,j)\in P} c_{ij}^{\pi}} = c_{s1} - \pi(s) + \cancel{\pi(1)} + c_{12} - \cancel{\pi(1)} + \pi(2) + c_{2t} - \cancel{\pi(2)} + \pi(t) = \boxed{\sum_{(i,j)\in P} c_{ij} - \pi(s) + \pi(t)}$$

<u>Difference depending on the potential of terminals s and t</u>

Another basic concept in designing network flow algorithms is <u>residual network</u> (or graph): is an auxiliary network which measures how we can "move flow" w.r.t. a feasible solution x^0.

Given a flow x^0 on G, replace each arc (i,j) in G by two arcs:

- (i,j) with cost c_{ij} and <u>residual capacity</u> $r_{ij} = u_{ij} - x^0_{ij}$

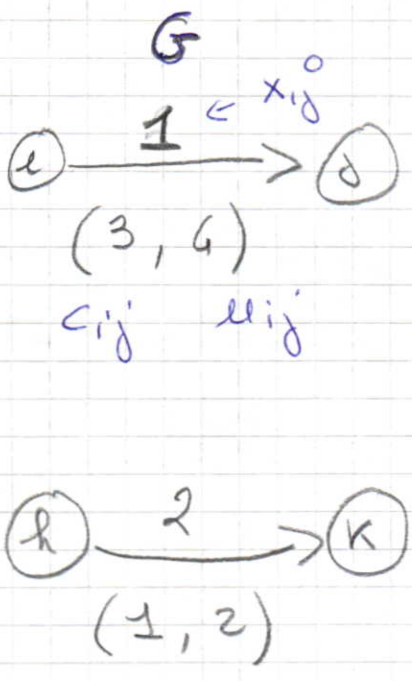
- (j,i) with cost $-c_{ij}$ and residual capacity $r_{ji} = x_{ij}^0$

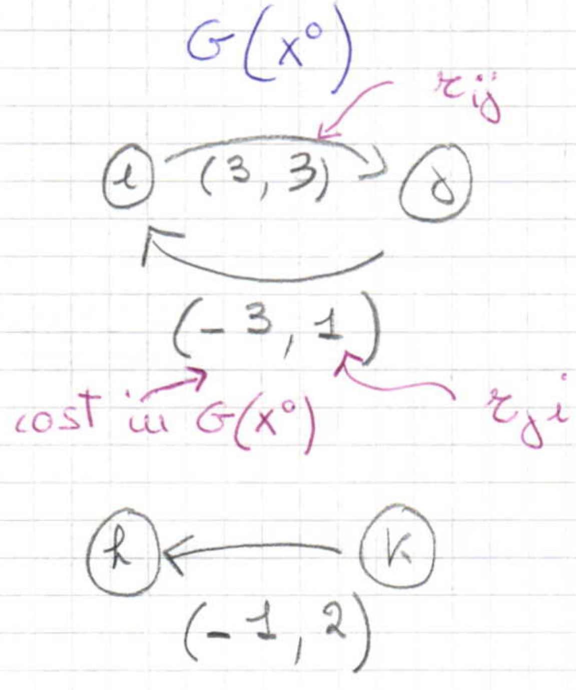
<u>Def</u> : <u>the residual network w.r.t. x^0</u>, <u>$G(x^0)$</u> $= (N, A(x^0))$, where $A(x^0)$ contains the arcs with a positive residual capacity

<u>NB</u> : we assume that G does not contain both (i,j) and (j,i), $\forall i, j \in N$

<u>NB</u> : in the maximum flow case, no cost is associated with the arcs in $G(x^0)$

<u>example</u>





The maximum flow: basic ideas

(Ahuja - Magnanti - Orlin: Chapter 6 (6.1, 6.3, 6.4 (until page 181, row 5), 6.5)

- already covered by the course RO

Let $G = (N, A)$ be a directed network

- $u_{ij} \in \mathbb{Z}^+$ capacity of (i,j), $\forall (i,j) \in A$
- $s \in N$ source node
- $t \in N$ destination node

We wish to push the maximum amount of flow from s to t, by satisfying the arc capacities and the flow conservation constraints.

We can state a direct LP formulation (in place of the one introduced before, as special minimum cost flow):

$$\text{Max } v$$

value of the flow

flow

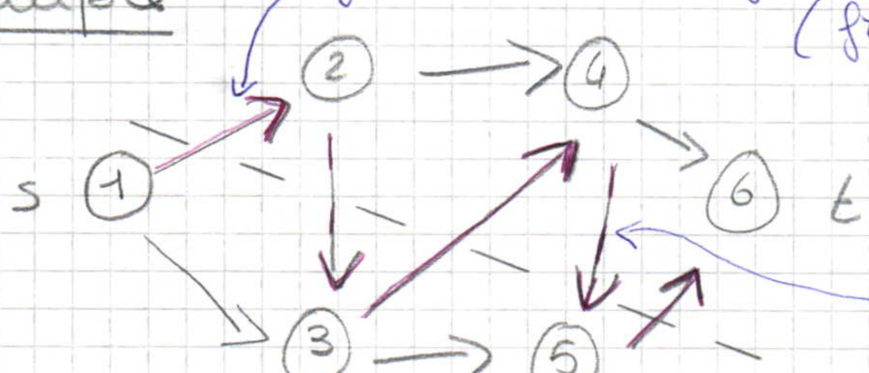
$$\sum_{(i,j)\in FS(i)} x_{ij} - \sum_{(j,i)\in BS(i)} x_{ji} = \begin{cases} v & \text{if } i=s \\ -v & \text{if } i=t \\ 0 & \text{otherwise} \end{cases}$$

$$0 \leq x_{ij} \leq u_{ij} \quad , \forall (i,j)\in A$$

A basic concept is s-t cut: given a partition of N into S and $\bar{S} = N \setminus S$ (cut, denoted by $[S, \bar{S}]$), an s-t cut is a cut s.t. $s \in S$ and $t \in \bar{S}$

example

(1,2): forward arc of the cut (from S to $\bar{S}$)



(4,5): backward arc of the cut (from $\bar{S}$ to S)

$S = \{1, 3, 5\}$ $\quad \bar{S} = \{2, 4, 6\}$

$[S, \bar{S}]$: s-t cut

<u>Capacity of</u> $[S, \bar{S}]$:

$$u[S, \bar{S}] = \sum_{\substack{(i,j) \\ \text{forward of the cut}}} u_{ij}$$

<u>Minimum cut</u>: s-t cut with minimum capacity among all s-t cuts.

<u>Property</u>: for any flow x (of value v) and any s-t cut $[S, \bar{S}]$:

$$\boxed{v \leq u[S, \bar{S}]}$$

Indeed, the <u>max-flow min-cut theorem</u> states that, for some flow x^* and some s-t cut $[S^*, \bar{S}^*]$: $v^* = u[S^*, \bar{S}^*]$

When this happens $\rightarrow$ x^* is a maximum flow and $[S^*, \bar{S}^*]$ is a minimum cut.