

# Business Processes Modelling

## MPB (6 cfu, 295AA)

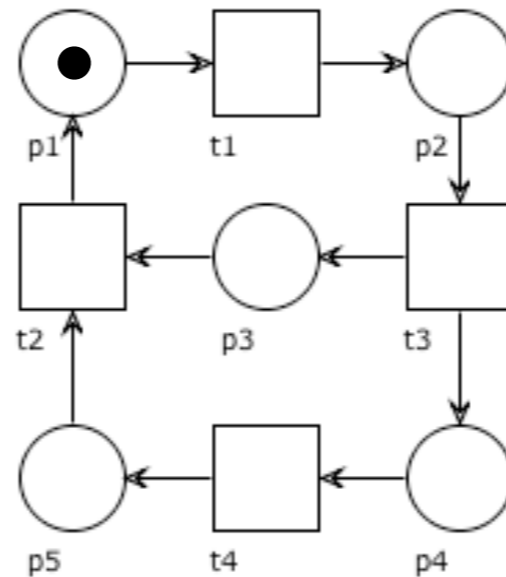
Roberto Bruni

<http://www.di.unipi.it/~bruni>

17 - T-systems



# Object



We study some “good” properties of T-systems

Free Choice Nets (book, optional reading)

<https://www7.in.tum.de/~esparza/bookfc.html>

T-systems

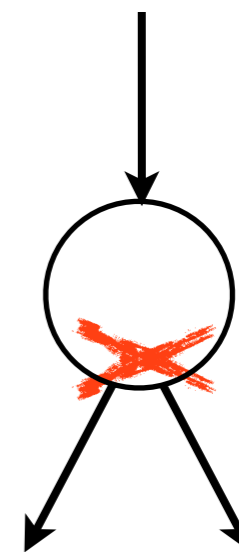
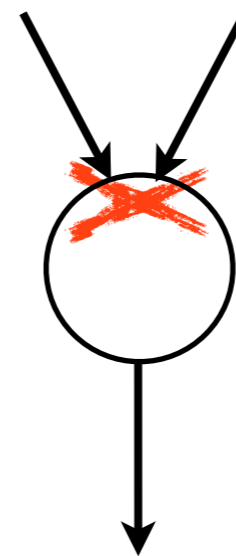
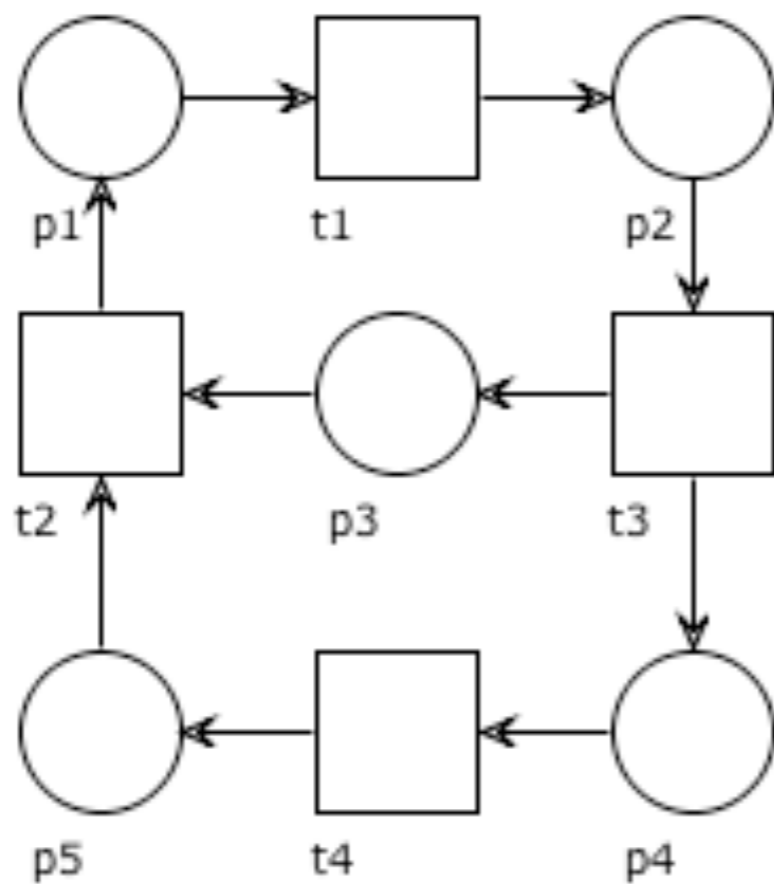
# T-system

**Definition:** We recall that a net  $N$  is a **T-net** if each place has exactly one input transition and exactly one output transition

$$\forall p \in P, \quad |\bullet p| = 1 = |p \bullet|$$

A system  $(N, M_0)$  is a **T-system** if  $N$  is a T-net

# T-net: example



# T-systems: an observation

Notably, computation in T-systems is concurrent,  
but essentially deterministic:

the firing of a transition  $t$  in  $M$  cannot disable  
another transition  $t'$  enabled at  $M$

# T-net $N^*$

Is the following conjecture true?  
A workflow net  $N$  is a T-net  
iff  $N^*$  is a T-net

# T-net $N^*$

Is the following conjecture true?

A workflow net  $N$  is a T-net  
iff  $N^*$  is a T-net

No, a workflow net cannot be a T-net because  
the place  $i$  has no incoming arc  
and the place  $o$  has no outgoing arc

( $N^*$  can be a T-net)



# T-systems: another observation

Determination of control:

the transitions responsible for enabling  $t$  are  
one for each input place of  $t$

# Notation: token count of a circuit

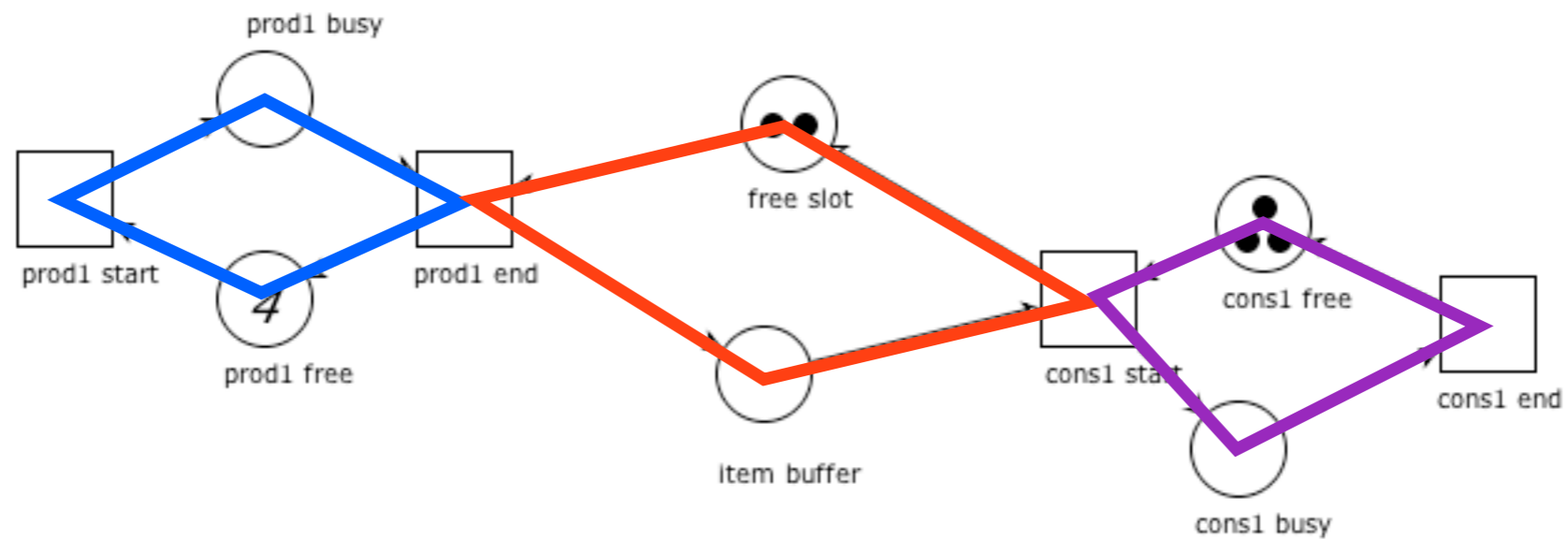
Let  $\gamma = (x_1, y_1)(y_1, x_2)(x_2, y_2)\dots(x_n, y_n)$  be a circuit.

Let  $P|_{\gamma} \subseteq P$  be the set of places in  $\gamma$ .

$$M(\gamma) = M(P|_{\gamma}) = \sum_{p \in P|_{\gamma}} M(p)$$

We say that  $\gamma$  is **marked at**  $M$  if  $M(\gamma) > 0$

# Example



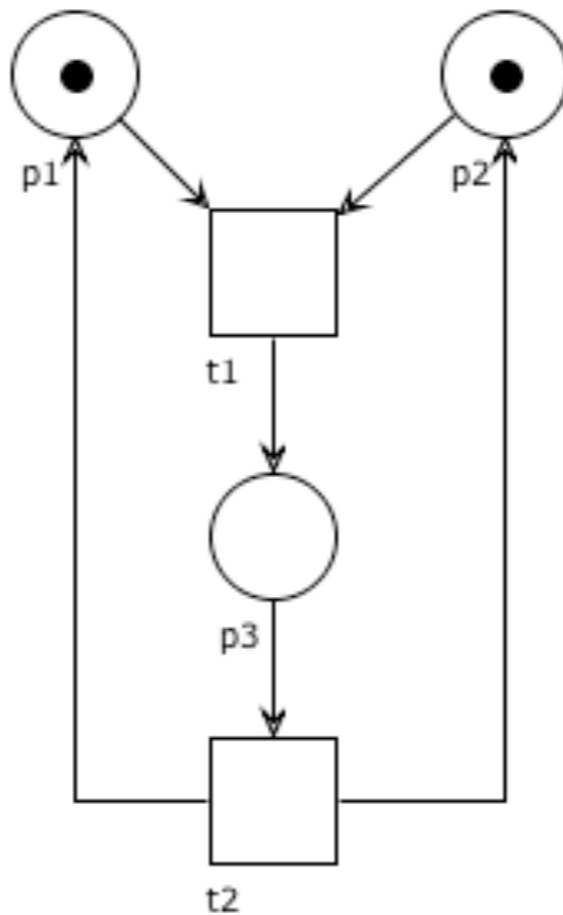
$$M(\gamma_1) = 4$$

$$M(\gamma_2) = 2$$

$$M(\gamma_3) = 3$$

# Question time

Trace two circuits over the T-system below



# Fundamental property of T-systems

The token count of a circuit is invariant under any firing.

# Fundamental property of T-systems

**Proposition:** Let  $\gamma$  be a circuit of a T-system  $(P, T, F, M_0)$ .  
If  $M$  is a reachable marking, then  $M(\gamma) = M_0(\gamma)$

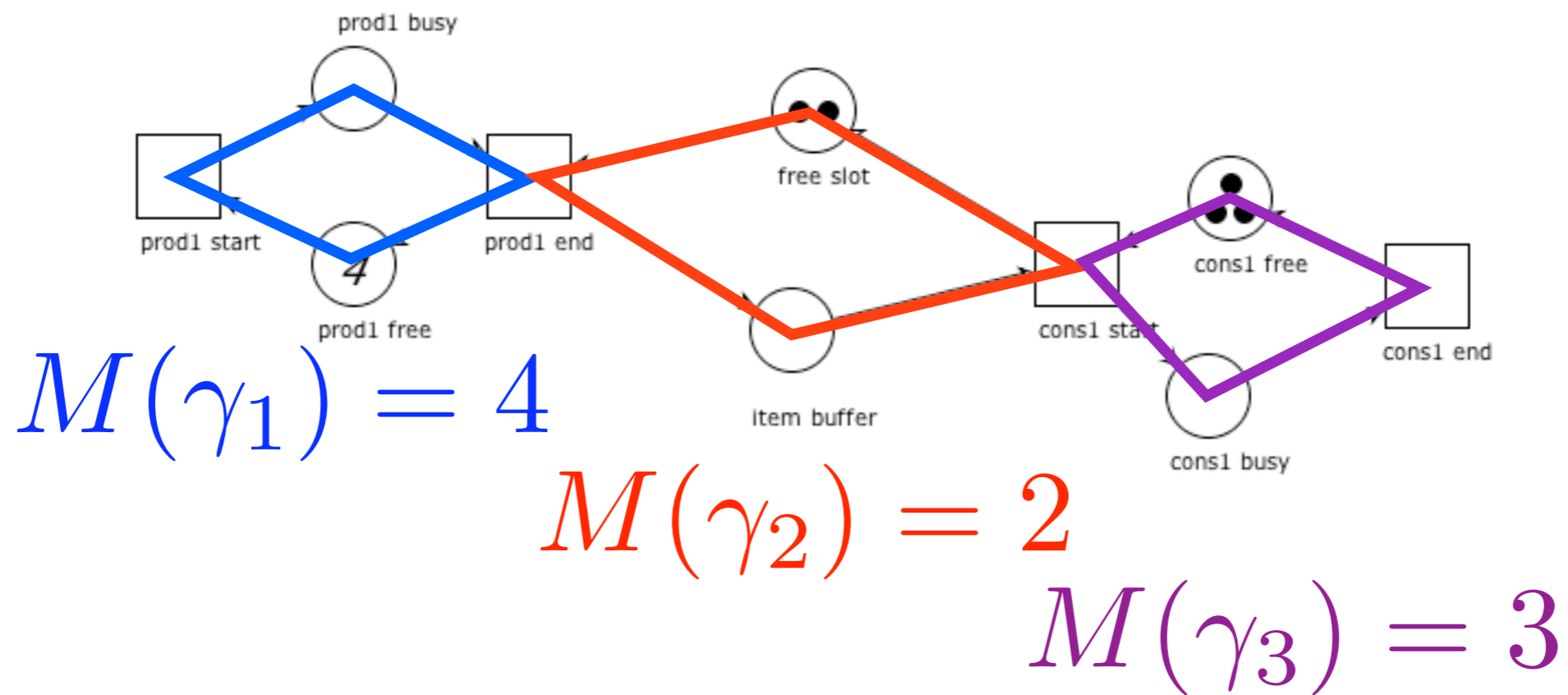
Take any  $t \in T$ : either  $t \notin \gamma$  or  $t \in \gamma$ .

If  $t \notin \gamma$ , then no place in  $\bullet t \cup t \bullet$  is in  $\gamma$   
(otherwise, by definition of T-nets,  $t$  would be in  $\gamma$ ).

Then, an occurrence of  $t$  does not change the token count of  $\gamma$ .

If  $t \in \gamma$ , then exactly one place in  $\bullet t$  and one place in  $t \bullet$  are in  $\gamma$ .  
Then, an occurrence of  $t$  does not change the token count of  $\gamma$ .

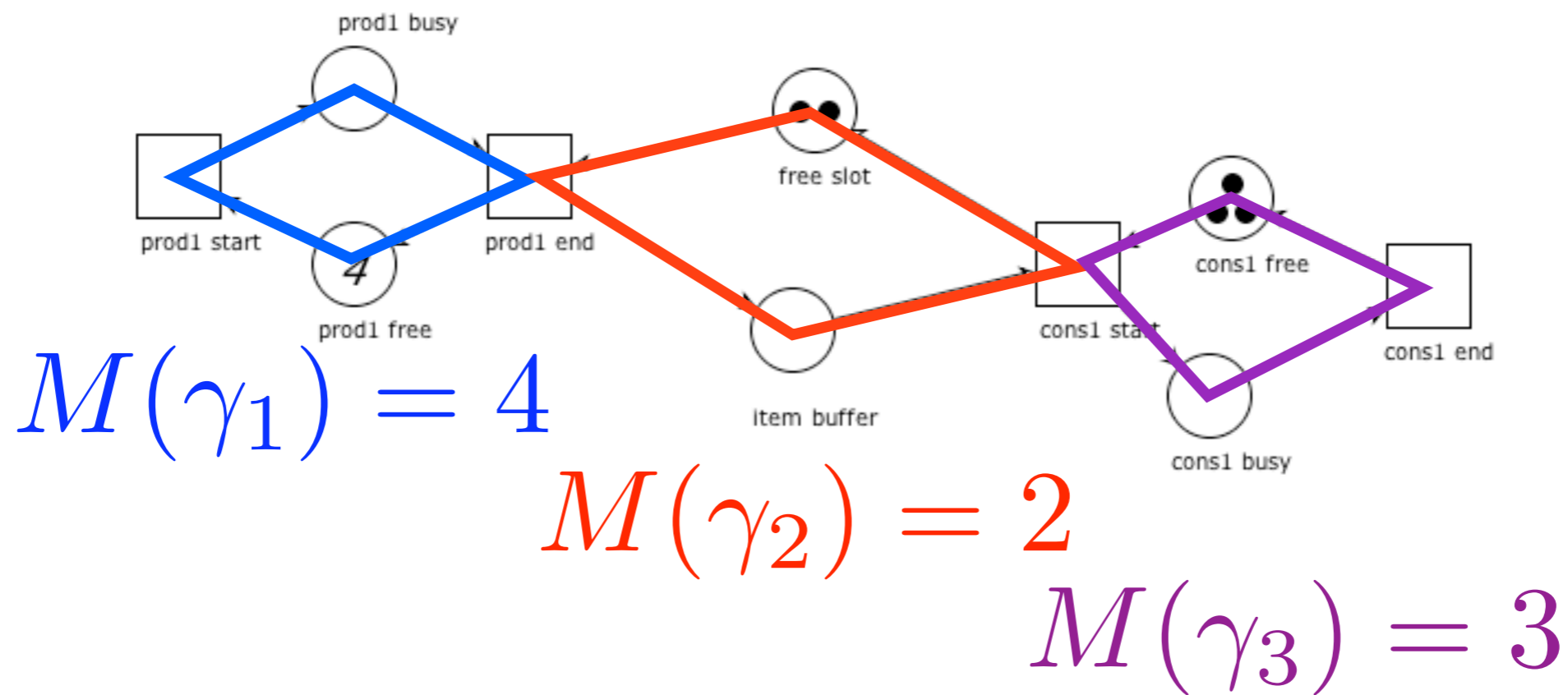
# Example



$$M_0 = [0 \ 4 \ 2 \ 0 \ 3 \ 0]$$

$$M = [2 \ 2 \ 1 \ 2 \ 2 \ 1]$$

# Example



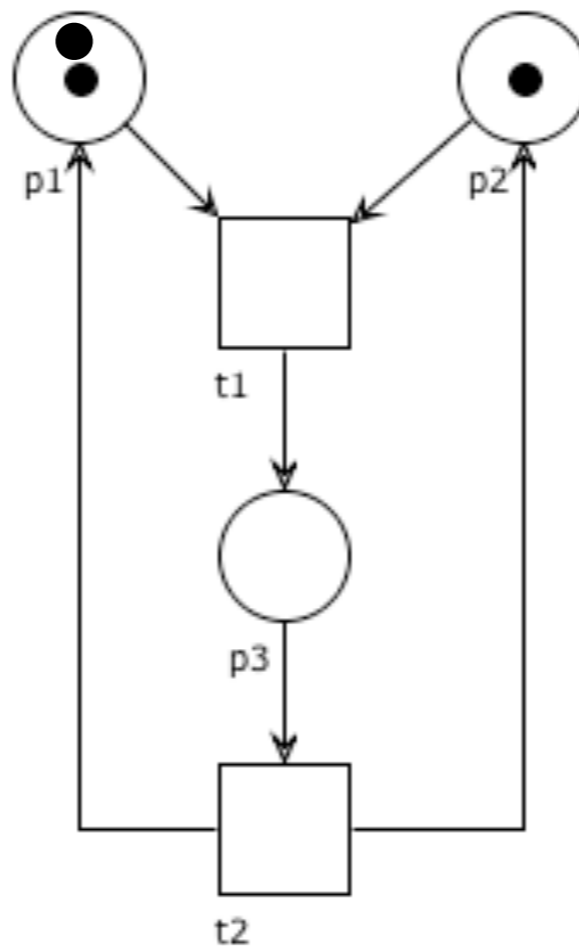
$$M_0 = [0 \ 4 \ 2 \ 0 \ 3 \ 0]$$

$$M' = [2 \ 1 \ 1 \ 1 \ 2 \ 2]$$



# Question time

Is the marking  $p_1 + 2p_2$  reachable? (why?)



# T-invariants of T-nets

**Proposition:** Let  $N=(P,T,F)$  be a (connected) T-net.  $J$  is a T-invariant of  $N$  iff  $J=[k \dots k]$  for some value  $k$

(the proof is dual to the analogous proposition for S-invariants of S-nets)

# Boundedness in strongly connected T-systems

**Lemma:** If a T-system  $(N, M_0)$  is strongly connected, then it is bounded

Let  $\Gamma$  be the set of the circuits of  $N$  and let  $k = \max_{\gamma \in \Gamma} M_0(\gamma)$ .

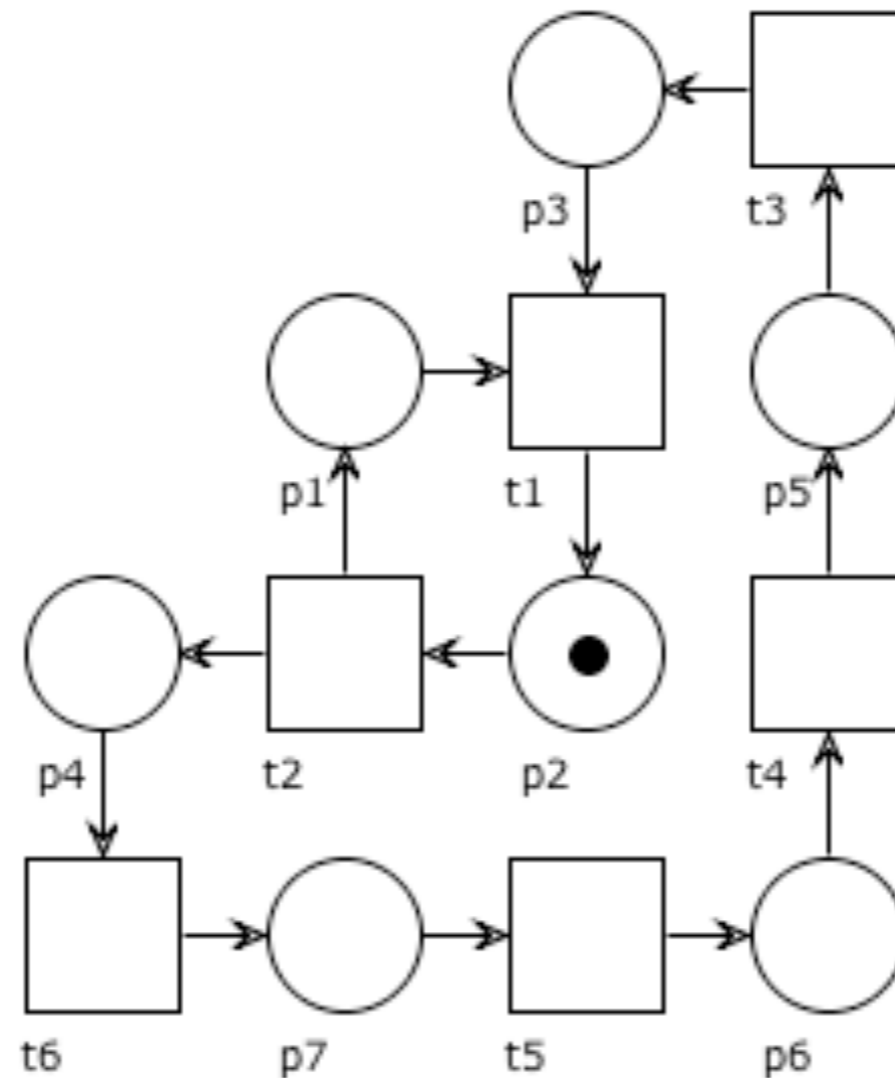
Since  $N$  is strongly connected, every place  $p$  belongs to some circuit  $\gamma_p$ .

By the fundamental property of T-systems: token count of  $\gamma_p$  is invariant.

Thus, for any reachable marking  $M$ , we have  $M(p) \leq M(\gamma_p) = M_0(\gamma_p) \leq k$ .  
Hence the net is  $k$ -bounded.

# Question time

Is the T-systems below bounded? (why?)



# Liveness theorem for T-systems

**Theorem:** A T-system  $(N, M_0)$  is live  
**iff** every circuit of  $N$  is marked at  $M_0$

$\Rightarrow$ ) (quite obvious)

By contradiction, let  $\gamma$  be a circuit with  $M_0(\gamma) = 0$ .

By the fundamental property of T-systems:  $\forall M \in [M_0 \rangle, M(\gamma) = 0$ .

Take any  $t \in T|_\gamma$  and  $p \in P|_\gamma \cap \bullet t$ .

For any  $M \in [M_0 \rangle$ , we have  $M(p) = 0$ .

Hence  $t$  is never enabled and the T-system is not live.

# Liveness theorem for T-systems

**Theorem:** A T-system  $(N, M_0)$  is live  
**iff** every circuit of  $N$  is marked at  $M_0$

$\Leftarrow$ ) (more involved)

Take any  $t \in T$  and  $M \in [M_0 \rangle$ .

We need to show that some marking  $M'$  reachable from  $M$  enables  $t$ .

The key idea is to collect the places that control the firing of  $t$ :

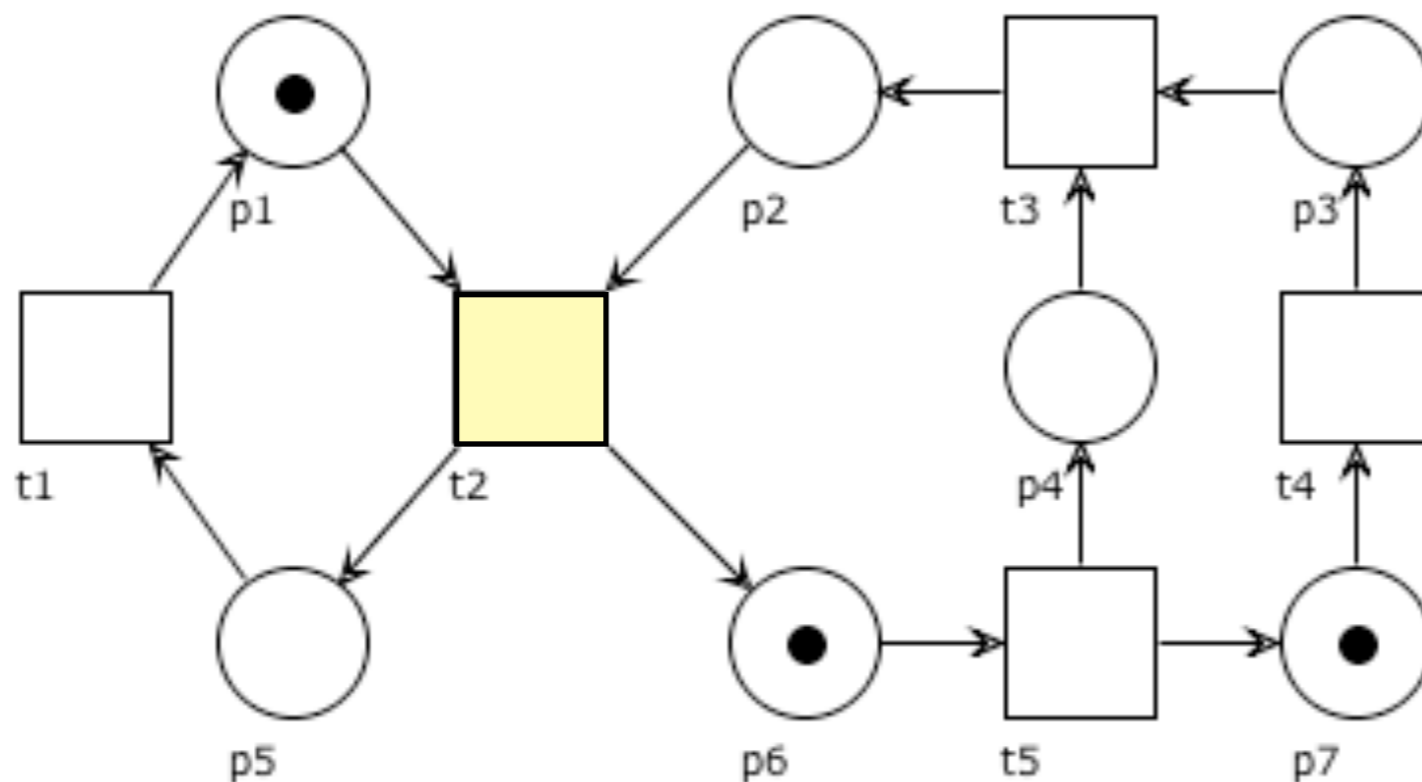
$p \in P_{M,t}$  if there is a path from  $p$  to  $t$  through places unmarked at  $M$ .

We then proceed by induction on the size of  $P_{M,t}$ .

We just sketch the key idea of the proof over a T-system.

# Liveness theorem for T-systems

**Theorem:** A T-system  $(N, M_0)$  is live  
 $\Leftrightarrow$  every circuit of  $N$  is marked at  $M_0$



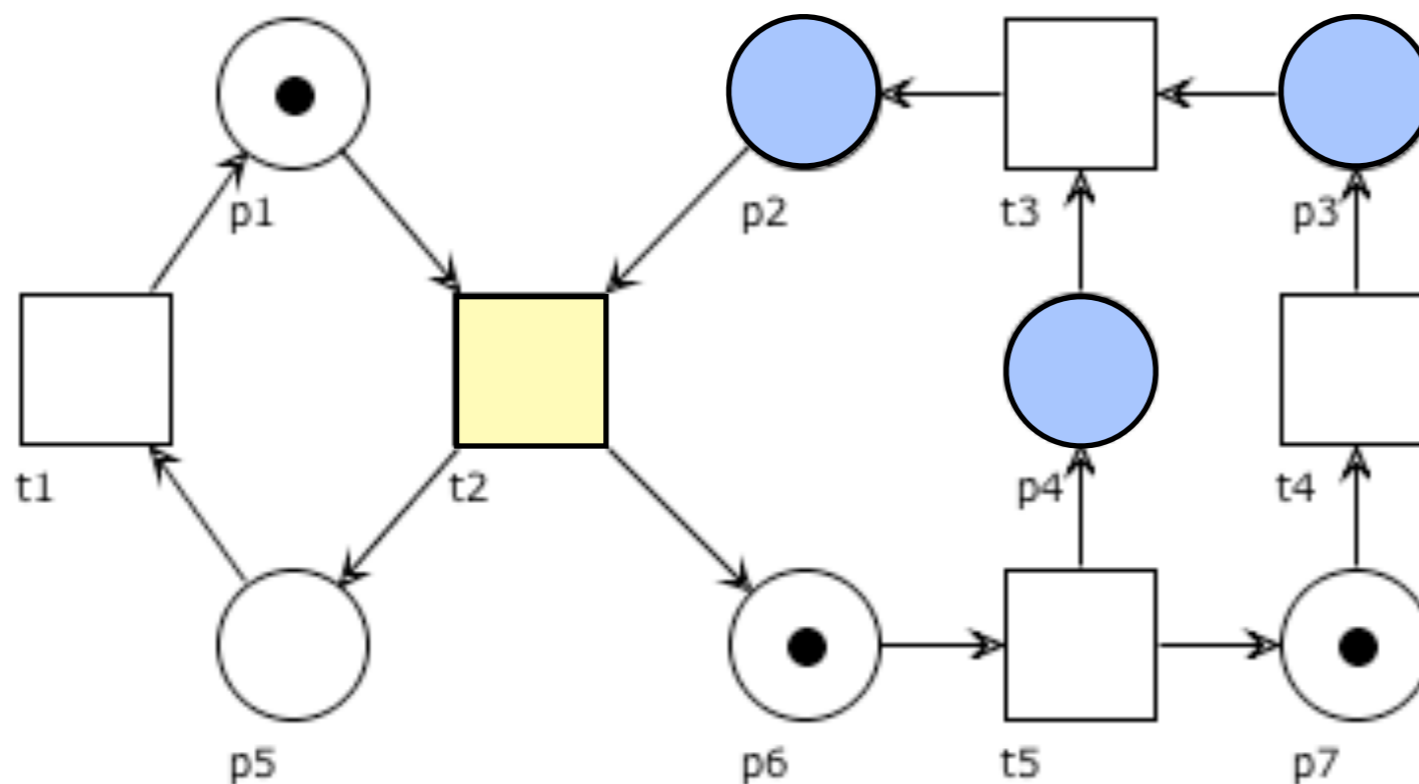
$$M = p_1 + p_6 + p_7$$

$M'$  enabling  $t_2$ ?

$p \in P_{M,t}$  if there is a path from  $p$  to  $t$  through places unmarked at  $M$ .

# Liveness theorem for T-systems

**Theorem:** A T-system  $(N, M_0)$  is live  $\Leftrightarrow$  every circuit of  $N$  is marked at  $M_0$



$$P_{M, t_2} = \{ p_2, p_3, p_4 \}$$



# Liveness theorem for T-systems

**Theorem:** A T-system  $(N, M_0)$  is live  
 $\Leftrightarrow$  every circuit of  $N$  is marked at  $M_0$

$\Leftarrow$ ) (continued proof sketch)

Base case:  $|P_{M,t}| = 0$ .

Every place in  $\bullet t$  is already marked at  $M$ .

Hence  $t$  is enabled at  $M$ .

# Liveness theorem for T-systems

**Theorem:** A T-system  $(N, M_0)$  is live  
 $\Leftrightarrow$  every circuit of  $N$  is marked at  $M_0$

$\Leftarrow$ ) (continued proof sketch)

Inductive case:  $|P_{M,t}| > 0$ .

Therefore  $t$  is not enabled at  $M$ .

We look for a path  $\pi$  of maximal length necessary for firing  $t$ .

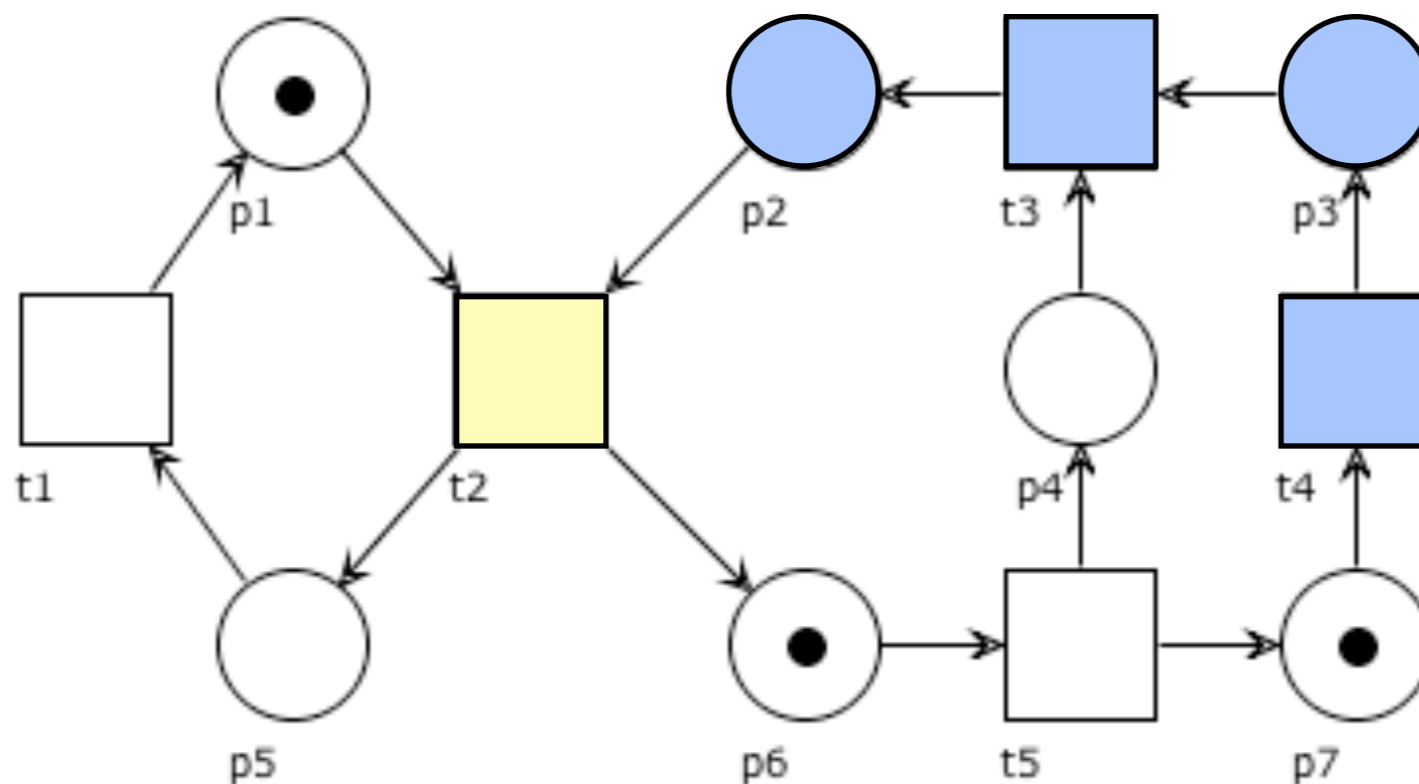
$\pi$  must contain only places unmarked at  $M$ .

By the fundamental property of T-systems: all circuits are marked at  $M$ .

$\pi$  is not necessarily unique, but exists (no cycle in it).

# Liveness theorem for T-systems

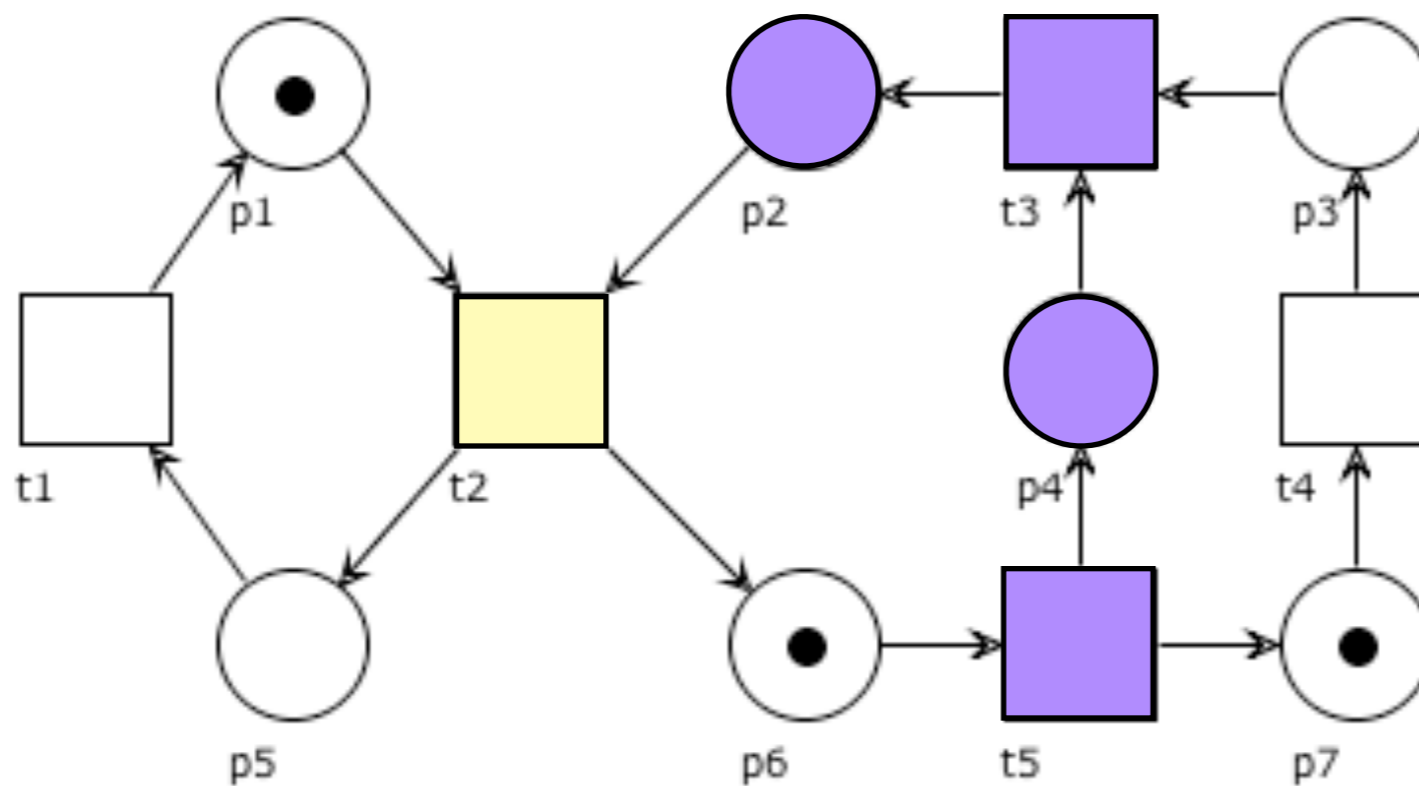
**Theorem:** A T-system  $(N, M_0)$  is live  $\Leftrightarrow$  every circuit of  $N$  is marked at  $M_0$



$$\pi = t_4 p_3 t_3 p_2 t_2$$

# Liveness theorem for T-systems

**Theorem:** A T-system  $(N, M_0)$  is live  
 $\Leftrightarrow$  every circuit of  $N$  is marked at  $M_0$



$$\pi = t_5 p_4 t_3 p_2 t_2$$

# Liveness theorem for T-systems

**Theorem:** A T-system  $(N, M_0)$  is live  
 $\Leftrightarrow$  every circuit of  $N$  is marked at  $M_0$

$\Leftarrow$ ) (Inductive case:  $|P_{M,t}| > 0$ , continued proof sketch)

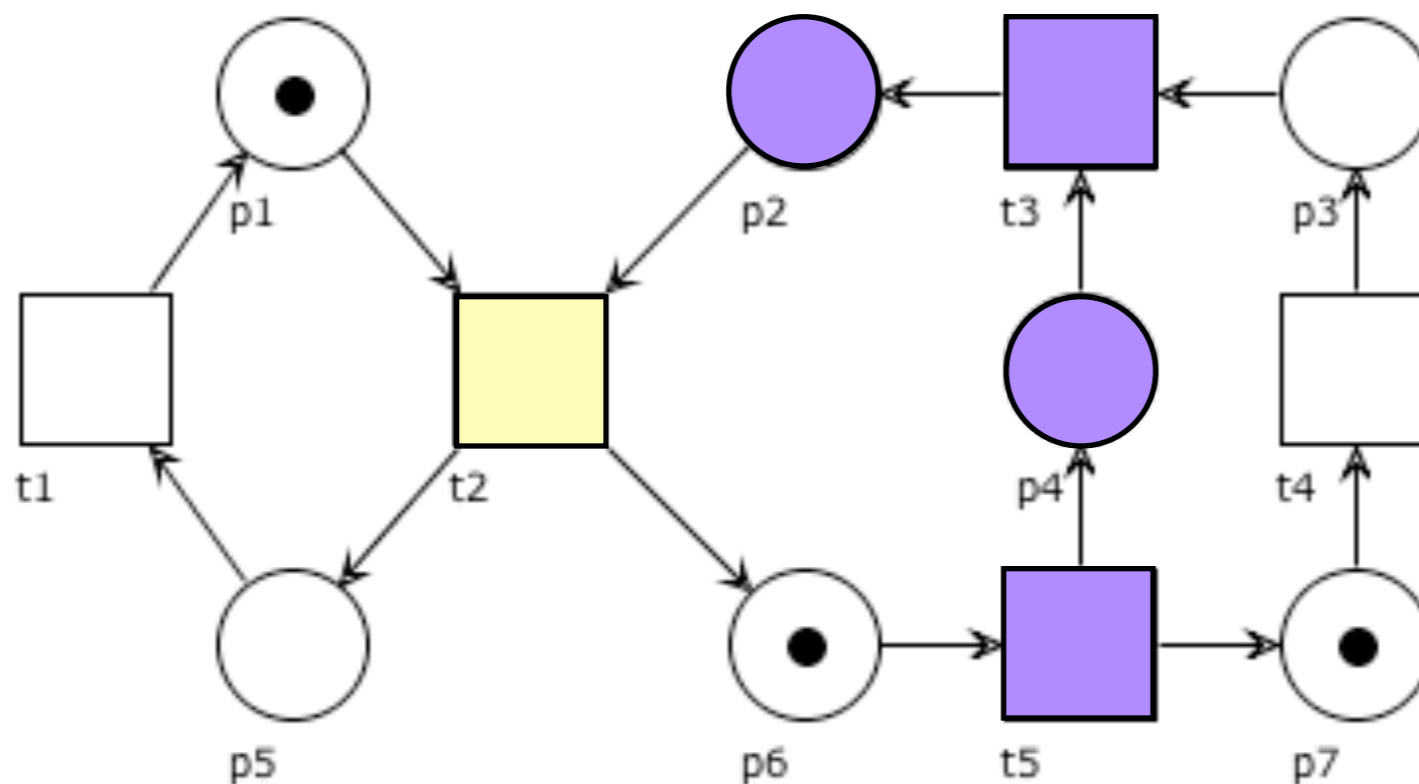
$\pi$  begins with a transition  $t'$  enabled at  $M$ .  
(otherwise a longer path could be found).

By firing  $t'$  we reach a marking  $M''$  such that  $P_{M'',t} \subset P_{M,t}$ .

Hence  $|P_{M'',t}| < |P_{M,t}|$  and we conclude by inductive hypothesis.

# Liveness theorem for T-systems

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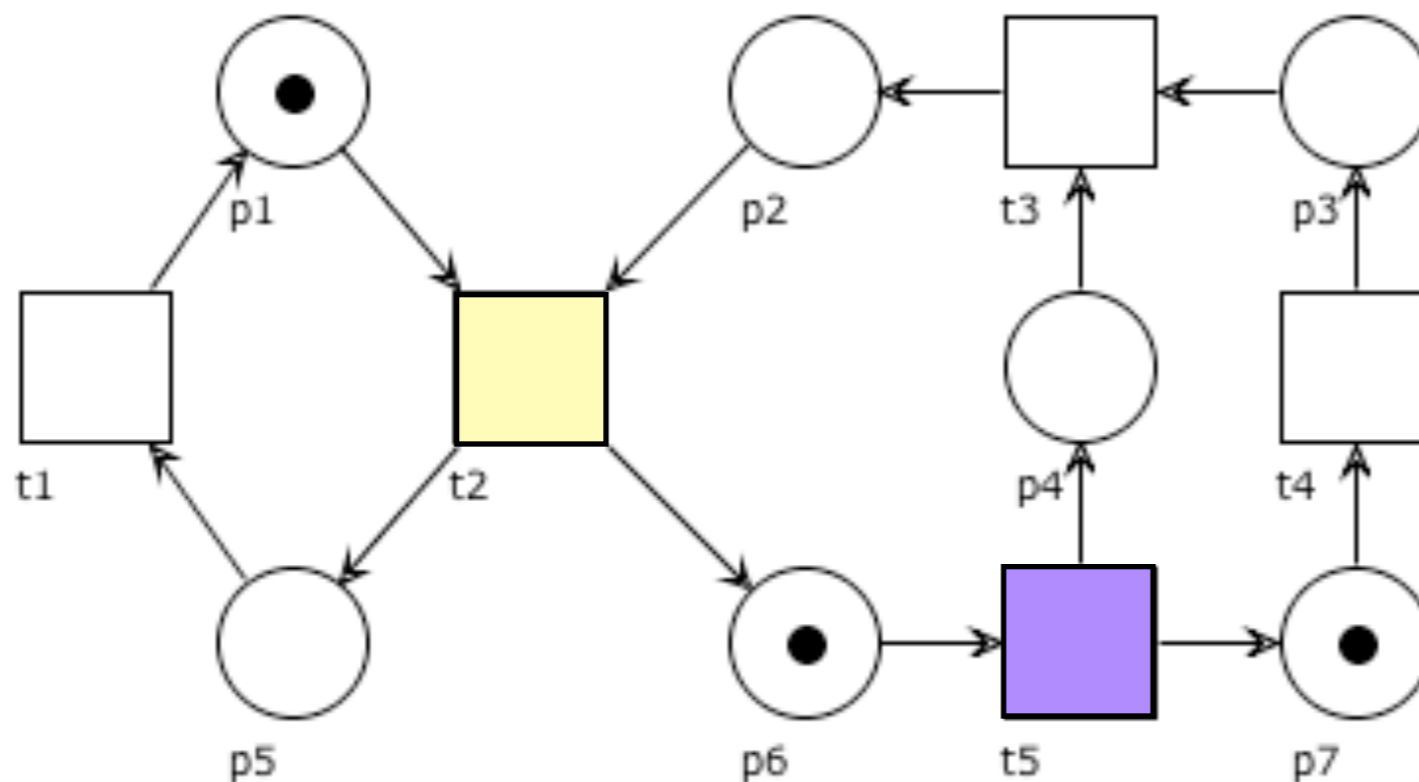


$$P_{M, t_2} = \{ p_2, p_3, p_4 \}$$

$$\pi = t_5 p_4 t_3 p_2 t_2$$

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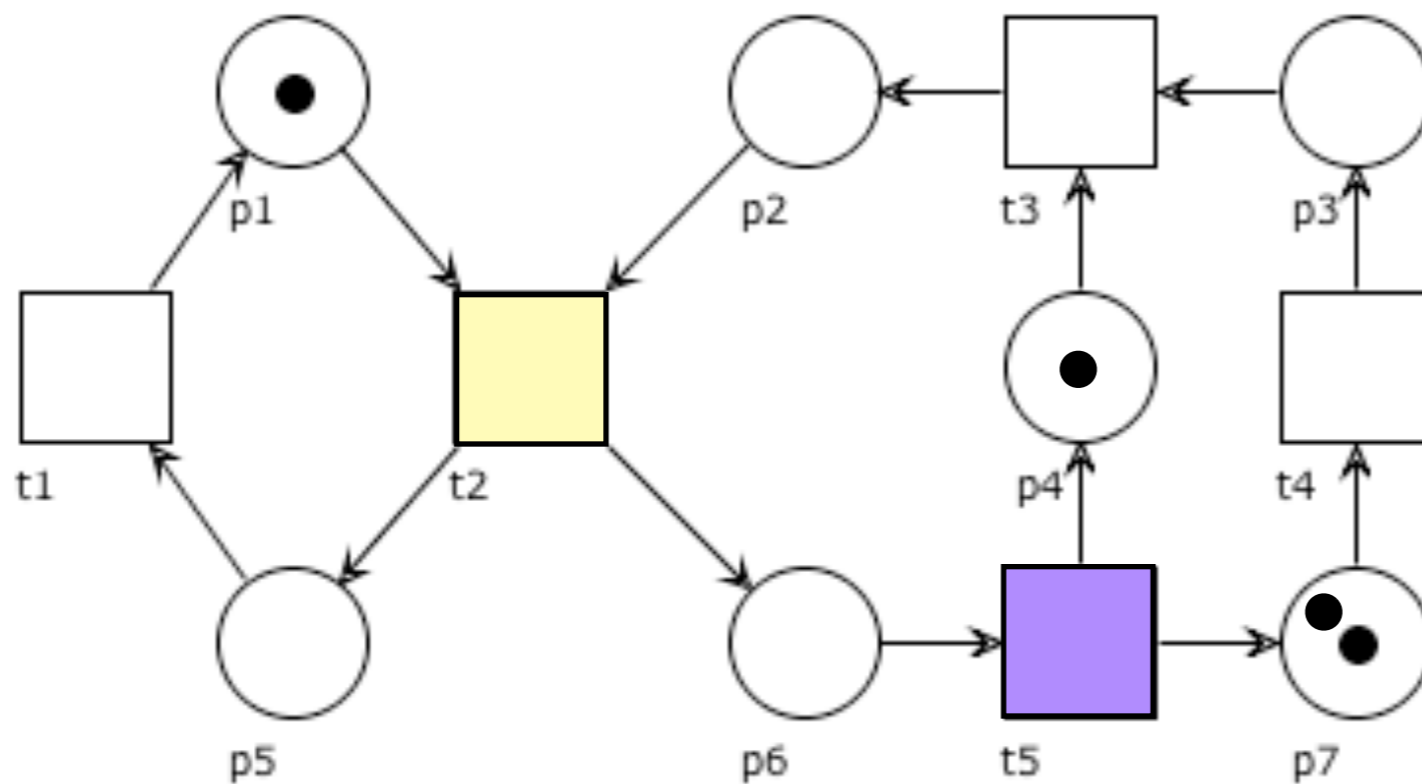


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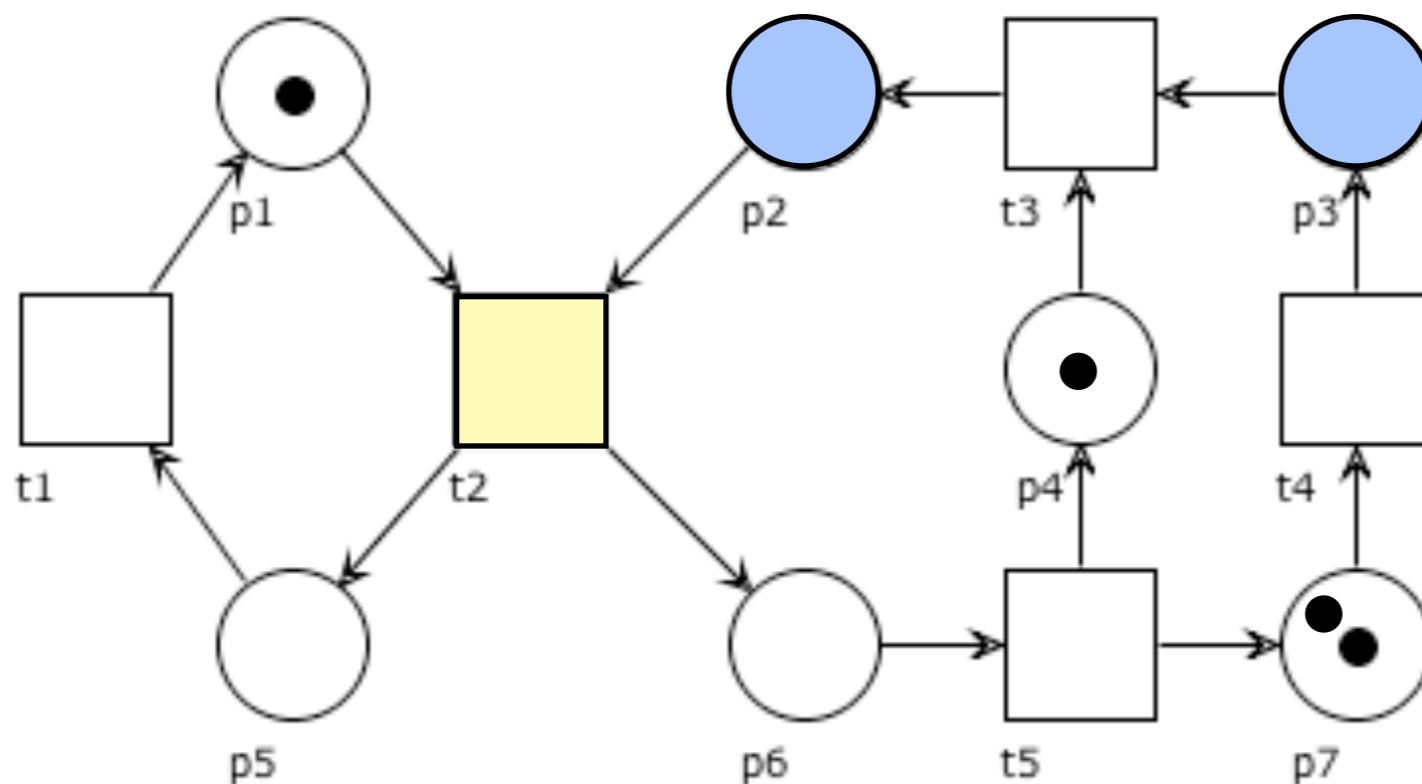
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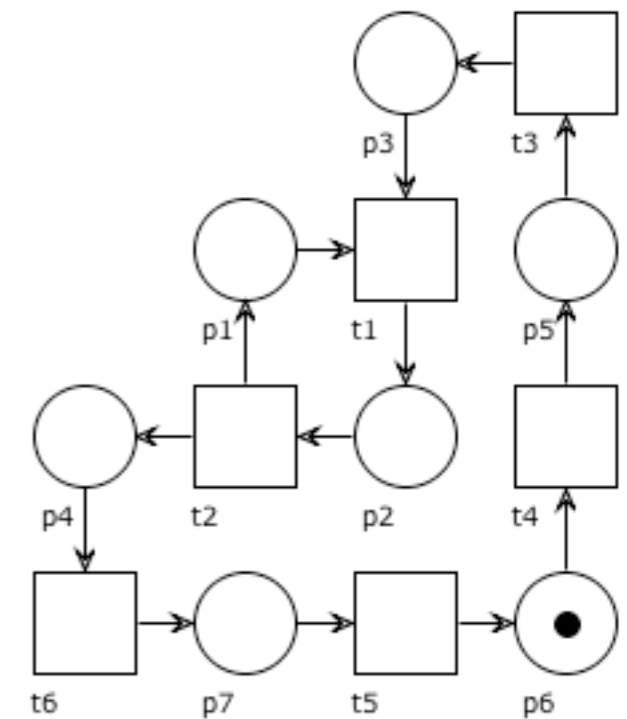
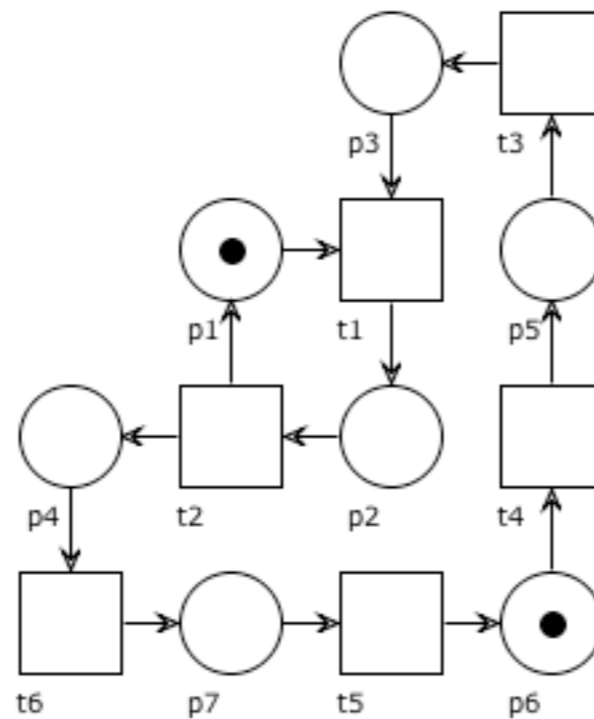
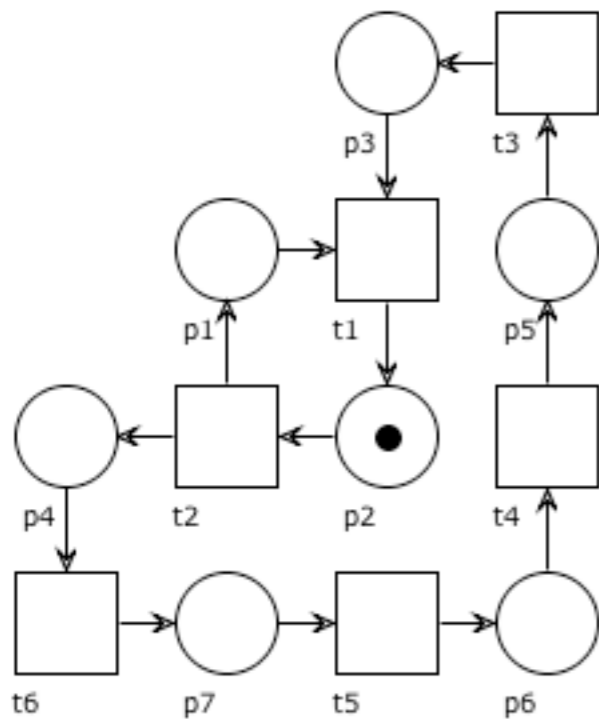


$$P_{M, t_2} = \{ p_2, p_3, p_4 \}$$

$$P_{M'', t_2} = \{ p_2, p_3 \}$$

# Question time

Which of the T-systems below is live? (why?)



# Boundedness theorem for live T-systems

**Theorem:** A live T-system  $(P, T, F, M_0)$  is  $k$ -bounded iff every place  $p \in P$  belongs to a circuit  $\gamma_p$  with  $M_0(\gamma_p) \leq k$ .

$\Leftarrow$ ) Let  $M \in [M_0 \rangle$  and take any  $p \in P$ .

By the fundamental property of T-systems:  
 $M(p) \leq M(\gamma_p) = M_0(\gamma_p) \leq k$

# Boundedness theorem for live T-systems

**Theorem:** A live T-system  $(P, T, F, M_0)$  is  $k$ -bounded iff every place  $p \in P$  belongs to a circuit  $\gamma_p$  with  $M_0(\gamma_p) \leq k$ .

$\Rightarrow$ ) Let  $k_p \leq k$  be the bound of  $p$ .  
Take  $M \in [M_0]$  with  $M(p) = k_p$ .

Define  $L = M - k_p p$  and note that the T-system  $(N, L)$  is not live.  
(otherwise  $L \xrightarrow{\sigma} L'$  with  $L'(p) > 0$  for enabling  $t \in p\bullet$ . But then:  
 $M = L + k_p p \xrightarrow{\sigma} L' + k_p p = M'$  with  $M'(p) = L'(p) + k_p > k_p!$ )

By the liveness theorem: some circuit  $\gamma$  is not marked at  $L$ .  
Since  $(N, M)$  is live, the circuit  $\gamma$  is marked at  $M \supset L$ .  
Since  $M - L = k_p p$ , the circuit  $\gamma$  contains  $p$  and  
 $M_0(\gamma) = M(\gamma) = M(p) = k_p \leq k$ .

# Place bounds in live T-systems

Let  $(P, T, F, M_0)$  be a **live** T-system.

We can draw some easy consequences of the above results:

- 1) If  $p \in P$  is bounded, then it belongs to some circuit.  
(see part  $\Rightarrow$  of the proof of the boundedness theorem)
- 2) If  $p \in P$  belongs to some circuit, then it is bounded.  
(by the fundamental property of T-systems)
- 3) If  $(N, M_0)$  is bounded, then it is strongly connected.  
(by strong connectedness theorem, holding for any system)
- 4) If  $N$  is strongly connected, then  $(N, M_0)$  is bounded.  
(by 1, since any  $p \in P$  belongs to a circuit by strong connectdness)

# Place bounds in live T-systems

Let  $(P, T, F, M_0)$  be a **live** T-system.

We can draw some easy consequences of the above results:

1+2)  $p \in P$  is bounded iff it belongs to some circuit.

3+4)  $(N, M_0)$  is bounded iff it is strongly connected.

# T-systems: recap

T-system +  $\gamma$  circuit + M reachable  $\Rightarrow M(\gamma) = M_0(\gamma)$

T-system +  $\gamma$  circuit +  $M(\gamma) \neq M_0(\gamma) \Rightarrow M$  not reachable

T-system +  $\gamma_1 \dots \gamma_n$  circuits:  $\exists i. p \in \gamma_i \Leftrightarrow p$  bounded

T-system:  $M_0(\gamma) > 0$  for all circuits  $\gamma \Leftrightarrow$  live

T-system: strongly connected  $\Rightarrow$  bounded

T-system + live: strongly connected  $\Leftrightarrow$  bounded

T-system: T-invariant  $\mathbf{J} \Leftrightarrow \mathbf{J} = [k \ k \ \dots \ k]$

# Consequences on workflow nets

**Theorem:** If  $N$  is a workflow net s.t.  $N^*$  is a T-system then  
 $N$  is safe and sound **iff**  
every circuit of  $N^*$  is marked

$N$  workflow net  $\Rightarrow N^*$  strongly connected

all circuits of  $N^*$  are marked  $\Leftrightarrow N^*$  live

$N^*$  strongly connected +  $N^*$  T-system  $\Rightarrow N^*$  bounded

$i \in \gamma \Leftrightarrow M_0(\gamma)=1 \Leftrightarrow \gamma$  marked circuit

$\gamma$  marked circuit +  $M$  reachable  $\Rightarrow M(\gamma)=1$

$N^*$  bounded  $\Leftrightarrow$  any place  $p$  belongs to a circuit of  $N^*$

$p$  belongs to a circuit of  $N^*$   $\Rightarrow p$  is safe

all places belong to marked circuits  $\Rightarrow N^*$  safe  $\Rightarrow N$  safe



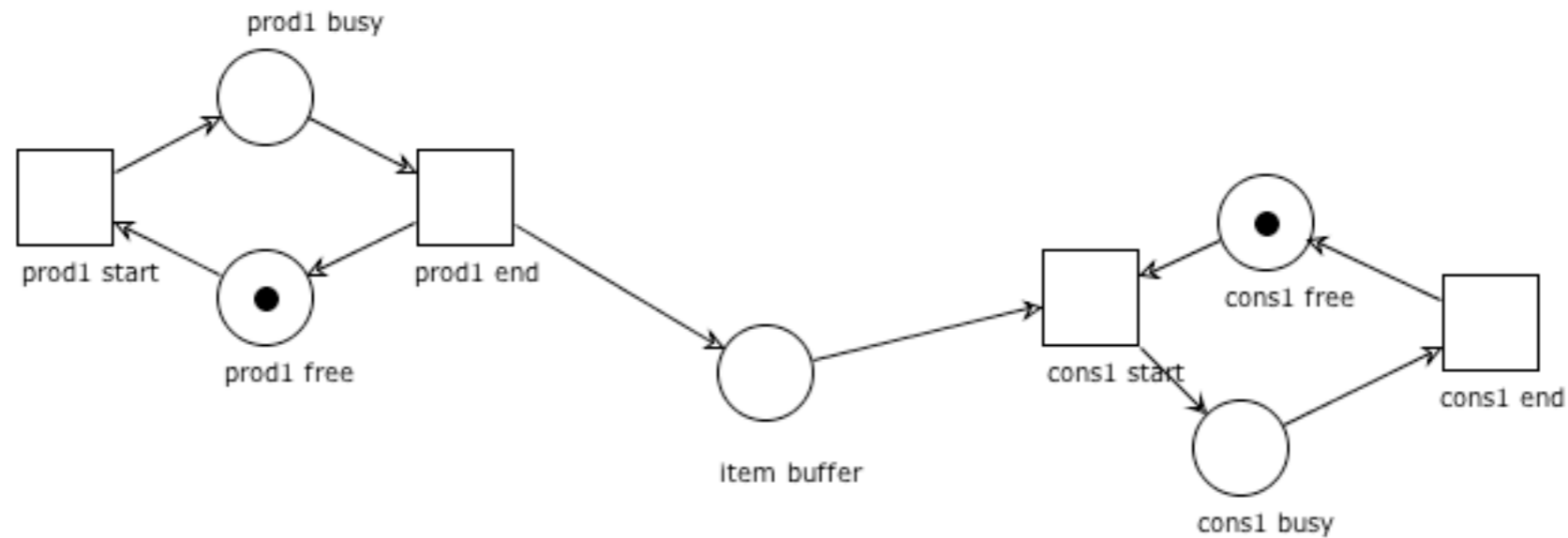
# Exercises

Which are the circuits of the T-system below?

Is the T-system below live? (why?)

Which places are bounded? (why?)

Assign a bound to each bounded place.



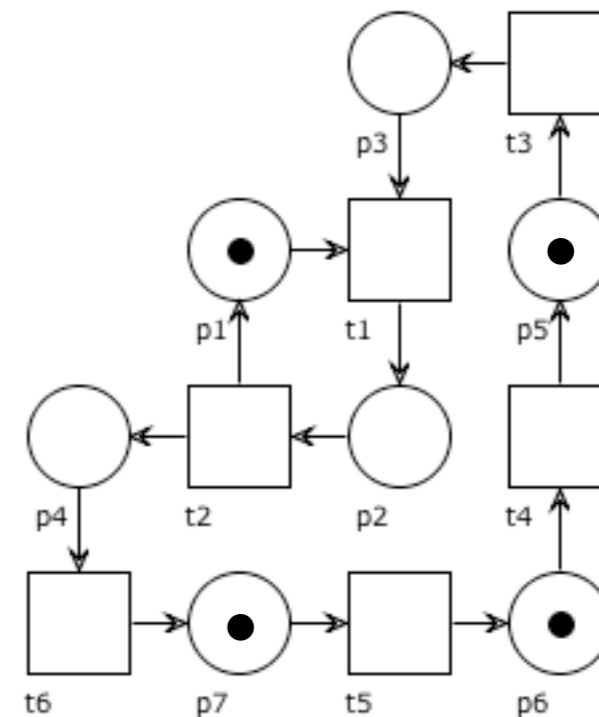
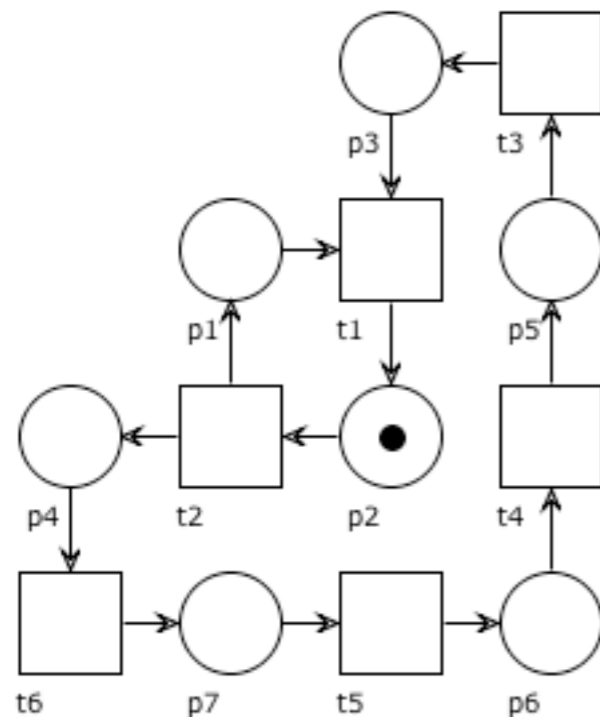
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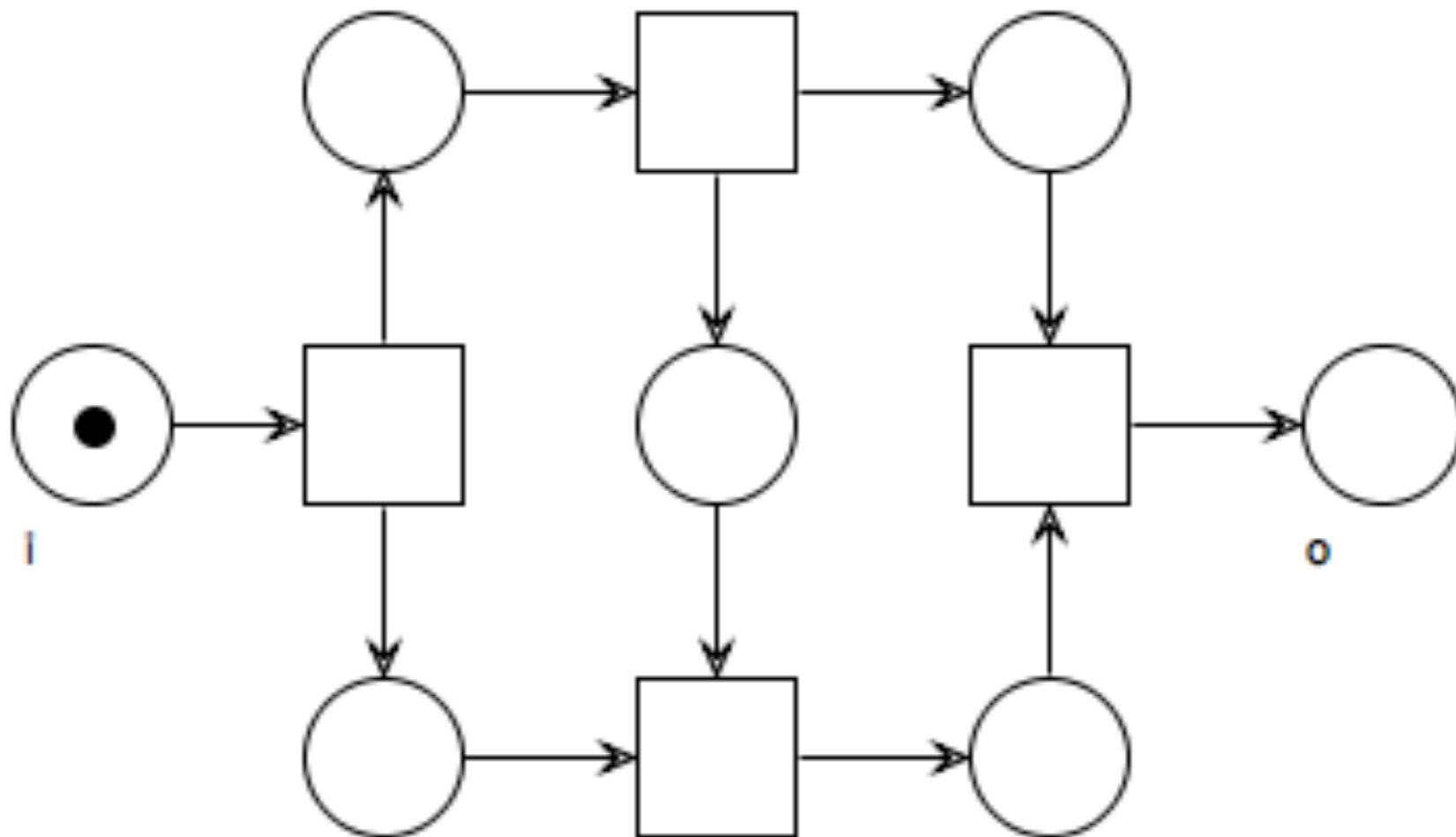
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# Exercise

Is the net below a workflow net?  
Is it sound?



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