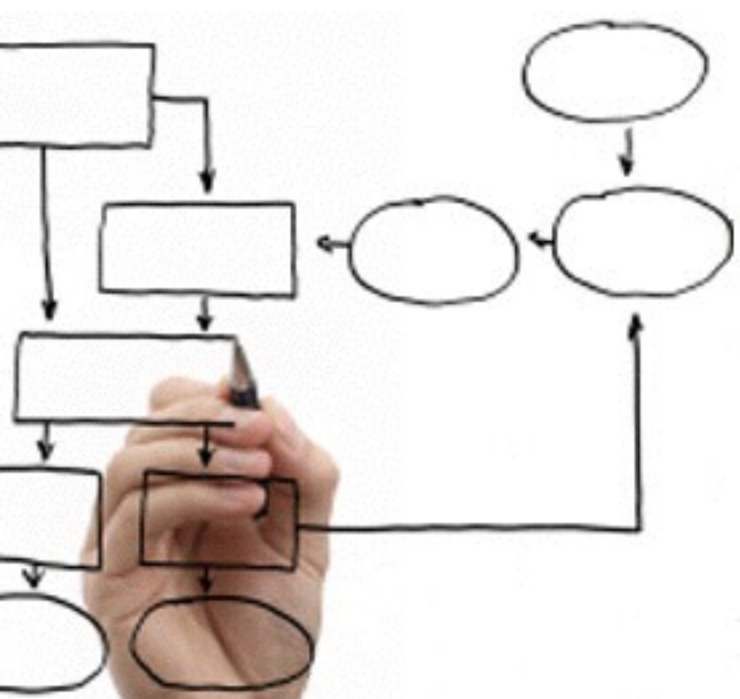
Methods for the specification and verification of business processes

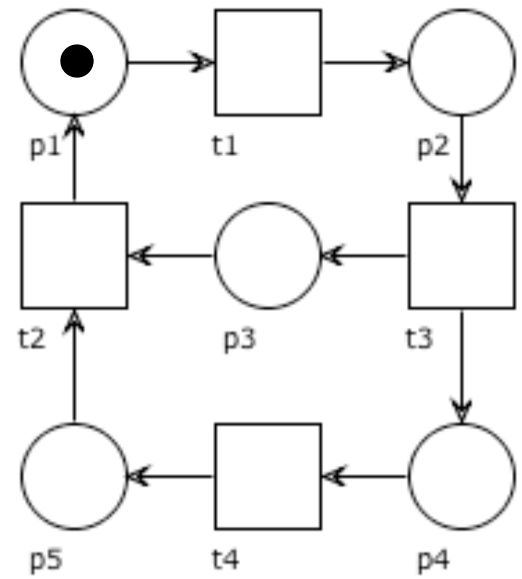
MPB (6 cfu, 295AA)

Roberto Bruni
http://www.di.unipi.it/~bruni

17 - T-systems



Object



We study some “good” properties of T-systems

Free Choice Nets (book, optional reading)
https://www7.in.tum.de/~esparza/bookfc.html

T-systems

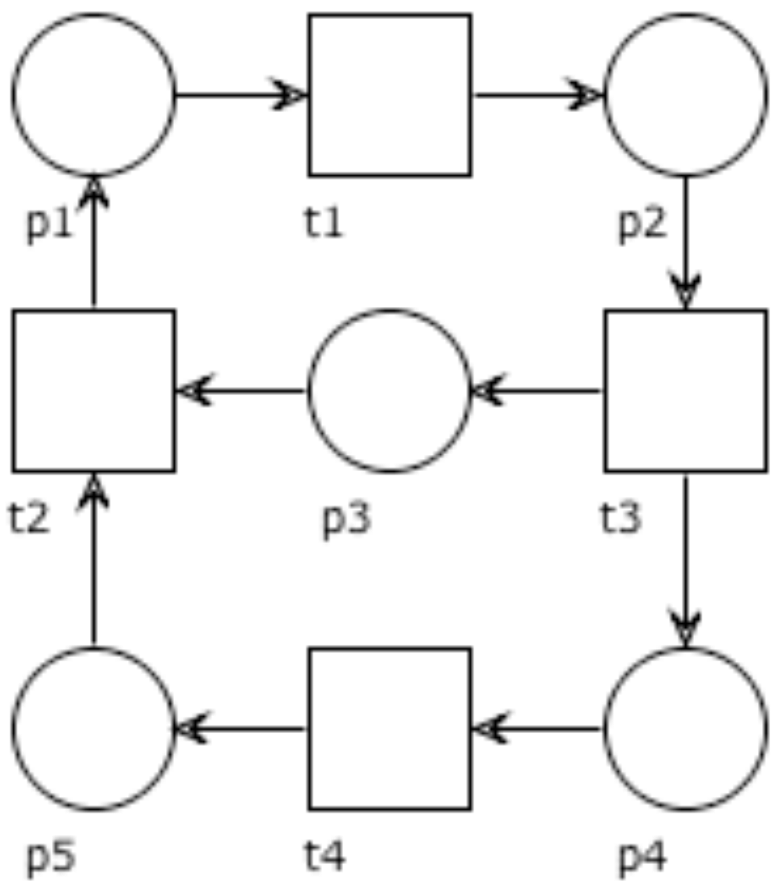
T-system

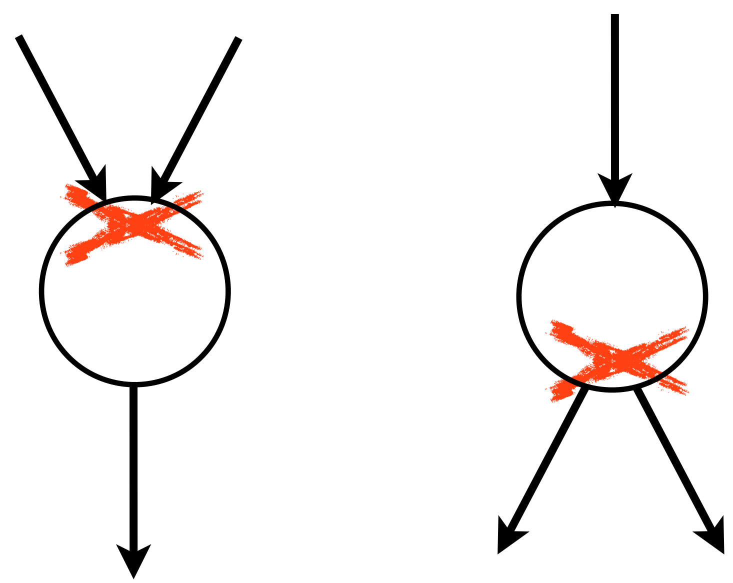
Definition: We recall that a net N is a **T-net** if each place has exactly one input transition and exactly one output transition

$$\forall p \in P, \qquad |\bullet p| = 1 = |p \bullet|$$

A system (N,M_0) is a **T-system** if N is a T-net

T-system: example





T-systems: an observation

Notably, computation in T-systems is concurrent, but essentially deterministic:

the firing of a transition t in M cannot disable another transition t' enabled at M

T-net N*

Is it true that: A workflow net N is a T-net iff N* is a T-net ?

T-net N*

Is the following conjecture true?
A workflow net N is a T-net
iff N* is a T-net

No, N can never be a T-net because
the place i has no incoming arc
and the place o has no outgoing arc

(N* can be a T-net)

T-systems: another observation

Determination of control:

the transitions responsible for enabling t are one for each input place of t

Notation: token count of a circuit

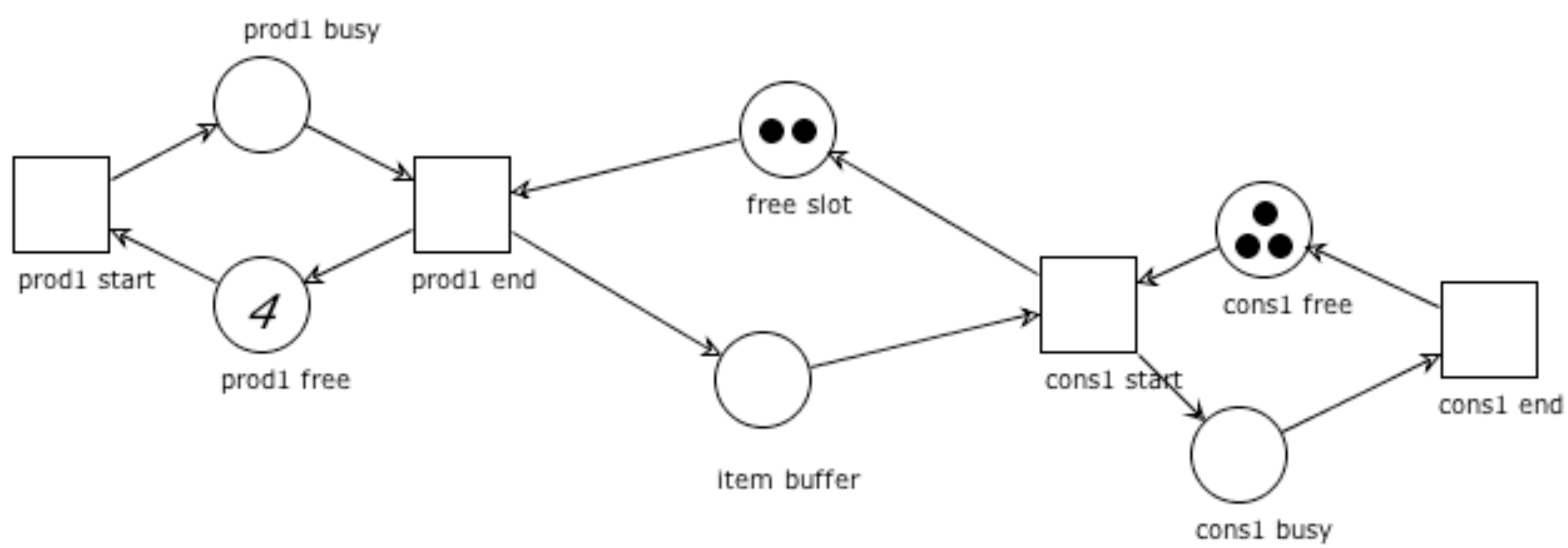
Let $\gamma = (x_1, y_1)(y_1, x_2)(x_2, y_2)...(x_n, y_n)$ be a circuit.

Let $P_{|\gamma} \subseteq P$ be the set of places in γ.

$$M(\gamma) = M(P_{|\gamma}) = \sum_{p \in P_{|\gamma}} M(p)$$

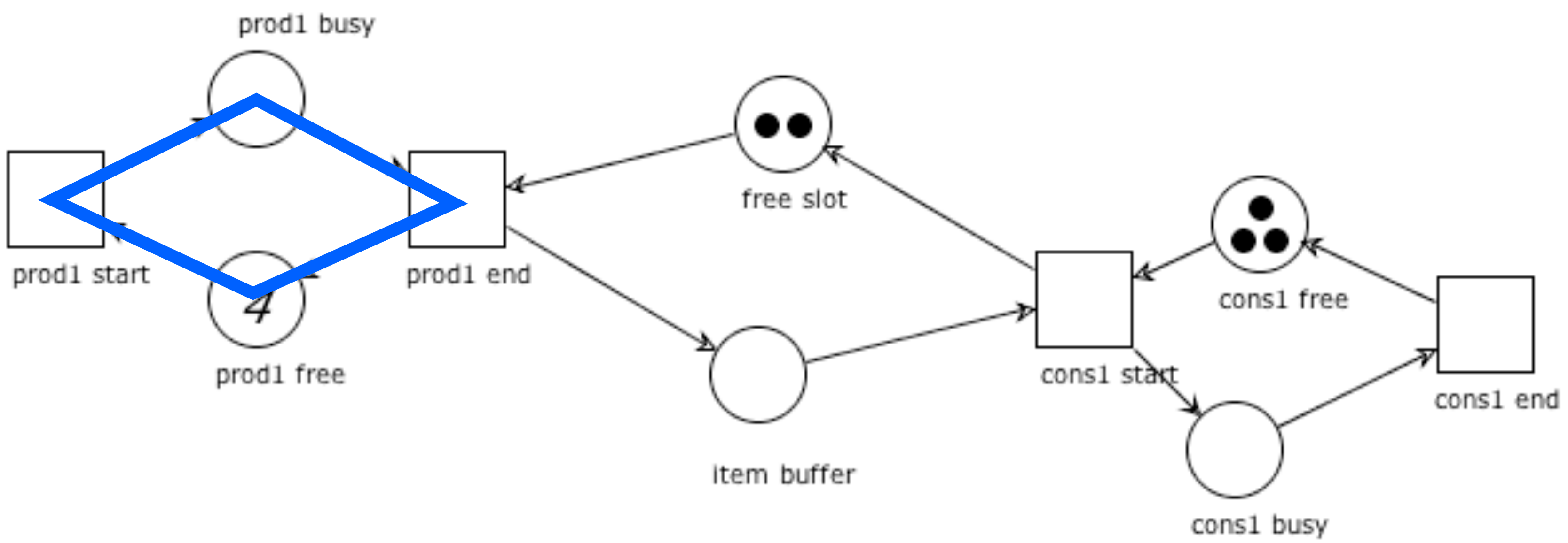
We say that γ is **marked at** M if $M(\gamma) > 0$

Example



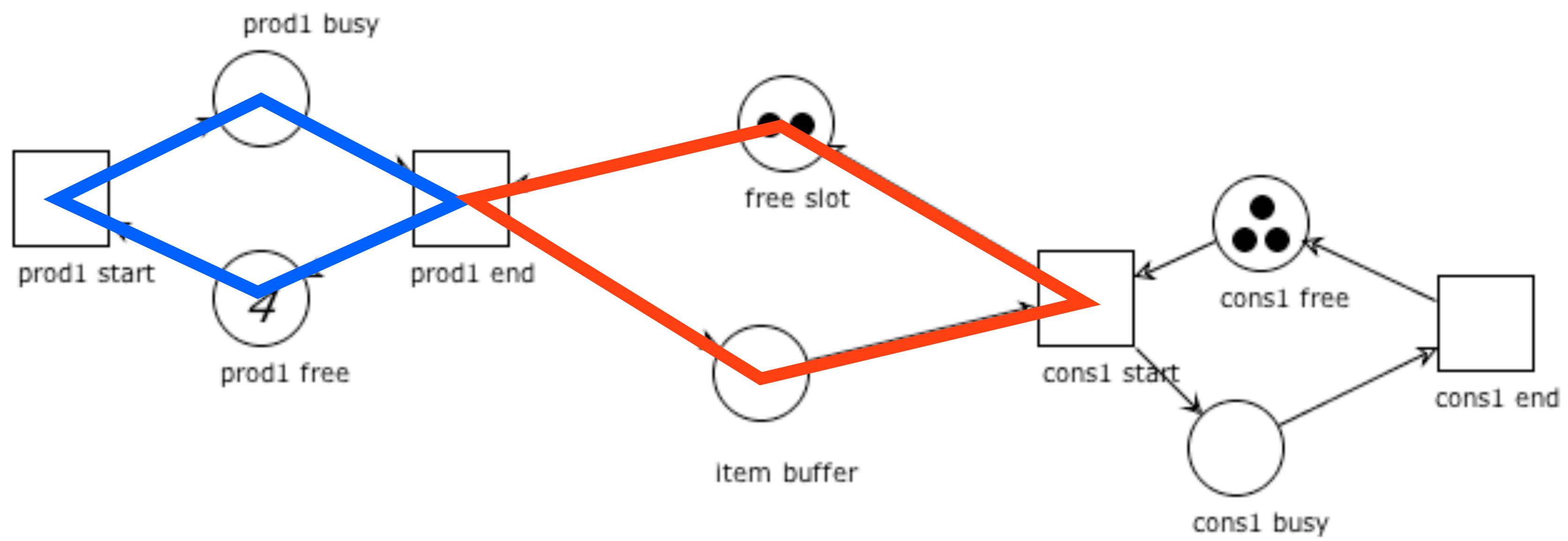
prod1 busy
prod1 start
4
prod1 free
prod1 end
free slot
item buffer
cons1 start
cons1 free
cons1 busy
cons1 end

Example



$$M(\gamma_1) = 4$$

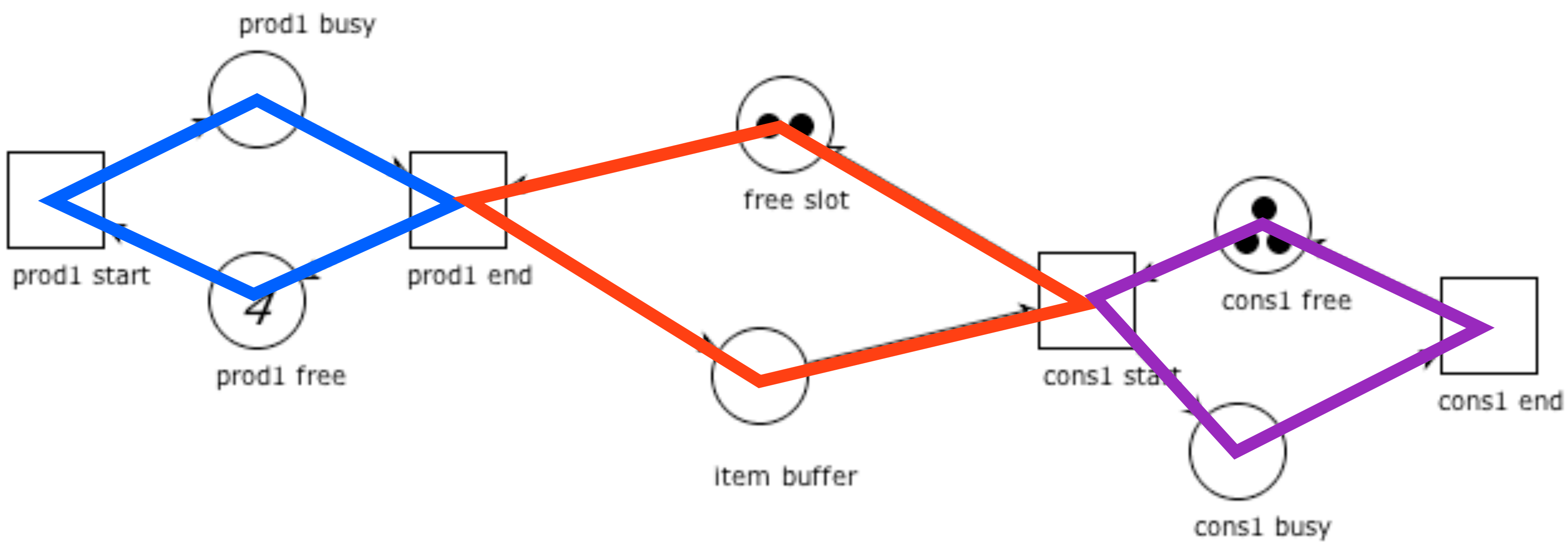
Example



$$M(\gamma_1) = 4$$

$$M(\gamma_2) = 2$$

Example



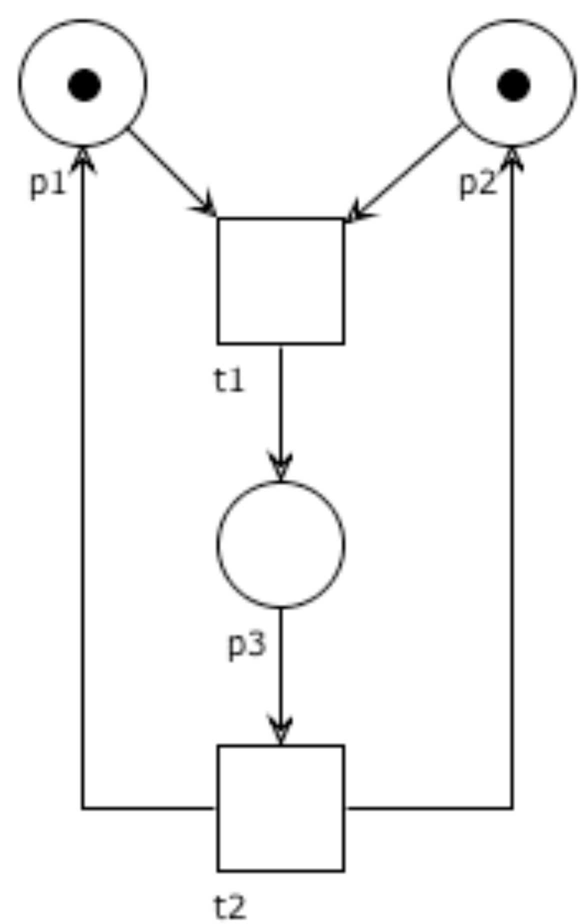
$$M(\gamma_1) = 4$$

$$M(\gamma_2) = 2$$

$$M(\gamma_3) = 3$$

Question time

Trace two circuits over the T-system below



Fundamental property of T-systems

The token count of a circuit is invariant under any firing.

Fundamental property of T-systems

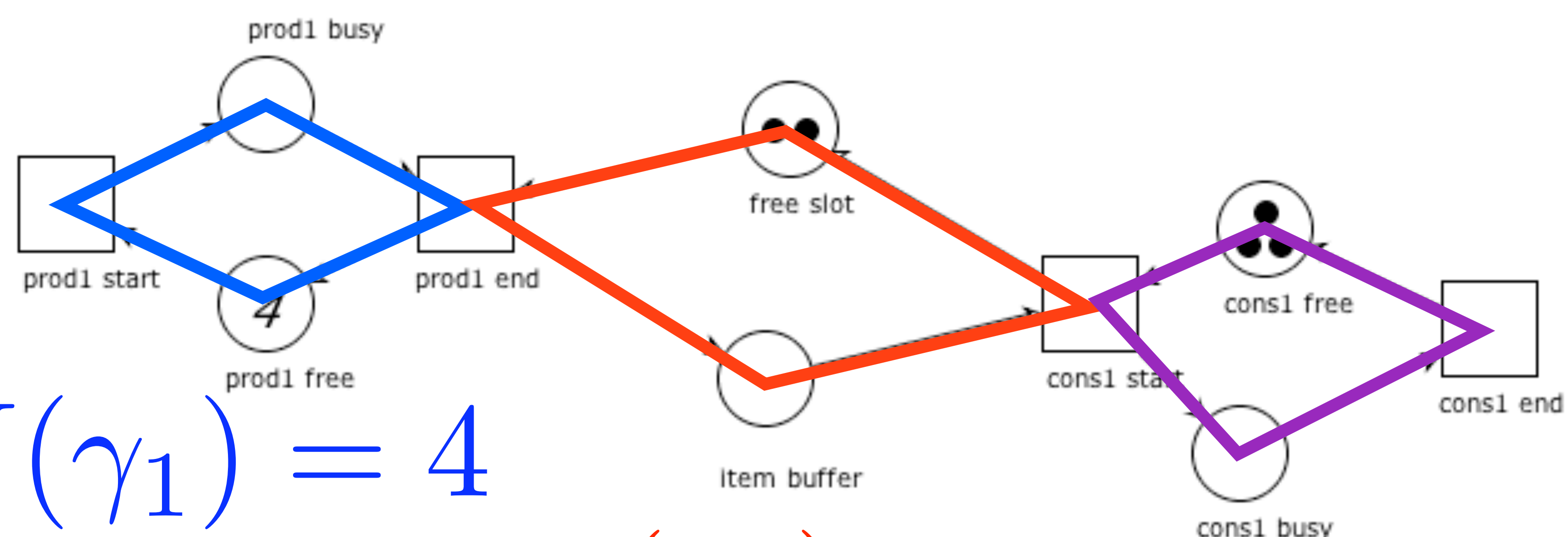
Proposition: Let γ be a circuit of a T-system (P, T, F, M_0).
If M is a reachable marking, then $M(\gamma) = M_0(\gamma)$

Take any $t \in T$: either $t \notin \gamma$ or $t \in \gamma$.

If $t \notin \gamma$, then no place in $\bullet t \cup t \bullet$ is in γ
(otherwise, by definition of T-nets, t would be in γ).
Then, an occurrence of t does not change the token count of γ.

If $t \in \gamma$, then exactly one place in $\bullet t$ and one place in $t \bullet$ are in γ.
Then, an occurrence of t does not change the token count of γ.

Example



$$M(\gamma_1) = 4$$

$$M(\gamma_2) = 2$$

$$M(\gamma_3) = 3$$

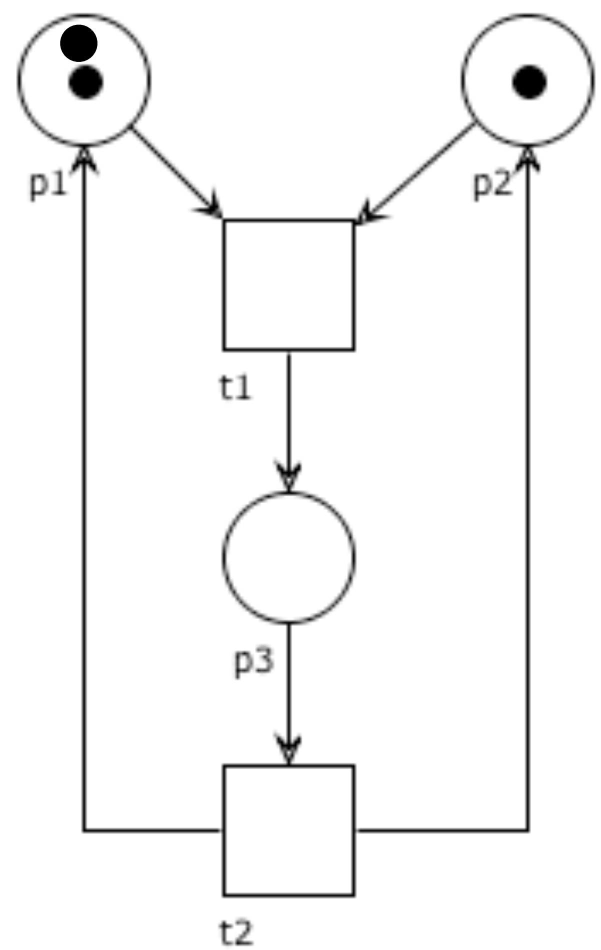
$M_0 = [\ 0\ \ 4\ \ 2\ \ 0\ \ 3\ \ 0\ ]$

$M = [\ 2\ \ 2\ \ 1\ \ 2\ \ 2\ \ 1\ ]$

$M' = [\ 2\ \ 1\ \ 1\ \ 1\ \ 2\ \ 2\ ]$

Question time

Is the marking $p_1 + 2p_2$ reachable? (why?)



T-invariants of T-nets

Proposition: Let N=(P,T,F) be a connected T-net.
J is a rational-valued T-invariant of N iff **J**=[x ... x]
for some rational value x

(the proof is dual to the analogous proposition for
S-invariants of S-nets)

Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live iff every circuit of N is marked at M_0

$\Rightarrow$) (quite obvious)
By contradiction, let γ be a circuit with $M_0(\gamma) = 0$.
By the fundamental property of T-systems: $\forall M \in [\, M_0 \,\rangle,\ M(\gamma) = 0$.

Take any $t \in T_{|\gamma}$ and $p \in P_{|\gamma} \cap \bullet t$.

For any $M \in [\, M_0 \,\rangle$, we have $M(p) = 0$.
Hence t is never enabled and the T-system is not live.

Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live iff every circuit of N is marked at M_0

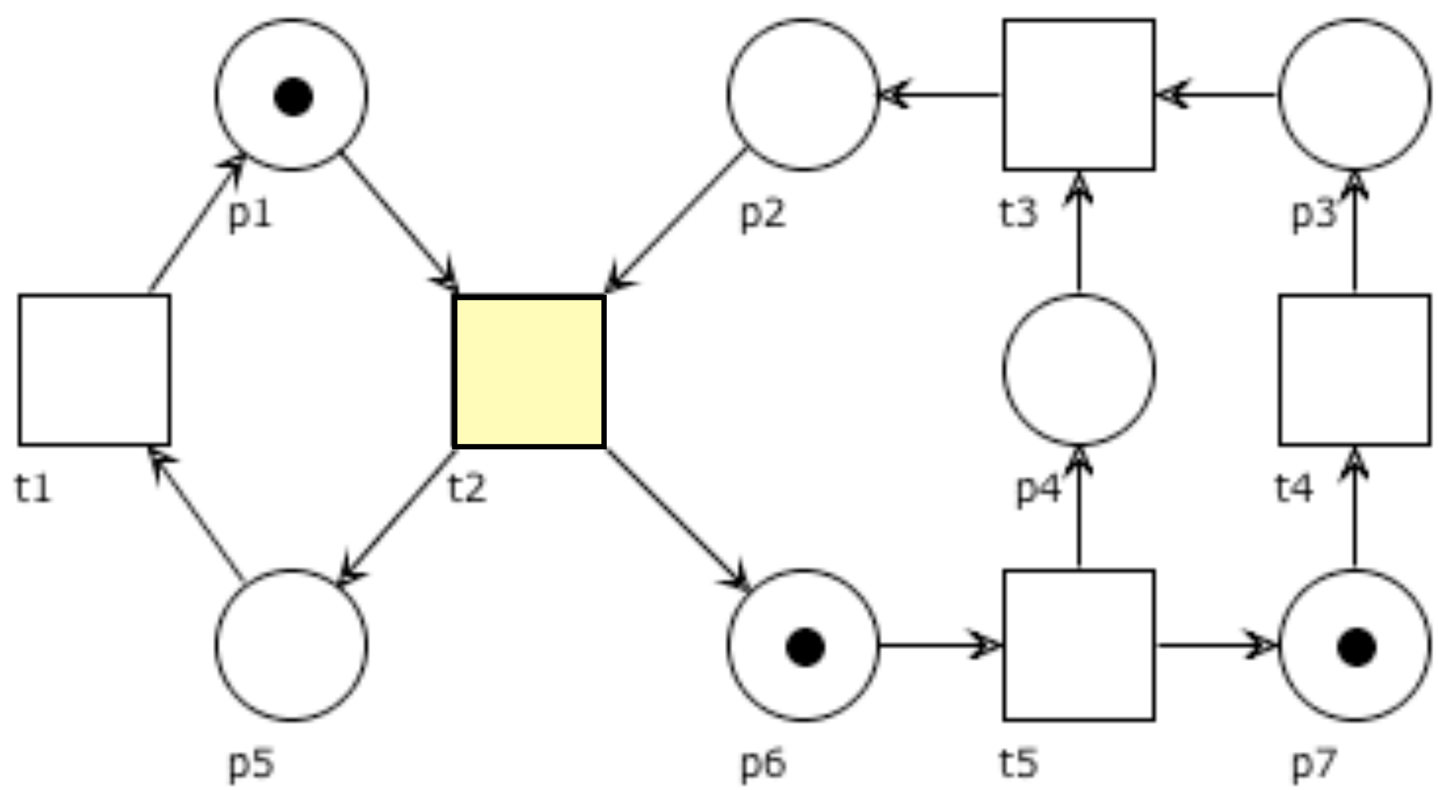
$\Leftarrow$) (more involved)
Take any $t \in T$ and $M \in [\, M_0 \,\rangle$.
We need to show that some marking M' reachable from M enables t.

The key idea is to collect the places that control the firing of t:
$p \in P_{M,t}$ if there is a path from p to t through places unmarked at M.
We then proceed by induction on the size of $P_{M,t}$.

We just sketch the key idea of the proof over a T-system.

Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0

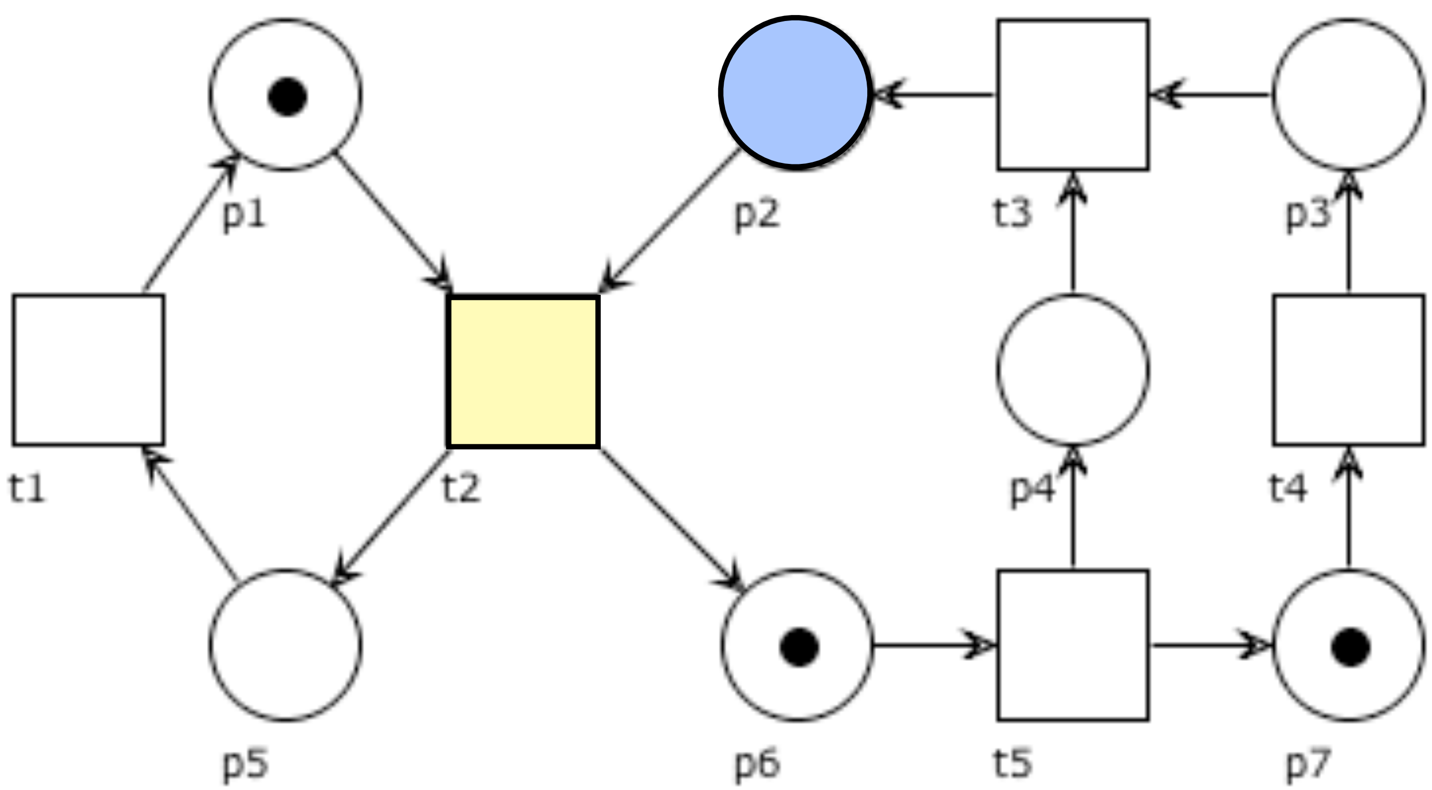


$M = p_1 + p_6 + p_7$

M' enabling t_2?

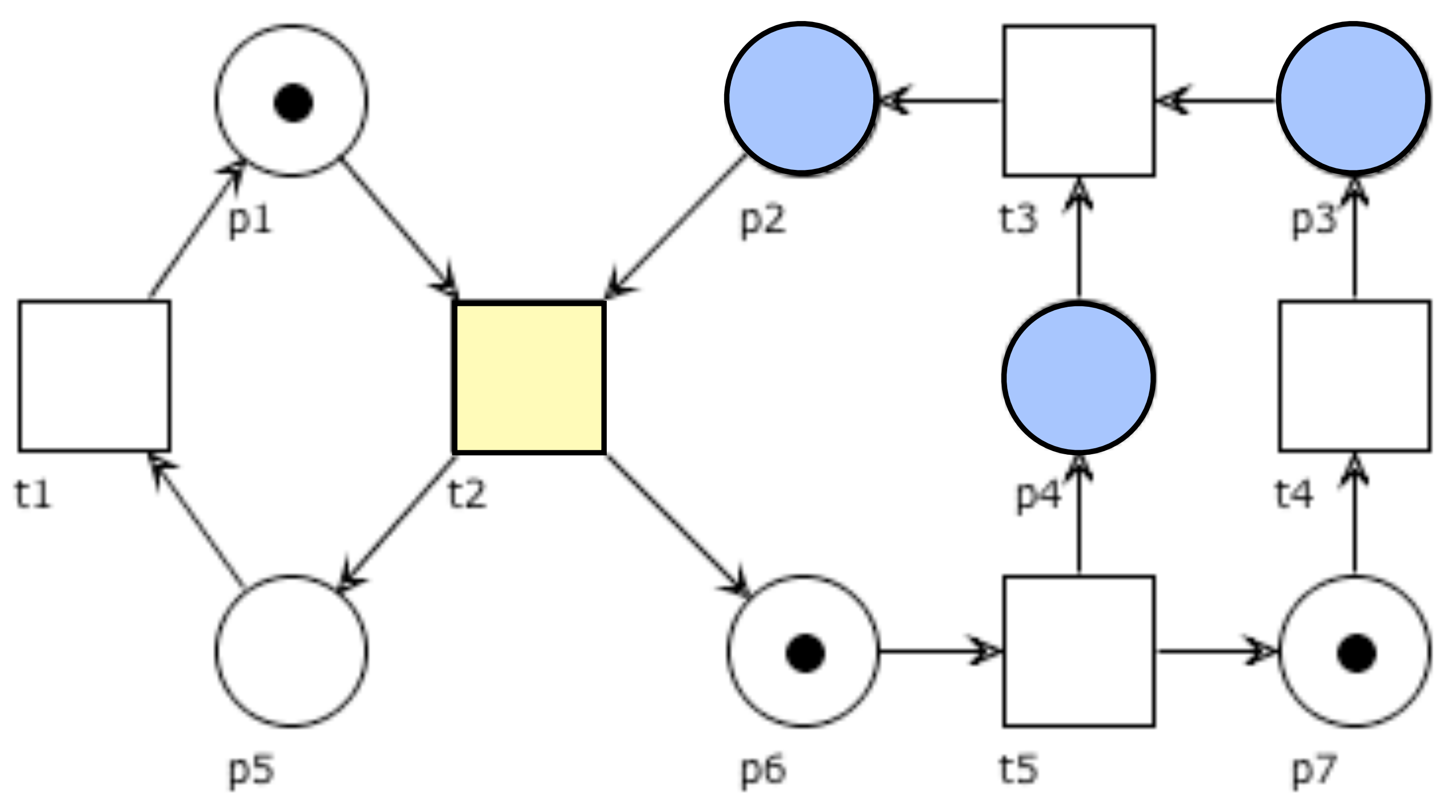
Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0



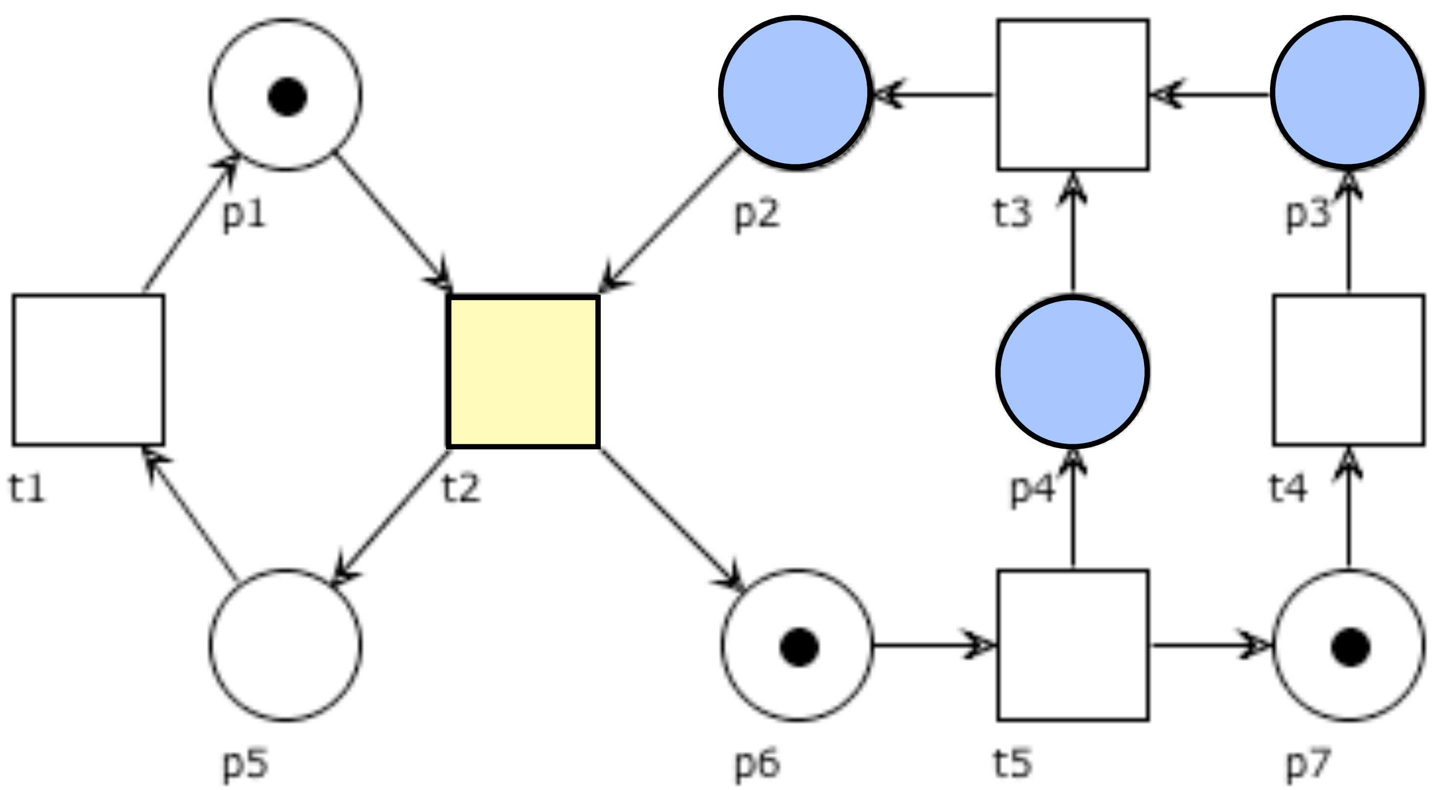
Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0



Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0



$P_{M,t_2} = \{ p_2, p_3, p_4 \}$

Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0

$\Leftarrow$) (continued proof sketch)

Base case: $|P_{M,t}| = 0$.

Every place in $\bullet t$ is already marked at M.

Hence t is enabled at M.

Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0

$\Leftarrow$) (continued proof sketch)

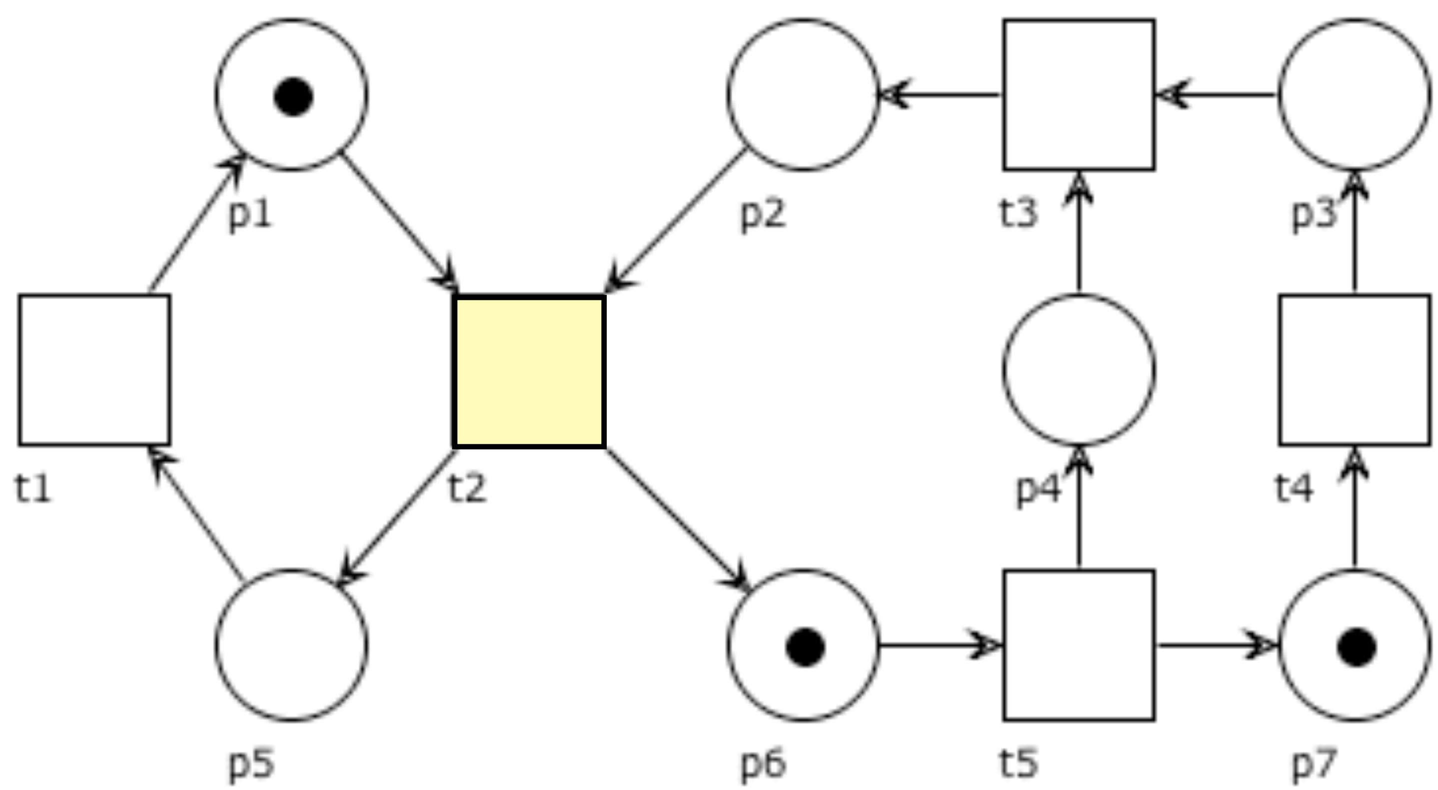
Inductive case: $|P_{M,t}| > 0$.
Therefore t is not enabled at M.

We look for a path π of maximal length necessary for firing t.
π must contain only places unmarked at M.

By the fundamental property of T-systems: all circuits are marked at M.
π is not necessarily unique, but exists (no cycle in it).

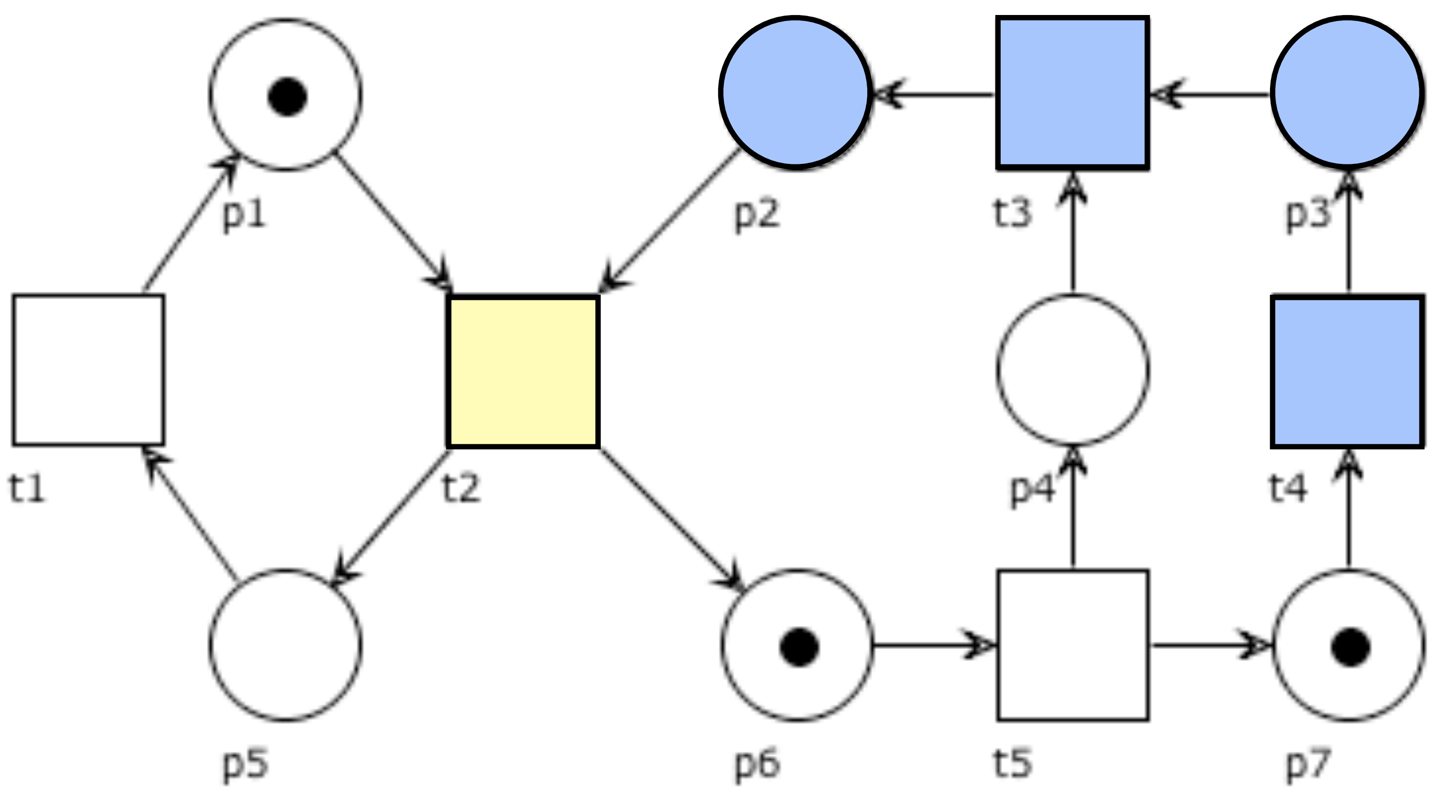
Liveness theorem for T-systems

Theorem: A T-system (N, M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0



Liveness theorem for T-systems

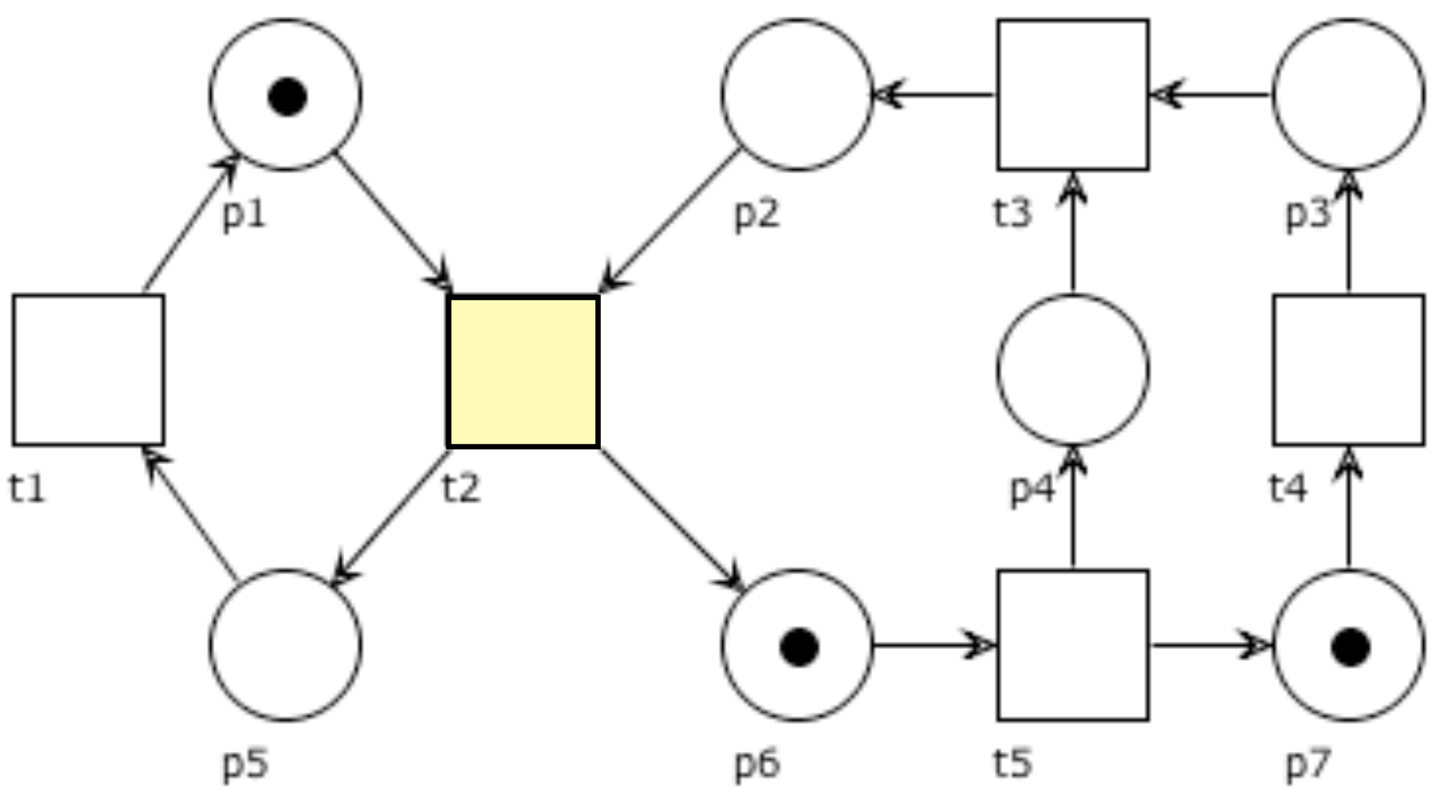
Theorem: A T-system (N, M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0



$$\pi \;=\; t_4\, p_3\, t_3\, p_2\, t_2$$

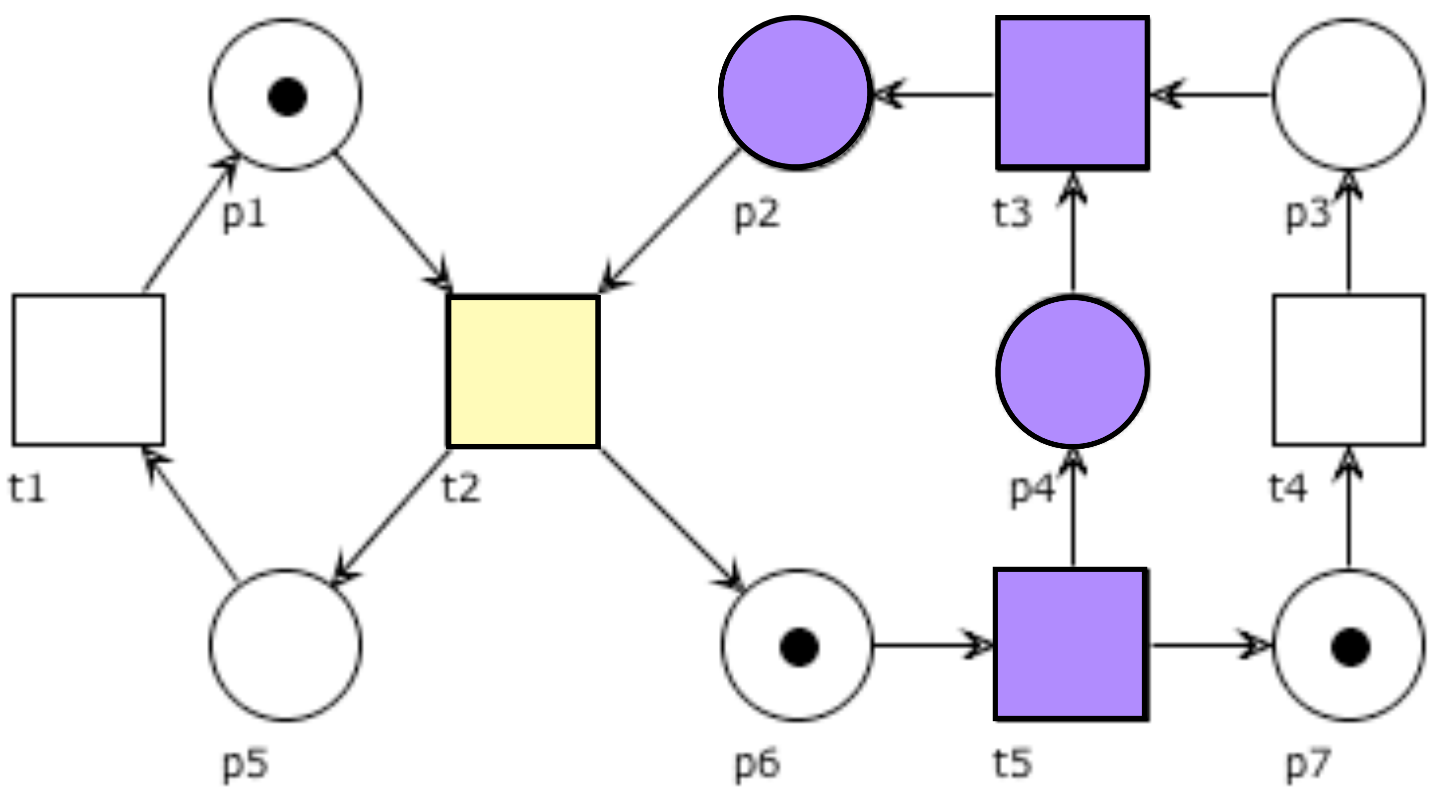
Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0



Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0



$$\pi = t_5\, p_4\, t_3\, p_2\, t_2$$

Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0

$\Leftarrow$) (Inductive case: $|P_{M,t}| > 0$, continued proof sketch)

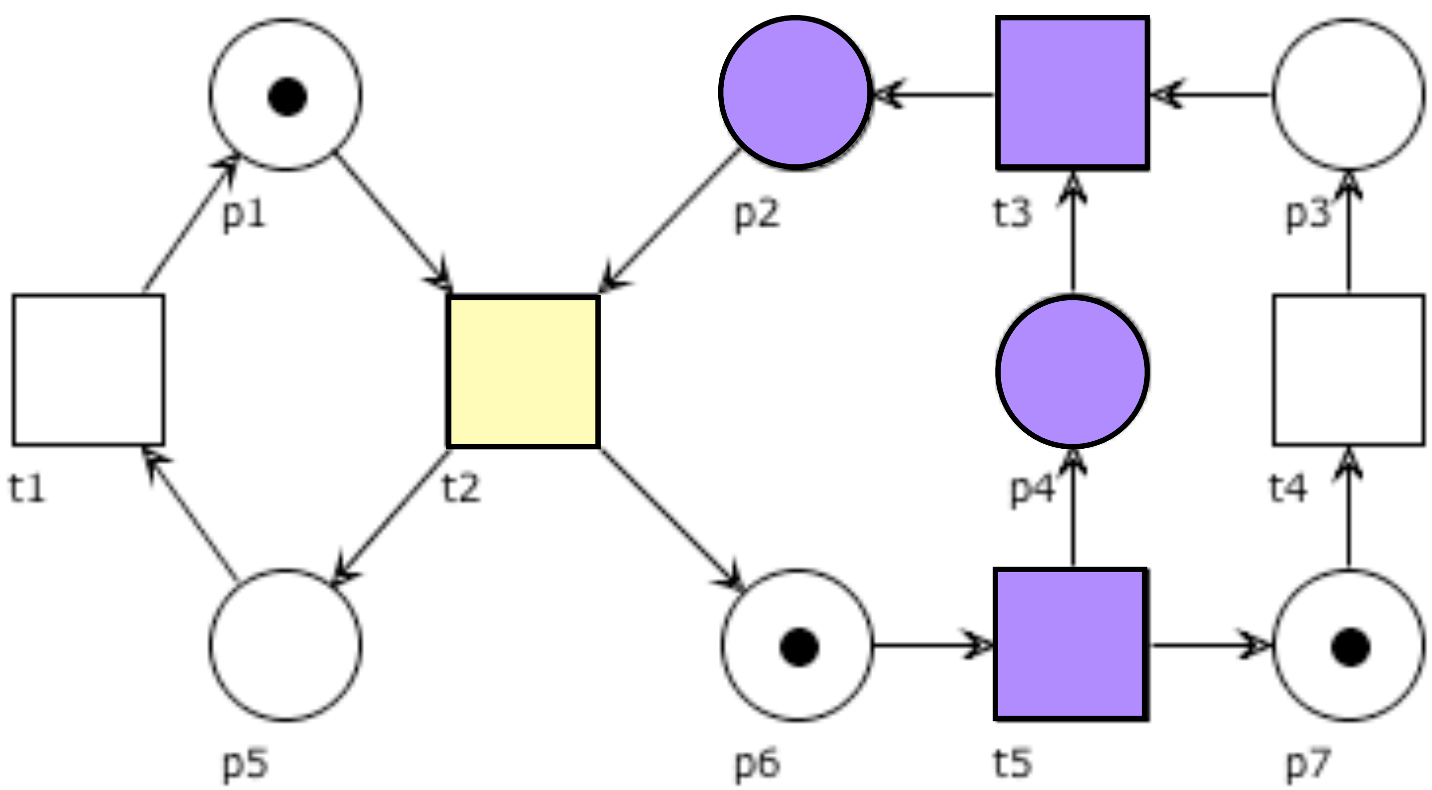
π begins with a transition t' enabled at M.
(otherwise a longer path could be found).

By firing t' we reach a marking M'' such that $P_{M'',t} \subset P_{M,t}$.

Hence $|P_{M'',t}| < |P_{M,t}|$ and we conclude by inductive hypothesis.

Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0

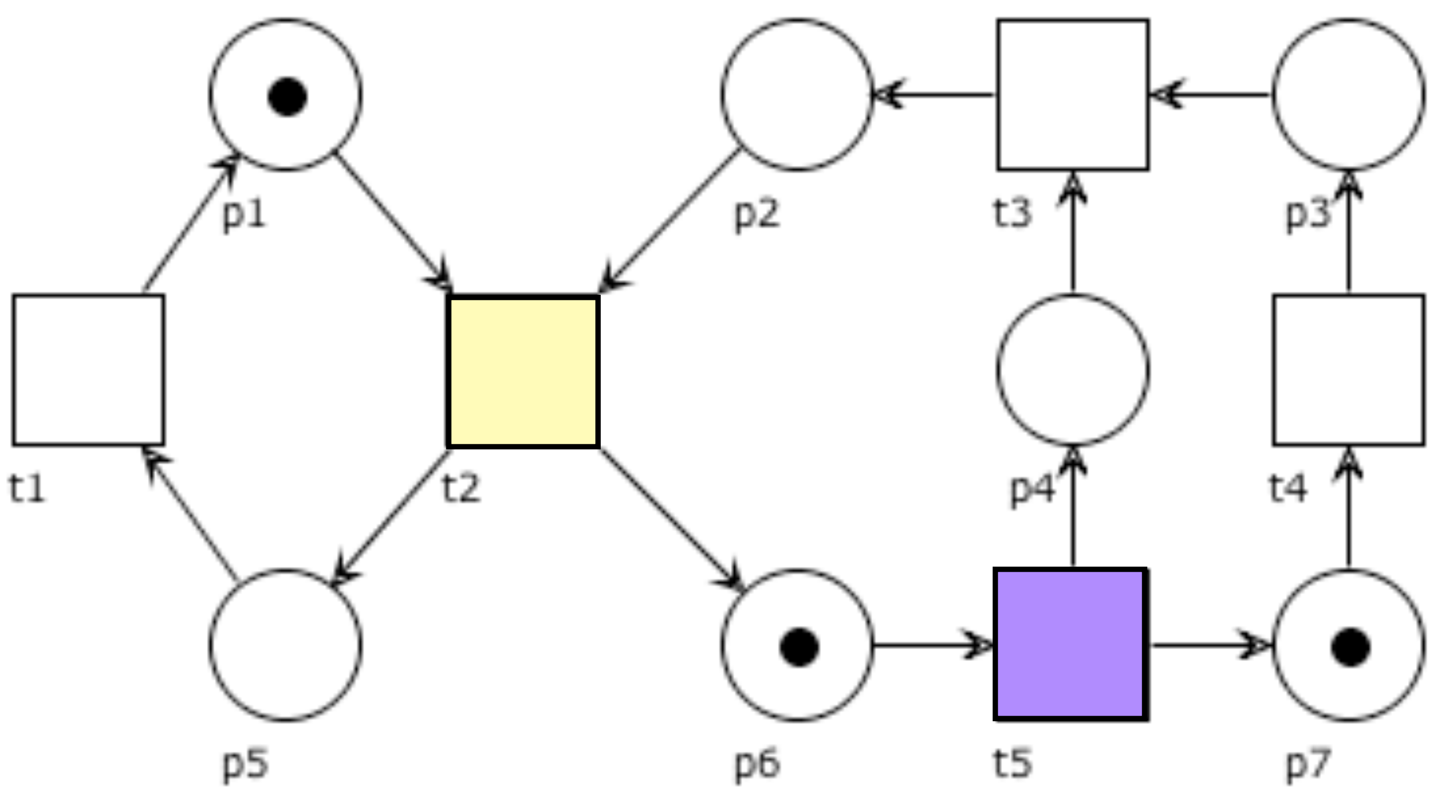


$P_{M,t_2} = \{ p_2, p_3, p_4 \}$

$$\pi = t_5\, p_4\, t_3\, p_2\, t_2$$

Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0

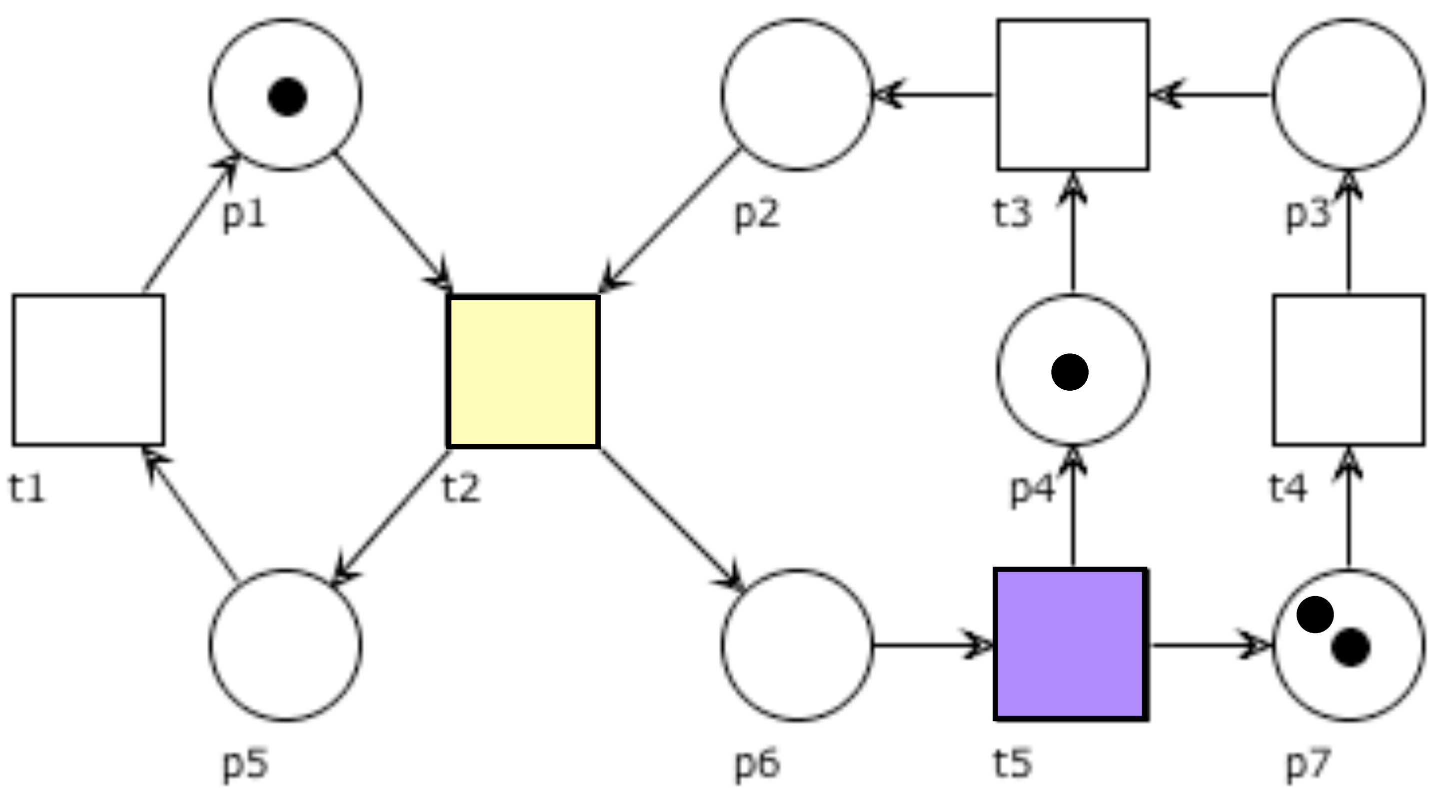


$P_{M,t_2} = \{ p_2, p_3, p_4 \}$

$$\pi = t_5\, p_4\, t_3\, p_2\, t_2$$

Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0

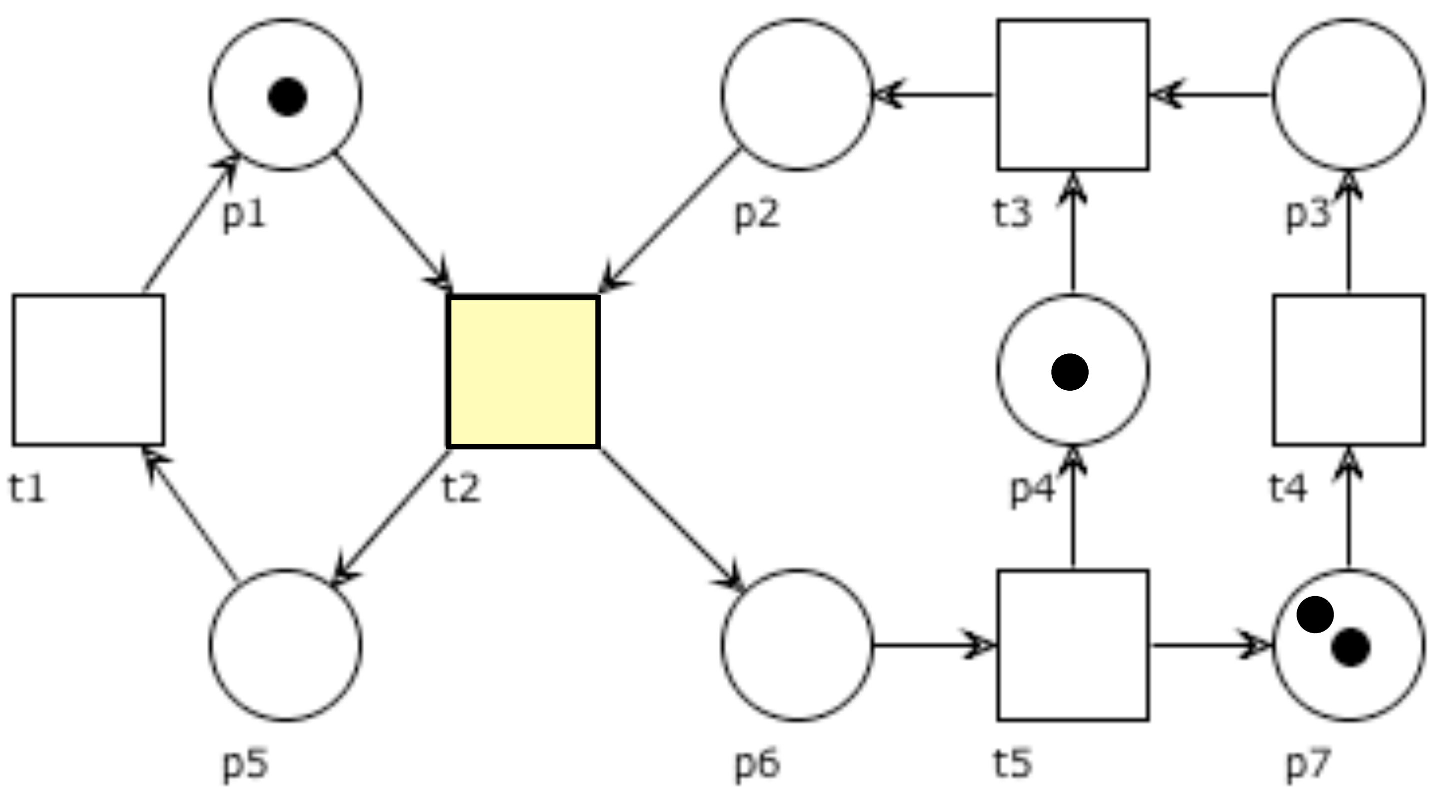


$P_{M,t_2} = \{ p_2, p_3, p_4 \}$

$$\pi \; = \; t_5 \, p_4 \, t_3 \, p_2 \, t_2$$

Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0

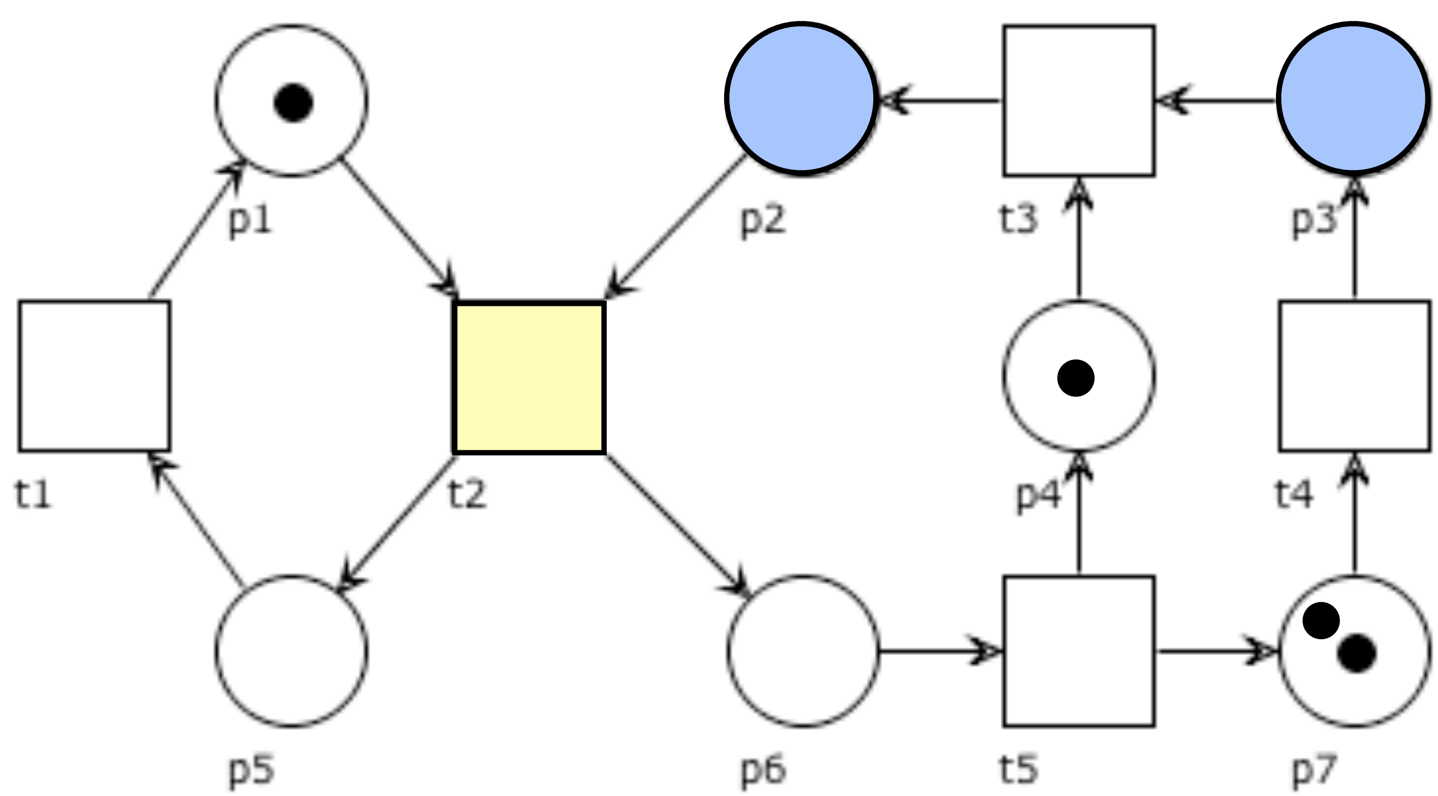


$P_{M,t_2} = \{ p_2, p_3, p_4 \}$

$P_{M'',t_2} = \{ p_2, p_3 \}$

Liveness theorem for T-systems

Theorem: A T-system (N,M_0) is live
$\Leftarrow$ every circuit of N is marked at M_0

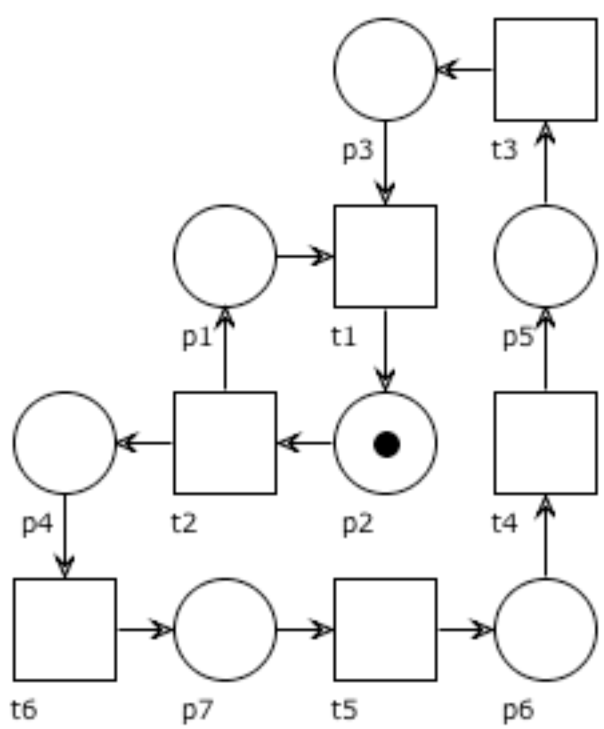


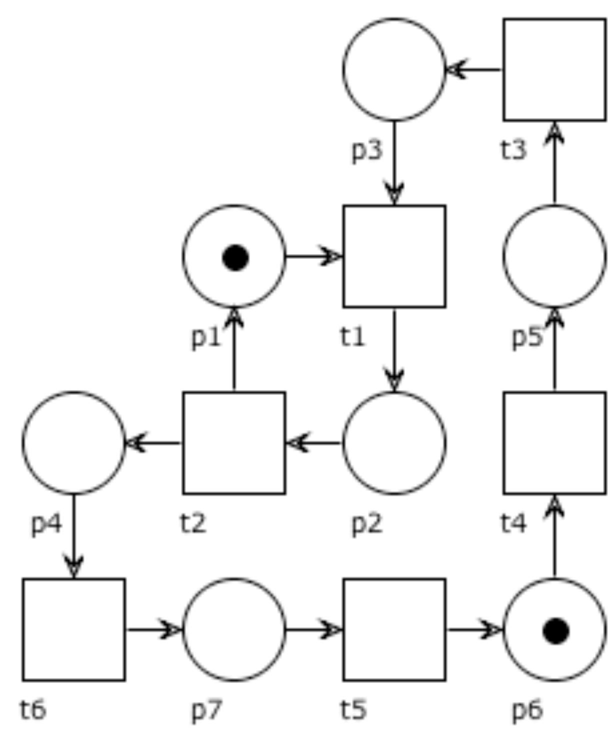
$P_{M,t_2} = \{ p_2, p_3, p_4 \}$

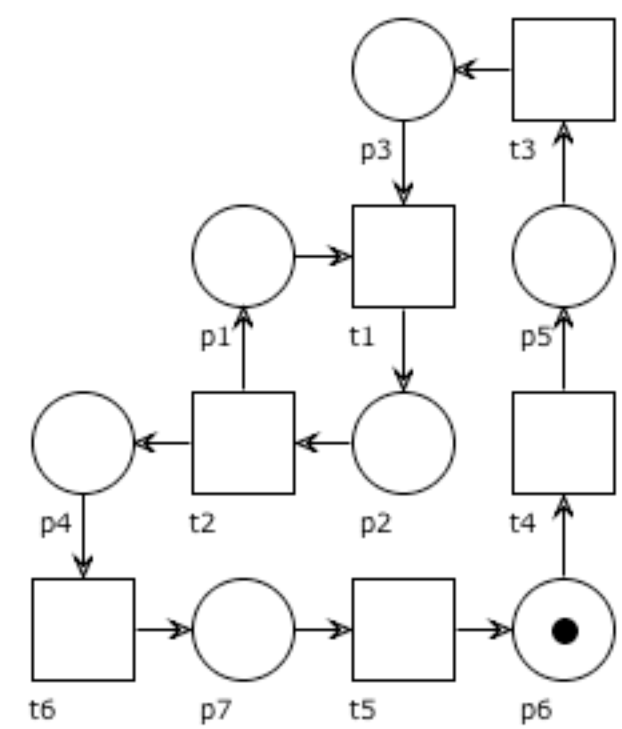
$P_{M'',t_2} = \{ p_2, p_3 \}$

Question time

Which of the T-systems below is live? (why?)







Boundedness theorem for live T-systems

Theorem: A live T-system (P, T, F, M_0) is k-bounded iff every place $p \in P$ belongs to a circuit γ_p with $M_0(\gamma_p) \leq k$.

$\Leftarrow$) Let $M \in [\, M_0 \,\rangle$ and take any $p \in P$.

By the fundamental property of T-systems:
$M(p) \leq M(\gamma_p) = M_0(\gamma_p) \leq k$

Boundedness theorem for live T-systems

Theorem: A live T-system (P, T, F, M_0) is k-bounded iff every place $p \in P$ belongs to a circuit γ_p with $M_0(\gamma_p) \leq k$.

$\Rightarrow$) Let $k_p \leq k$ be the bound of p.
Take $M \in [\, M_0 \,\rangle$ with $M(p) = k_p$.

Define $L = M - k_p p$ and note that the T-system (N, L) is not live.
(otherwise $L \xrightarrow{\sigma} L'$ with $L'(p) > 0$ for enabling $t \in p\bullet$. But then:
$M = L + k_p p \xrightarrow{\sigma} L' + k_p p = M'$ with $M'(p) = L'(p) + k_p > k_p$!)

By the liveness theorem: some circuit γ is not marked at L.
Since (N, M) is live, the circuit γ is marked at $M \supset L$.
Since $M - L = k_p p$, the circuit γ contains p and
$M_0(\gamma) = M(\gamma) = M(p) = k_p \leq k$.

Boundedness in strongly connected T-systems

Lemma: If a T-system (N, M_0) is strongly connected, then it is bounded

Let Γ be the set of the circuits of N and let $k = \max_{\gamma \in \Gamma} M_0(\gamma)$.

Since N is strongly connected, every place p belongs to some circuit γ_p.

By the fundamental property of T-systems: token count of γ_p is invariant.

Thus, for any reachable marking M, we have $M(p) \leq M(\gamma_p) = M_0(\gamma_p) \leq k$.
Hence the net is k-bounded.

Liveness in strongly connected T-systems

Lemma: If a T-system (N,M_0) is strongly connected, then it is live iff it is deadlock-free iff it has an infinite run

$\Longrightarrow$ $\Longrightarrow$

It is obvious that (for any net):
Liveness implies deadlock freedom.
Deadlock freedom implies the existence of an infinite run.

We show that (for strongly connected T-systems):
The existence of an infinite run implies liveness.

Liveness in strongly connected T-systems

Lemma: Let (N,M_0) be a strongly connected T-system. If it has an infinite run σ, then it is live

Since the T-system is strongly connected then it is bounded.

By the Reproduction lemma (holding for any bounded net):
There is a semi-positive T-invariant $\mathbf{J}$.
The support of $\mathbf{J}$ is included in the set of transitions of the infinite run σ.

By T-invariance in T-systems: $\langle\, \mathbf{J} \,\rangle = T$
(σ is an infinite run that contains all transitions).

Hence every transition can occur from M_0.
Hence every place can become marked.
Hence every circuit can become marked.

By the fundamental property of T-systems: every circuit is marked at M_0.

By the liveness theorem, (N, M_0) is live.

Place bounds in live T-systems

Let (P, T, F, M_0) be a **live** T-system.
We can draw some easy consequences of the above results:

1) If $p \in P$ is bounded, then it belongs to some circuit.
(see part $\Rightarrow$ of the proof of the boundedness theorem)

2) If $p \in P$ belongs to some circuit, then it is bounded.
(by the fundamental property of T-systems)

3) If (N, M_0) is bounded, then it is strongly connected.
(by strong connectedness theorem, holding for any system)

4) If N is strongly connected, then (N, M_0) is bounded.
(by 1, since any $p \in P$ belongs to a circuit by strong connectdness)

Place bounds in live T-systems

Let (P, T, F, M_0) be a live T-system.
We can draw some easy consequences of the above results:

1+2) $p \in P$ is bounded iff it belongs to some circuit.

3+4) (N, M_0) is bounded iff it is strongly connected.

T-systems: recap

T-system + γ circuit + M reachable => $M(\gamma) = M_0(\gamma)$

T-system + γ circuit + $M(\gamma) \neq M_0(\gamma)$ => M not reachable

T-system + $\gamma_1 \ldots \gamma_n$ circuits: $\exists i.\ p \in \gamma_i$ <=> p bounded

T-system: $M_0(\gamma)>0$ for all circuits γ <=> live

T-system: strongly connected => bounded

T-system + live: strongly connected <=> bounded

T-system + str. conn.: deadlock-free <=> live

T-system + str. conn.: infinite run <=> live

T-system: T-invariant **J** <=> **J** = [x x ... x]

Consequences on workflow nets

Theorem: If N is a workflow net s.t. N* is a T-system then
N is safe and sound iff
every circuit of N* is marked

N workflow net => N* strong connected
N* strong connected + N* T-system => N* bounded
$M_0(\gamma)>0$ for all circuits γ of N* <=> N* live

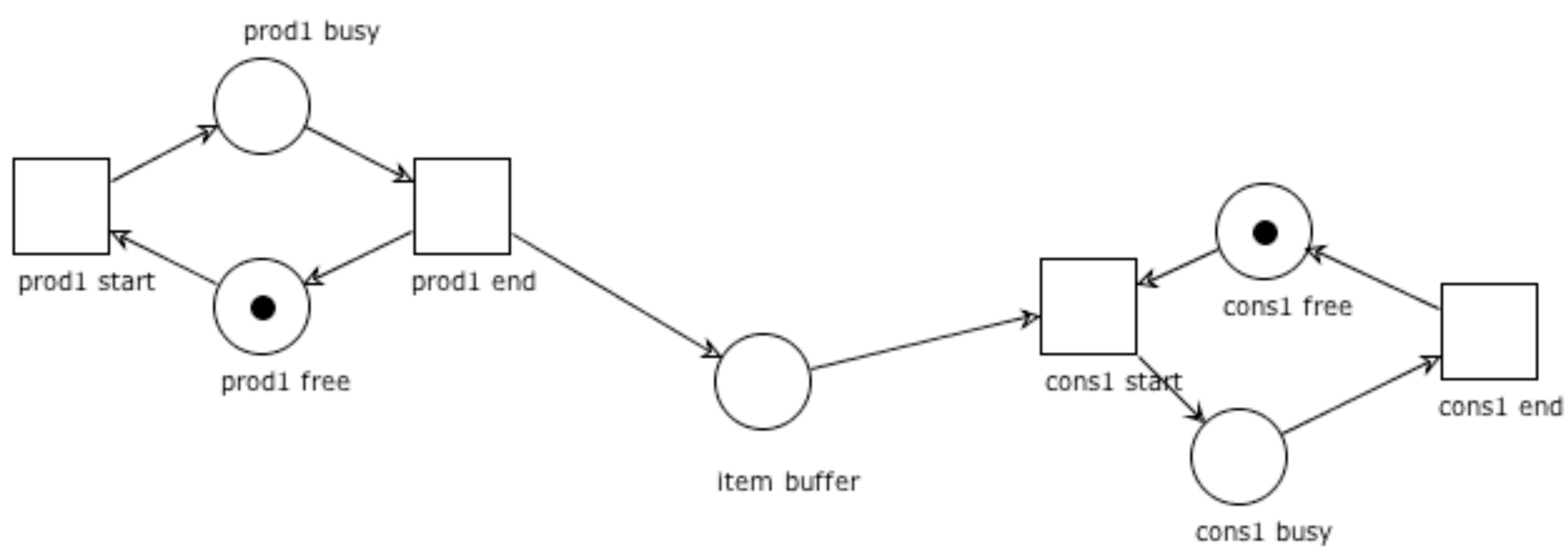
γ marked circuit <=> $i \in \gamma$ <=> $M_0(\gamma)=1$

γ marked circuit + M reachable => $M(\gamma)=1$

p belongs to a circuit of N* => p is safe
N* bounded <=> any place p belongs to a circuit of N*
all places belong to marked circuits => N* safe => N safe

Exercises

Which are the circuits of the T-system below?
Is the T-system below live? (why?)
Which places are bounded? (why?)
Assign a bound to each bounded place.



Exercises

Which are the circuits of the T-systems below?
Are the T-systems below live? (why?)
Which places are bounded? (why?)
Assign a bound to each bounded place.