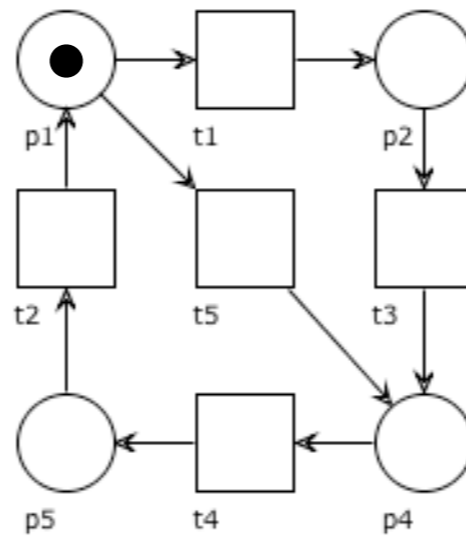


Object



We study some “good” properties of S-systems

Free Choice Nets (book, optional reading)

<https://www7.in.tum.de/~esparza/bookfc.html>

S-systems

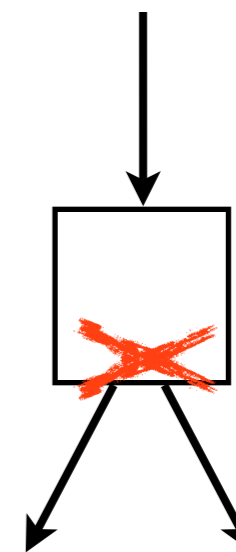
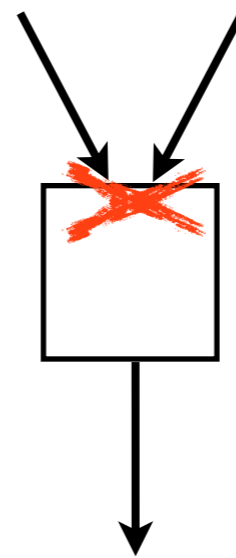
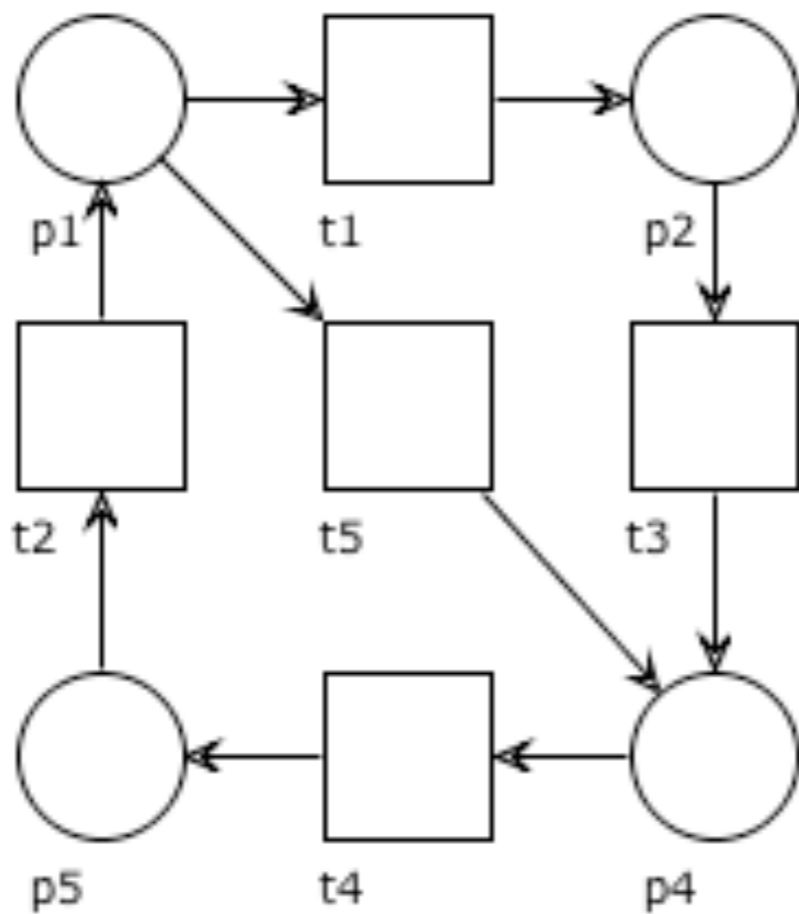
S-system

Definition: We recall that a net N is an **S-net** if each transition has exactly one input place and exactly one output place

$$\forall t \in T, \quad | \bullet t | = 1 = | t \bullet |$$

A system (N, M_0) is an **S-system** if N is an S-net

S-net: example



S-net N^*

Proposition: A workflow net N is an S-net
iff N^* is an S-net

N and N^* differ only for the reset transition,
that has exactly one incoming arc
and exactly one outgoing arc

S-systems: an observation

Observation: each transition t that fires removes exactly one token from some place p and inserts exactly one token in some place q (p and q can also coincide)

Notation: token count

$$M(P) = \sum_{p \in P} M(p)$$

Example

$$P = \{p_1, p_2, p_3\} \quad M = 2p_1 + 3p_2 \quad M(P) = 2 + 3 + 0 = 5$$

Fundamental property of S-systems

The overall number of tokens in the S-net is an invariant under any firing.

Fundamental property of S-systems

Proposition: Let (P, T, F, M_0) be an S-system.
If M is a reachable marking, then $M(P) = M_0(P)$

We show that for any $M \xrightarrow{\sigma} M'$ we have $M'(P) = M(P)$

base ($\sigma = \epsilon$): trivial ($M' = M$)

induction ($\sigma = \sigma' t$ for some $\sigma' \in T^*$ and $t \in T$):

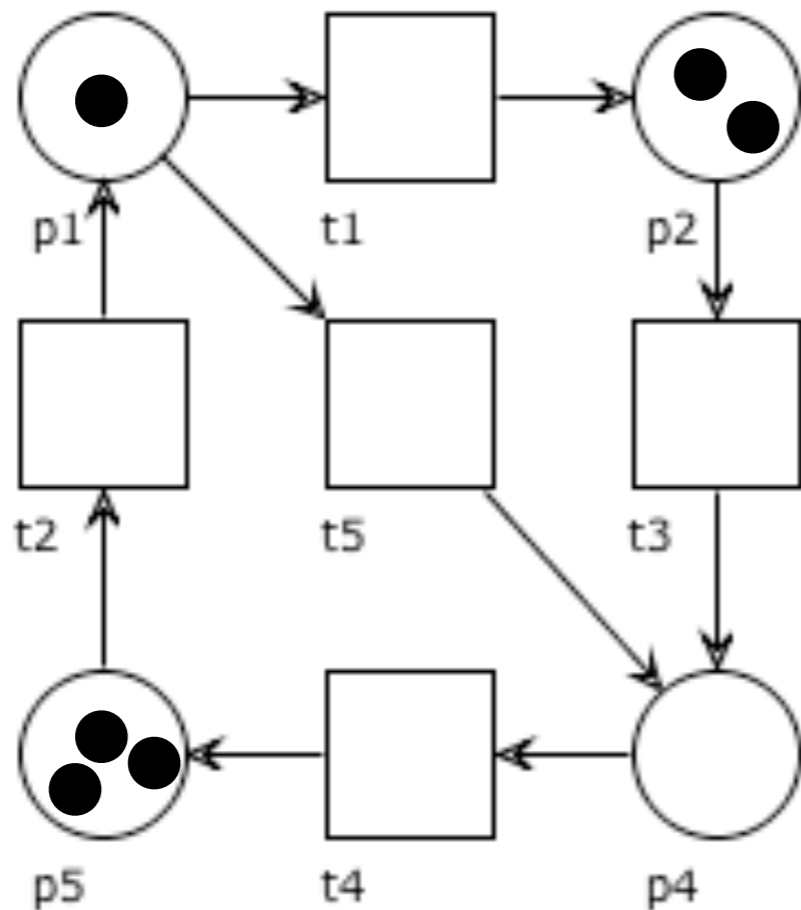
Let $M \xrightarrow{\sigma'} M'' \xrightarrow{t} M'$.

By inductive hypothesis: $M''(P) = M(P)$

By definition of S-system: $|\bullet t| = |t \bullet| = 1$

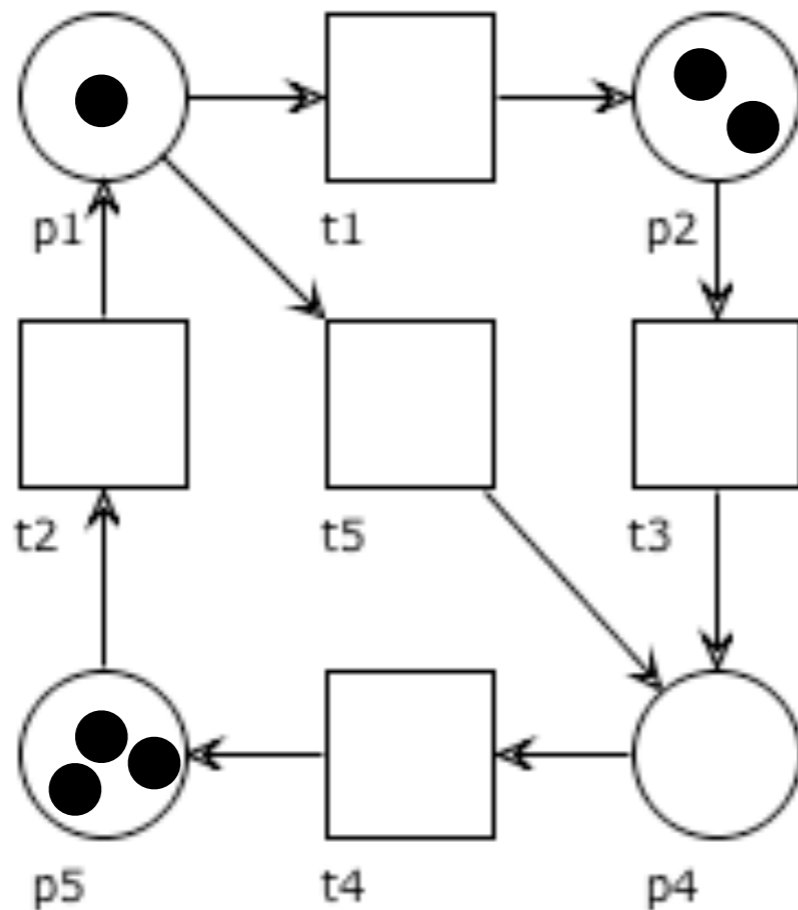
Thus, $M'(P) = M''(P) - |\bullet t| + |t \bullet| = M(P) - 1 + 1 = M(P)$

Question time



Is the marking
 $M = p_2 + 4p_4 + 2p_5$
reachable?

Question time



Is the marking
 $M = p_2 + 4p_4 + 2p_5$
reachable?

No:

S-system

$$M_0(P) = 6 \neq 7 = M(P)$$

A consequence of the fundamental property

Corollary: Any S-system is bounded

Let $M \in [M_0 \rangle$.

By the fundamental property of S-systems: $M(P) = M_0(P)$.

Then, for any $p \in P$ we have $M(p) \leq M(P) = M_0(P)$.

Thus the S-system is k -bounded for any $k \geq M_0(P)$.

$$M(P) = \sum_{p \in P} M(p)$$

S-invariants of S-nets

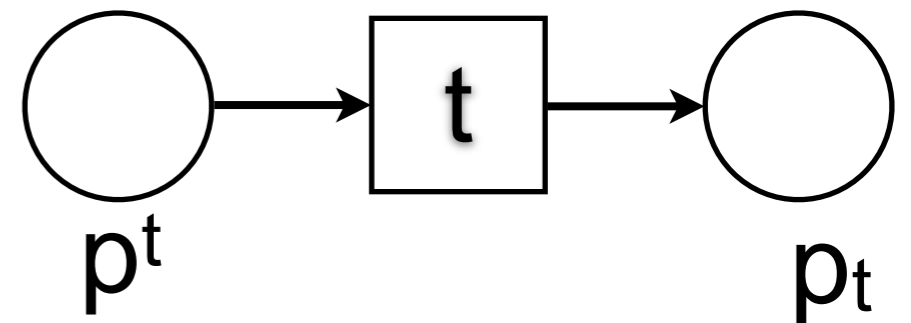
Proposition: Let $N=(P,T,F)$ be a (connected) S-net.
 I is an S-invariant of N **iff** $I=[k \dots k]$ for some value k

S-invariance $\forall t \in T, \sum_{p \in \bullet t} I(p) = \sum_{p \in t \bullet} I(p)$

S-nets $\forall t \in T, |\bullet t| = |t \bullet| = 1$

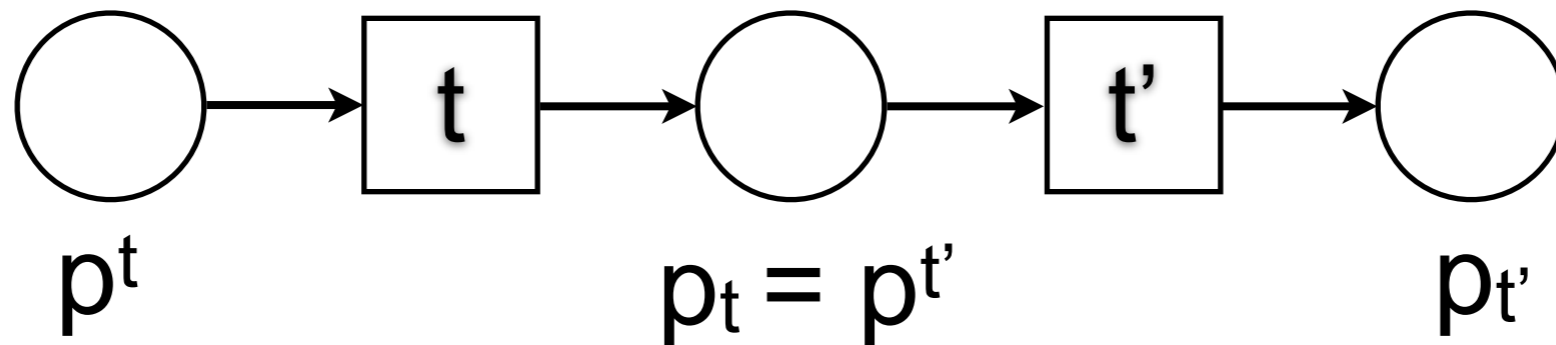
Let $\bullet t = \{p^t\}$ and $t \bullet = \{p_t\}$

$\forall t \in T, I(p^t) = I(p_t)$



S-invariants of S-nets

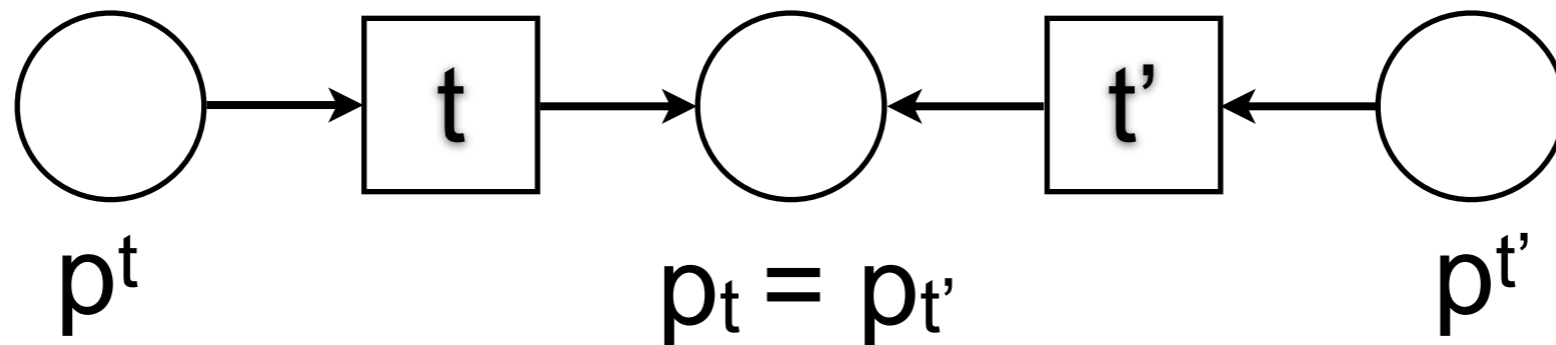
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$$I(p^t) = I(p_t) = I(p^{t'}) = I(p_{t'})$$

S-invariants of S-nets

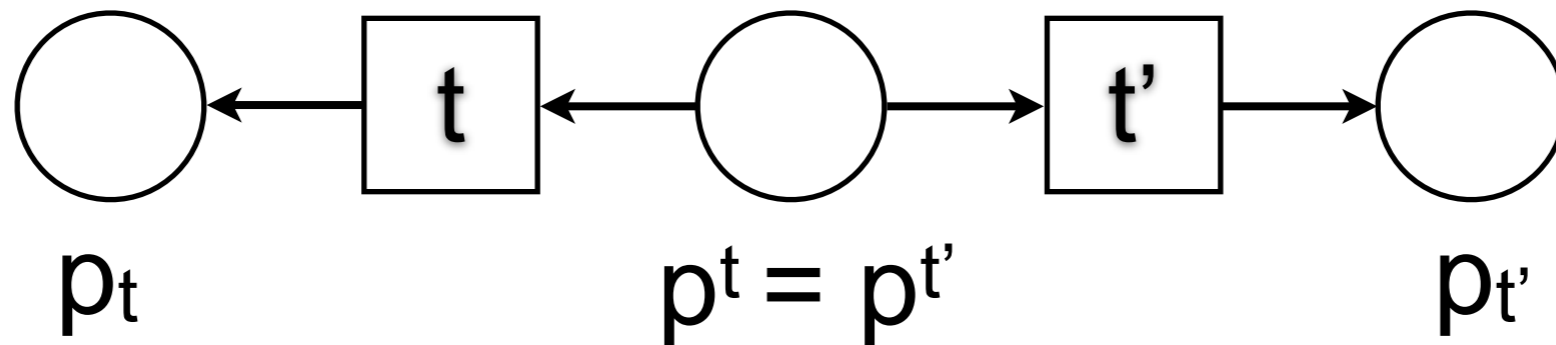
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$$I(p^t) = I(p_t) = I(p_{t'}) = I(p^{t'})$$

S-invariants of S-nets

Proposition: Let $N=(P,T,F)$ be a (connected) S-net.
 I is an S-invariant of N **iff** $I=[k \dots k]$ for some value k



$$I(p_t) = I(p^t) = I(p^{t'}) = I(p_{t'})$$

S-invariants of S-nets

Proposition: Let $N=(P,T,F)$ be a (connected) S-net.
 I is an S-invariant of N iff $I=[k \dots k]$ for some value k

$$\forall p_0, p_n \in P, \mathbf{I}(p_0) = \mathbf{I}(p_n)$$

Quick Recap

Let $N=(P,T,F,M_0)$ be an S-system

Prop. M reachable implies $M(P) = M_0(P)$

Corollary. N is bounded

Prop. I is an S-invariant iff $I=[k \dots k]$ for some k

A note on S-invariants and S-nets

S-invariance $\forall M \in [M_0 \rangle, \quad \mathbf{I} \cdot M = \mathbf{I} \cdot M_0$

S-invariant
of S-nets $\mathbf{I} = [1 \ 1 \ \dots \ 1]$

consequence $\forall M, \quad \mathbf{I} \cdot M = \sum_{p \in P} 1 \cdot M(p) = \sum_{p \in P} M(p) = M(P)$

We recover the
Fundamental
property of S-nets $\forall M \in [M_0 \rangle, \quad M(P) = \mathbf{I} \cdot M = \mathbf{I} \cdot M_0 = M_0(P)$

Liveness theorem for S-systems

Theorem: An S-system (N, M_0) is live iff
 N is strongly connected and M_0 marks at least one place

\Rightarrow) (quite obvious)

(N, M_0) is live by hypothesis and bounded (because S-system).
By the strong connectedness theorem, N is strongly connected.

Since (N, M_0) is live, then $M_0 \xrightarrow{t}$ for some t .

Assume $\bullet t = \{p\}$. Thus, $M_0(p) \geq 1$.

Liveness theorem for S-systems

Theorem: An S-system (N, M_0) is live iff N is strongly connected and M_0 marks at least one place

\Leftarrow) (more interesting)

Take any $M \in [M_0 \rangle$ and $t \in T$.

We want to find $M' \in [M \rangle$ such that $M' \xrightarrow{t}$.

Take $p_1 \in P$ such that $M(p_1) \geq 1$ (it exists, because $M(P) = M_0(P) \geq 1$).

By strong connectedness: there is a path from p_1 to $t_n = t$

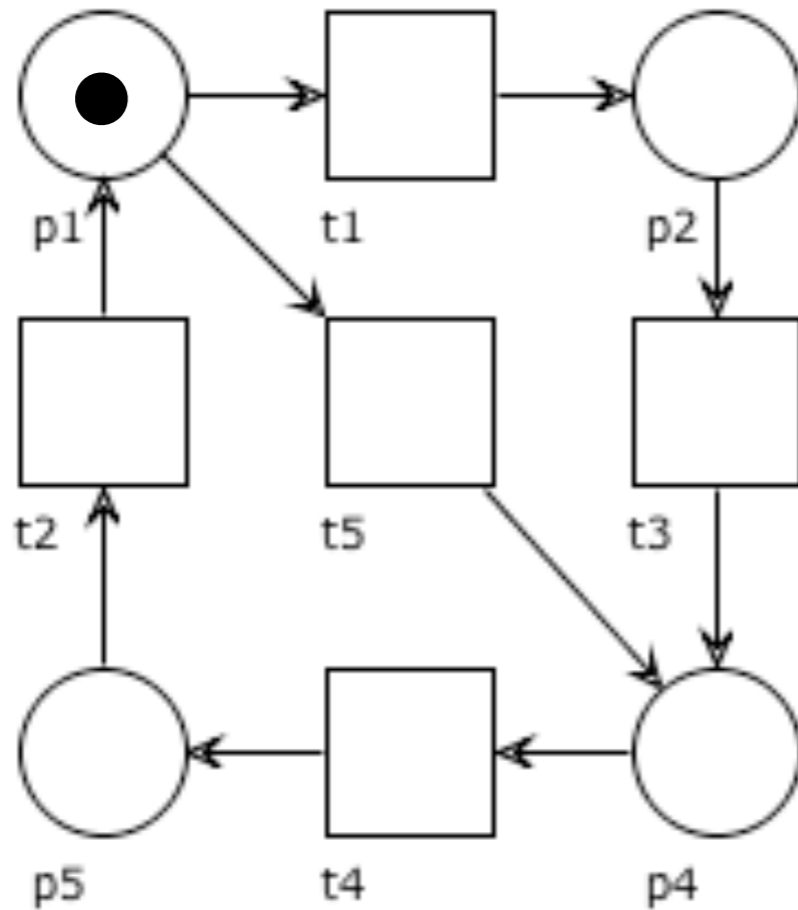
$(p_1, t_1)(t_1, p_2)(p_2, t_2) \dots (p_n, t_n)$

By definition of S-system: $\bullet t_i = \{p_i\}$ and $t_i \bullet = \{p_{i+1}\}$.

Thus, $M \xrightarrow{\sigma} M' \xrightarrow{t}$ for $\sigma = t_1 t_2 \dots t_{n-1}$.

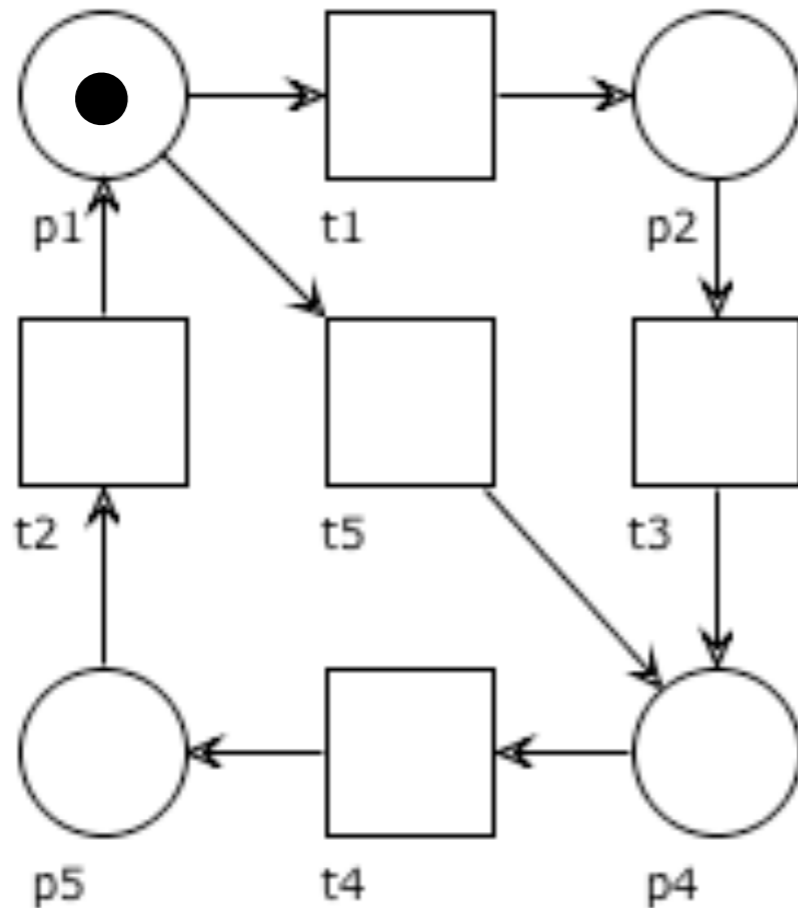
Question time

Is this system live?



Question time

Is this system live?

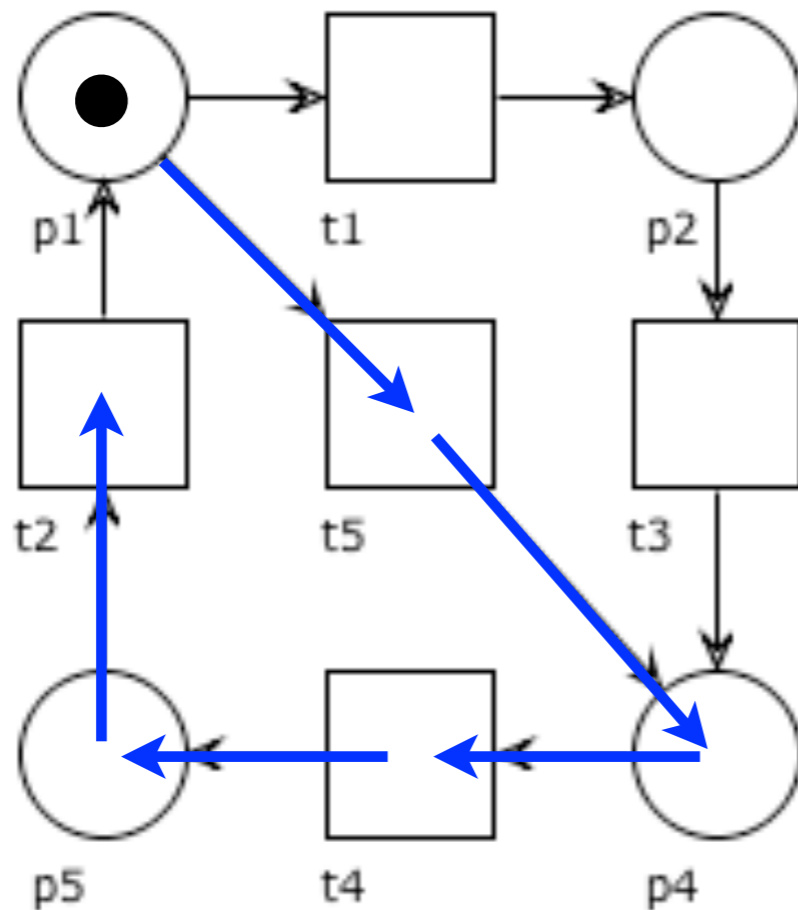


Yes:
S-system
strongly connected
 $M_0(P) > 0$

Question time

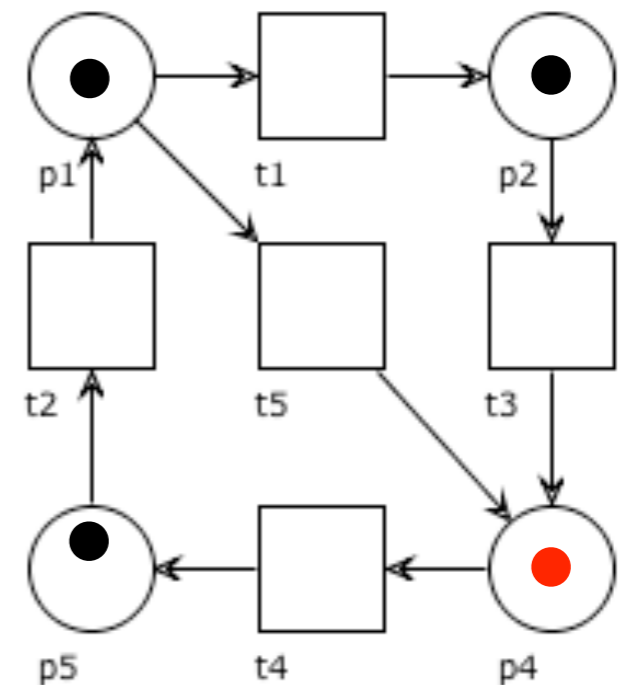
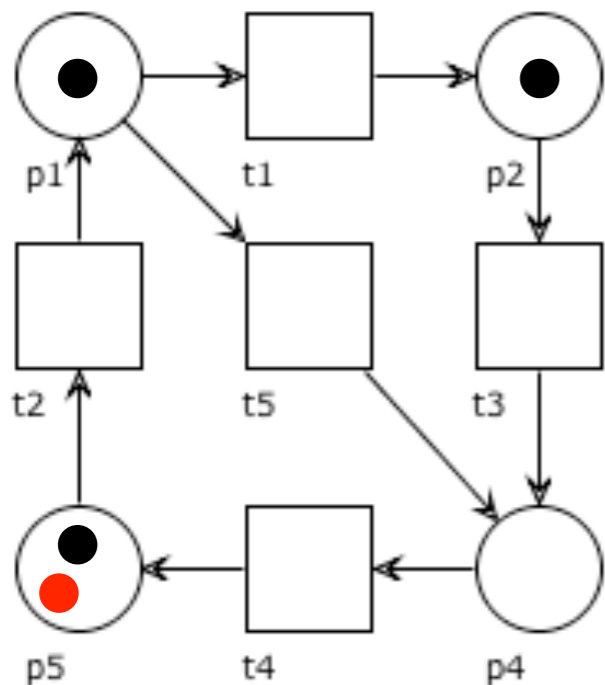
Is this system live?

How to enable t2?
just find a path
from p1 to t2



Reachability lemma for S-nets

Lemma: Let (P,T,F) be a strongly connected S-net.
If $M(P) = M'(P)$, then M' is reachable from M

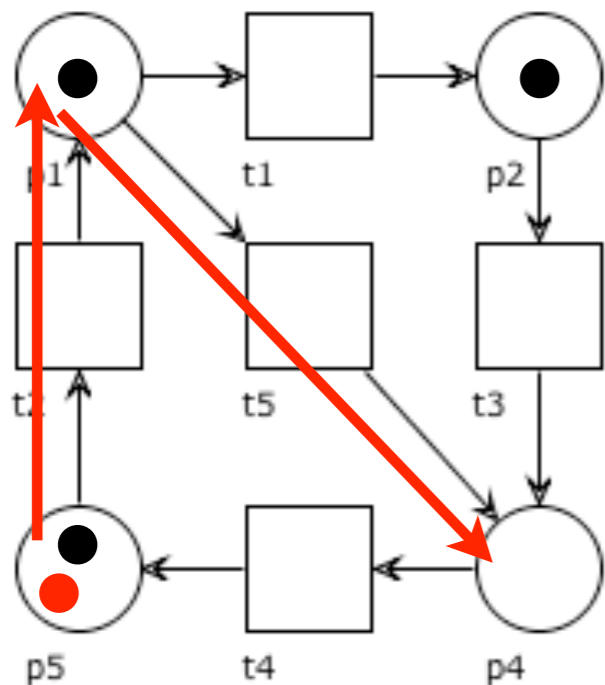


Intuitively:

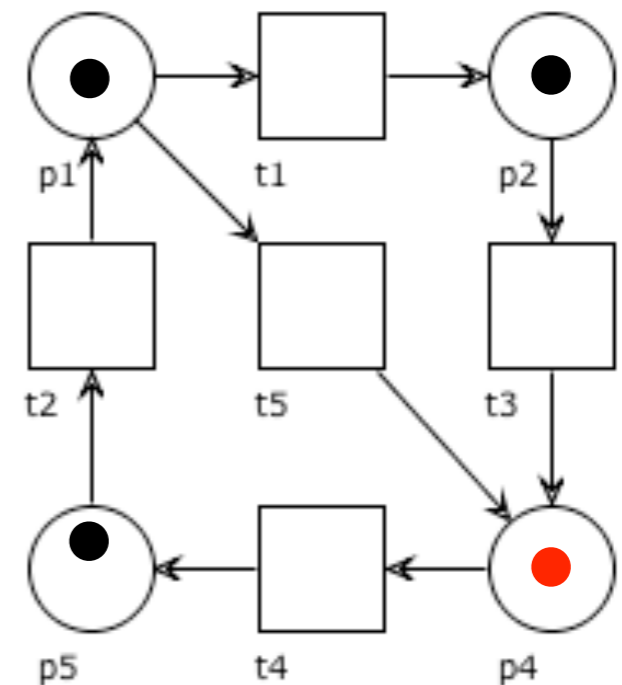
M and M' has the same number of tokens
but maybe they are in different places

Reachability lemma for S-nets

Lemma: Let (P,T,F) be a strongly connected S-net.
If $M(P) = M'(P)$, then M' is reachable from M



if there is only **one** token in the wrong place

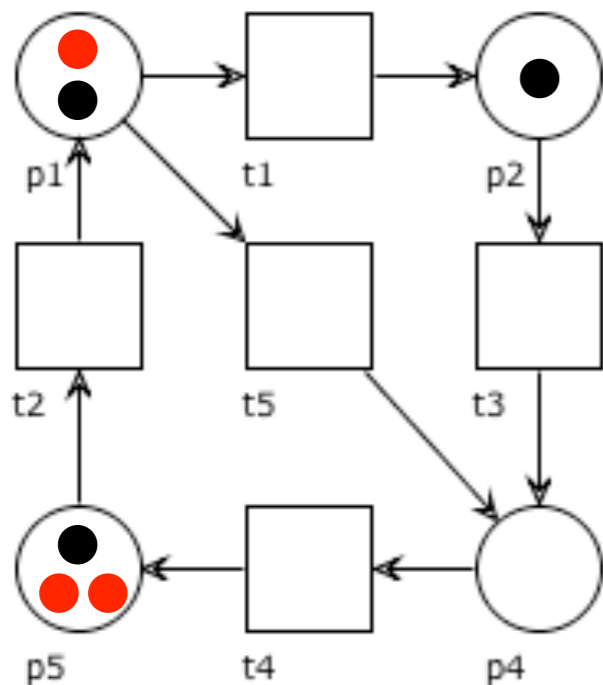


Intuitively:

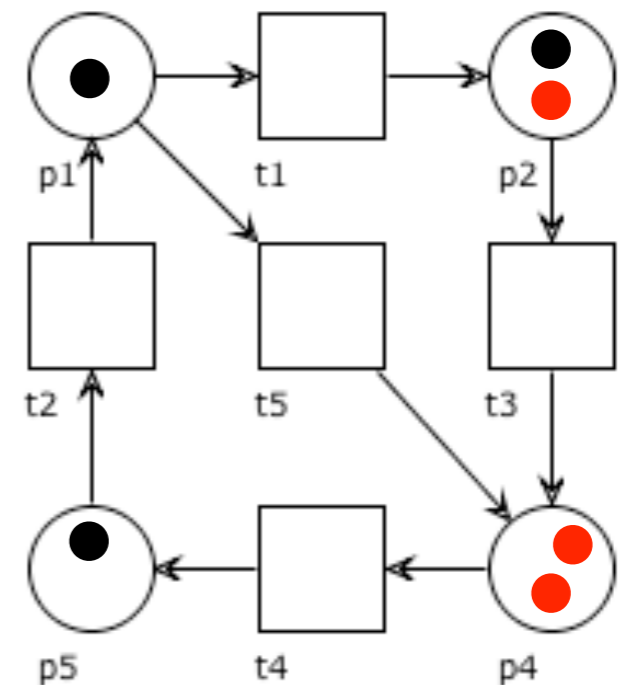
we need to move the token from p to q
by strong connectedness we have a path from p to q
by firing the transitions along the path we succeed

Reachability lemma for S-nets

Lemma: Let (P,T,F) be a strongly connected S-net.
 If $M(P) = M'(P)$, then M' is reachable from M

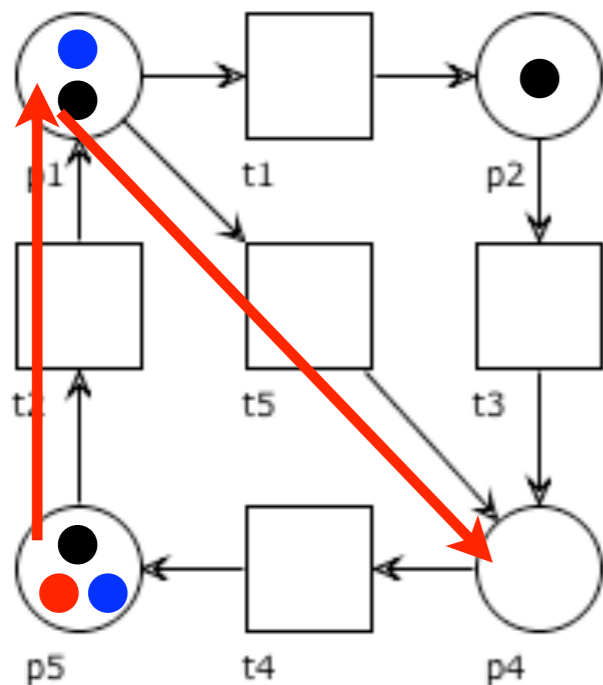


if there are **many**
(n) tokens in the
 wrong places

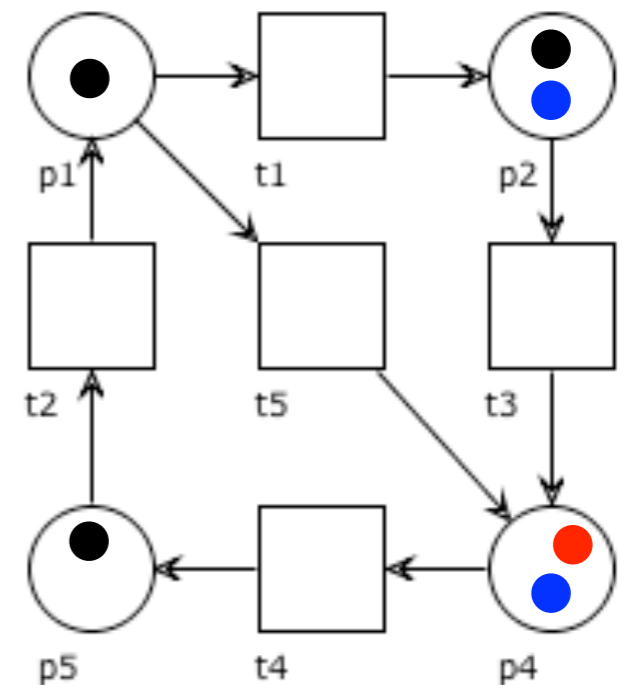


Reachability lemma for S-nets

Lemma: Let (P,T,F) be a strongly connected S-net.
 If $M(P) = M'(P)$, then M' is reachable from M



if there are **many**
(n) tokens in the
 wrong places



Intuitively:
 we move **one** of them...
 and inductively the remaining **n-1**

Reachability lemma for S-nets

Lemma: Let (P, T, F) be a strongly connected S-net.
 If $M(P) = M'(P)$, then M' is reachable from M

We proceed by induction on $M(P)$

base ($M(P) = M'(P) = 0$): trivial ($M' = M$)

induction ($M(P) = M'(P) > 0$):

Let $p, p' \in P$ be such that $M(p) > 0$ and $M'(p') > 0$.

Let $K = M - p$ and $K' = M' - p'$.

Clearly $K'(P) = K(P) < M(P) = M'(P)$.

By inductive hypothesis: $\exists \sigma, K \xrightarrow{\sigma} K'$

By strong connectedness: there is a path from $p_0 = p$ to $p_n = p'$

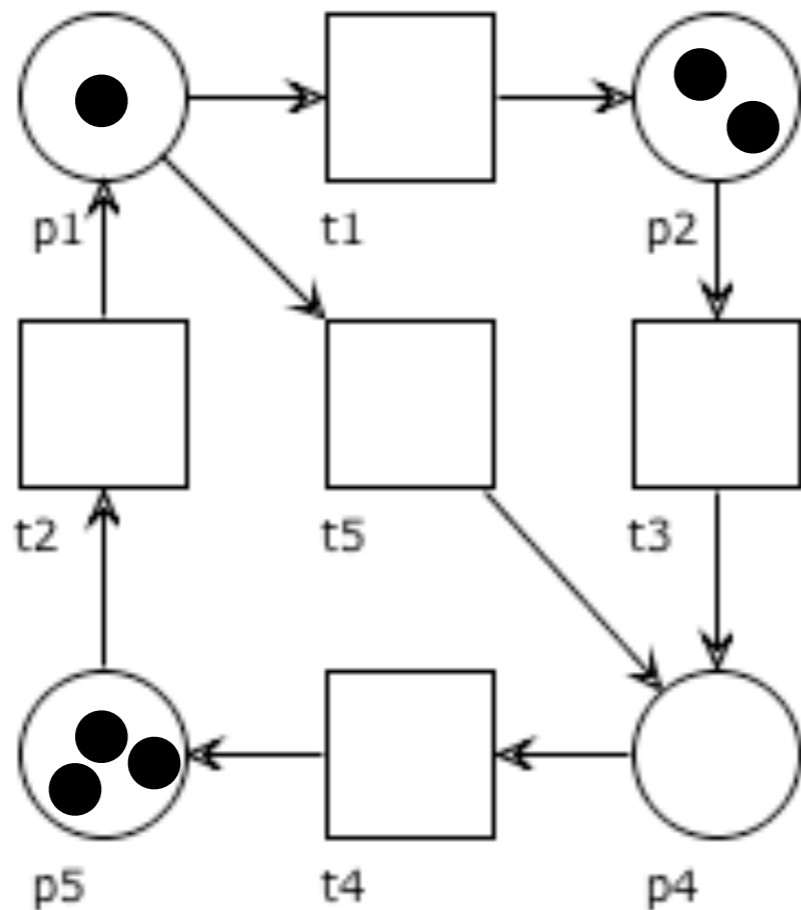
$(p_0, t_1)(t_1, p_1)(p_1, t_2) \dots (t_n, p_n)$

By definition of S-system: $\bullet t_i = \{p_{i-1}\}$ and $t_i \bullet = \{p_i\}$.

Thus, $p = p_0 \xrightarrow{\sigma'} p_n = p'$ for $\sigma' = t_1 t_2 \dots t_n$.

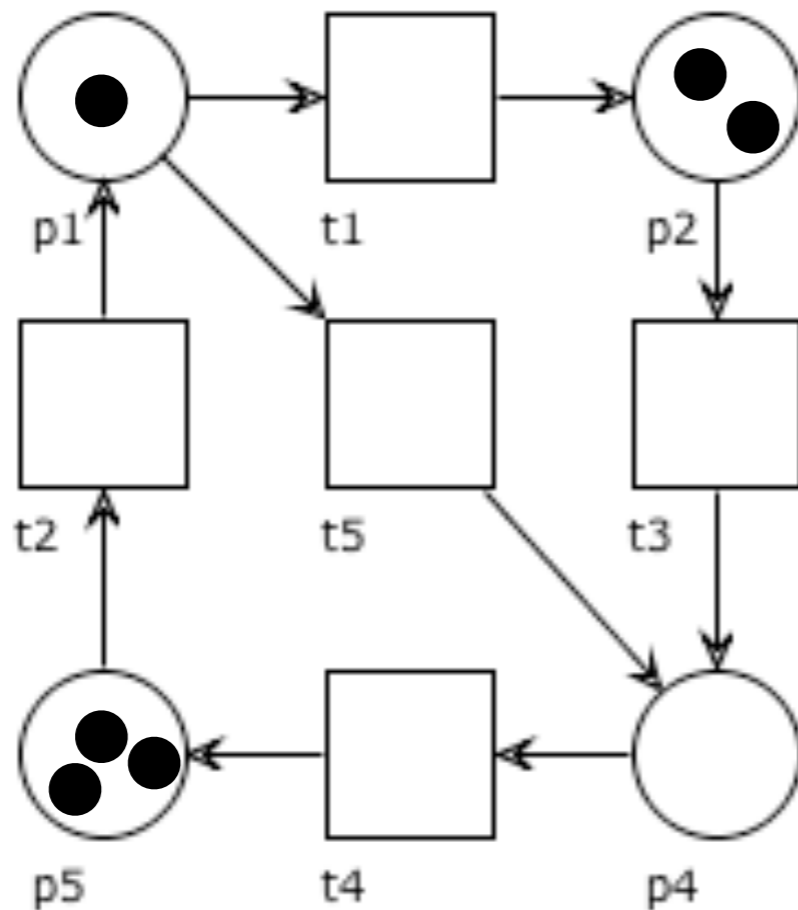
By the monotonicity lemma: $M = K + p \xrightarrow{\sigma} K' + p \xrightarrow{\sigma'} K' + p' = M'$

Question time



Is the marking
 $M = p_2 + 4p_4 + p_5$
reachable?

Question time



Is the marking
 $M = p_2 + 4p_4 + p_5$
reachable?

Yes:
S-system
strongly connected
 $M_0(P) = 6 = M(P)$

Reachability Theorem for S-systems

Theorem: Let (P, T, F, M_0) be a live S-system.
A marking M is reachable **iff** $M(P) = M_0(P)$

\Rightarrow) Follows from the fundamental property of S-systems

\Leftarrow) By the previous liveness theorem, the S-net is strongly connected.

We conclude by applying the reachability lemma for S-systems.

S-systems: recap

S-system \Rightarrow bounded

S-system: strong conn. + $M_0(P) > 0 \Leftrightarrow$ live

S-system + M reachable $\Rightarrow M(P) = M_0(P)$

S-system + str. conn.: $M(P) = M_0(P) \Leftrightarrow$ M reachable

S-system + live: $M(P) = M_0(P) \Leftrightarrow$ M reachable

S-system: S-invariant $I \Leftrightarrow I = [k \ k \ \dots \ k]$

Workflow S-nets

Theorem: If a workflow net N is an S-system
then it is sound

N is S-system $\Leftrightarrow N^*$ is S-system

N and N^* S-systems $\Rightarrow N$ and N^* bounded

N workflow net $\Rightarrow N^*$ strong connected

$M_0(P)=1$ (initially one token in place i)

N^* strong connected + $M_0(P) > 0 \Leftrightarrow N^*$ live

N^* bounded and live $\Leftrightarrow N$ sound

Workflow S-nets

Theorem: If a workflow net N is an S-system
then it is **safe** and sound

N is S-system $\Leftrightarrow N^*$ is S-system

$M_0(P)=1$ (initially one token in place i)

N and N^* S-systems + $M_0(P)=1 \Rightarrow N$ and N^* safe

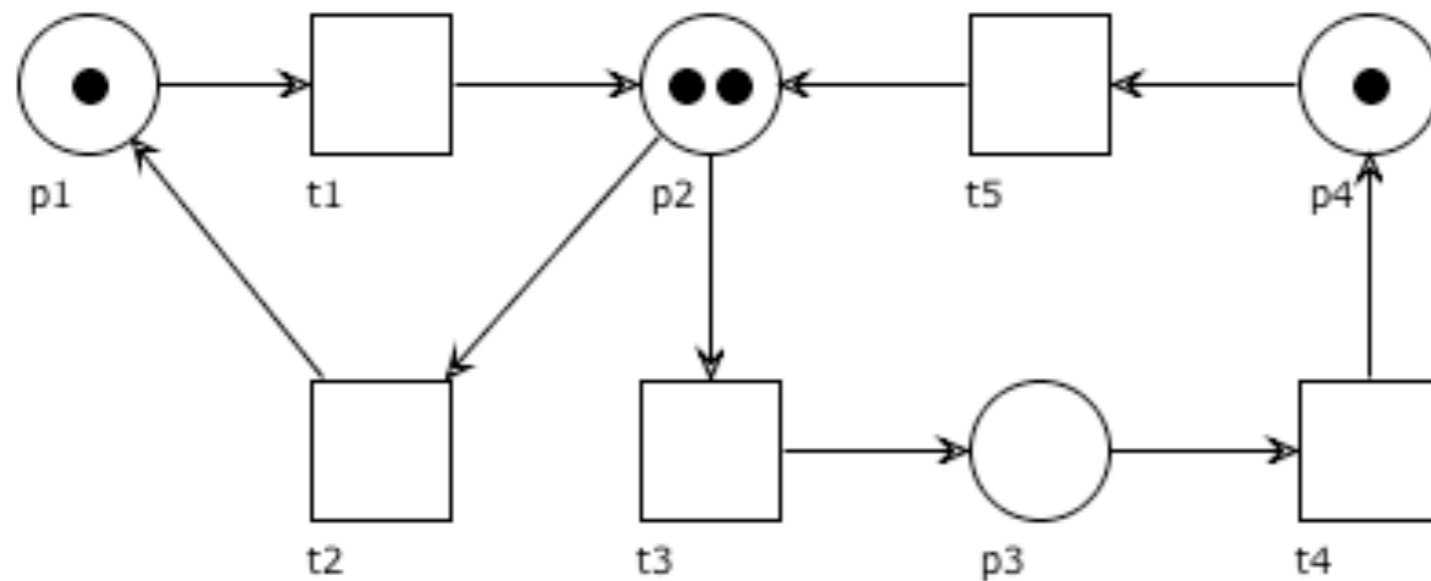
N workflow net $\Rightarrow N^*$ strong connected

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N^* bounded and live $\Leftrightarrow N$ sound

Question time

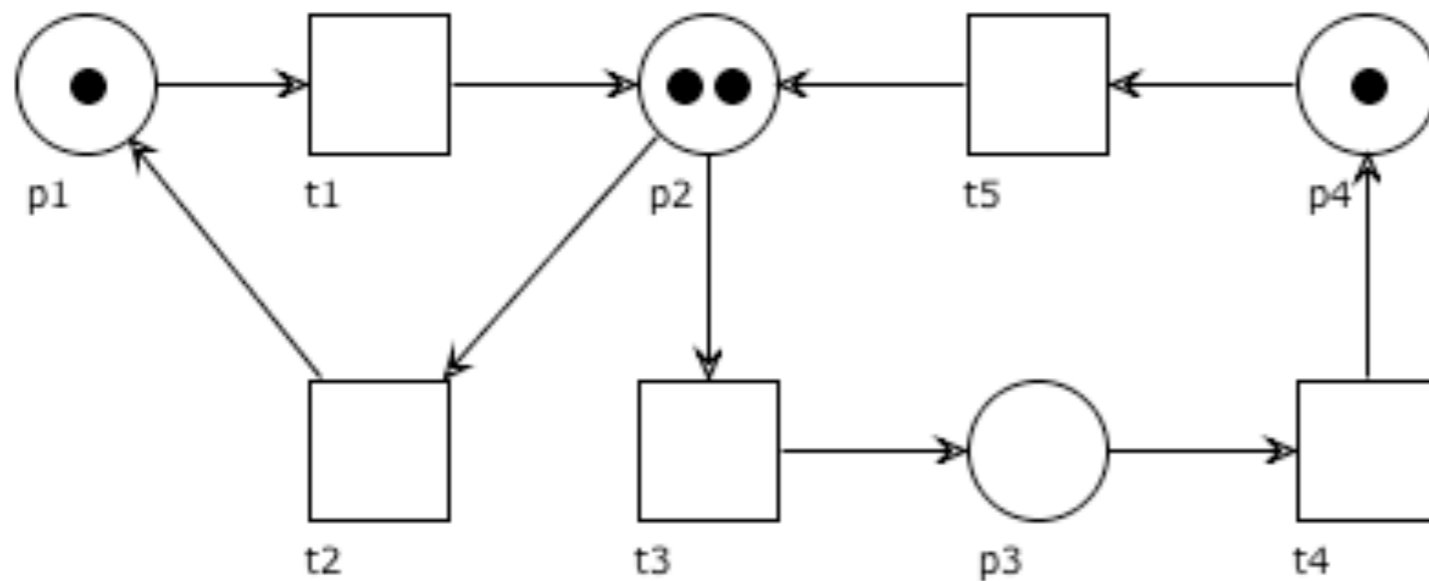
Which of the following markings are reachable? (why?)


$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & 2 & 1 & 2 \\ 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 \end{bmatrix}$$

Question time

Which of the following markings are reachable? (why?)

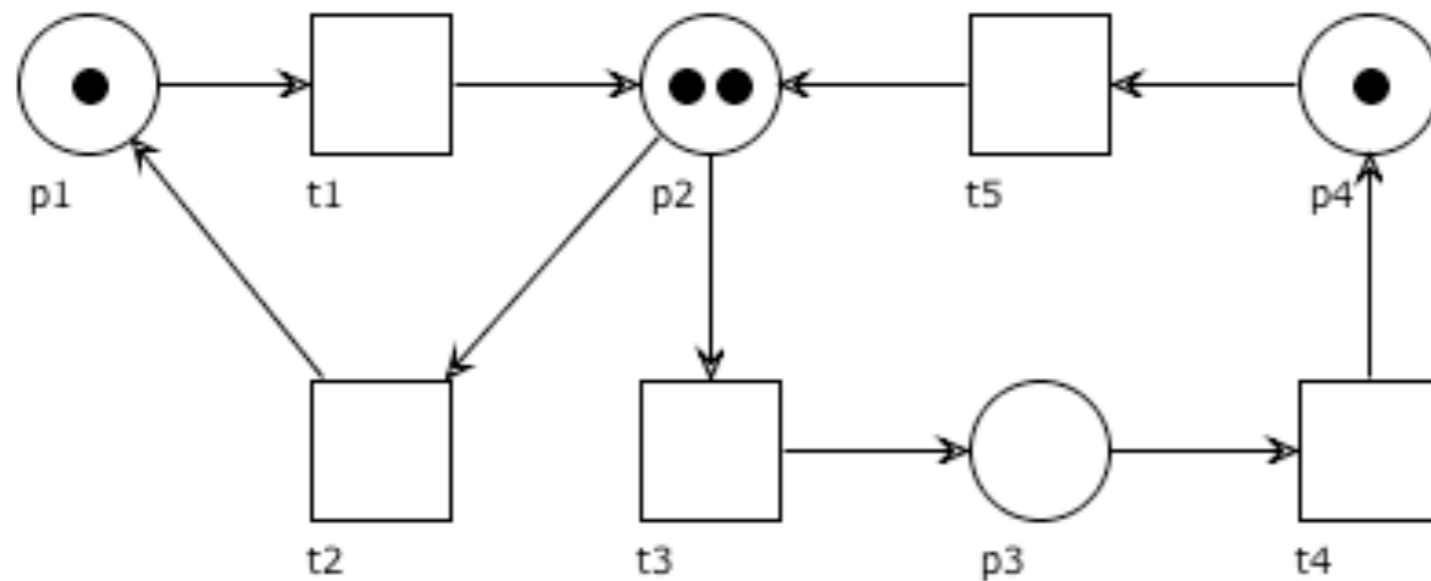
strong connected



1	1	1	1
2	0	2	0
1	2	1	2
4	0	0	0
0	4	0	4

Question time

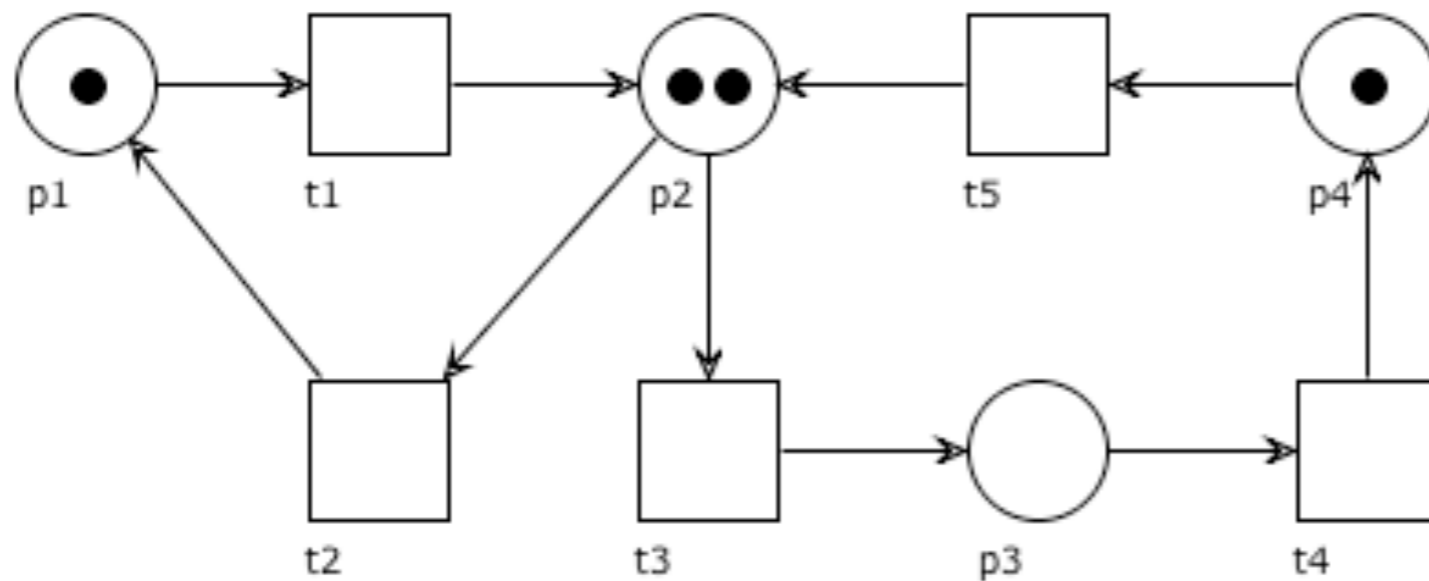
Which of the following markings are reachable? (why?)



$[1\ 1\ 1\ 1]$
 $[2\ 0\ 2\ 0]$
 $[1\ 2\ 1\ 2]$
 $[4\ 0\ 0\ 0]$
 $[0\ 4\ 0\ 4]$

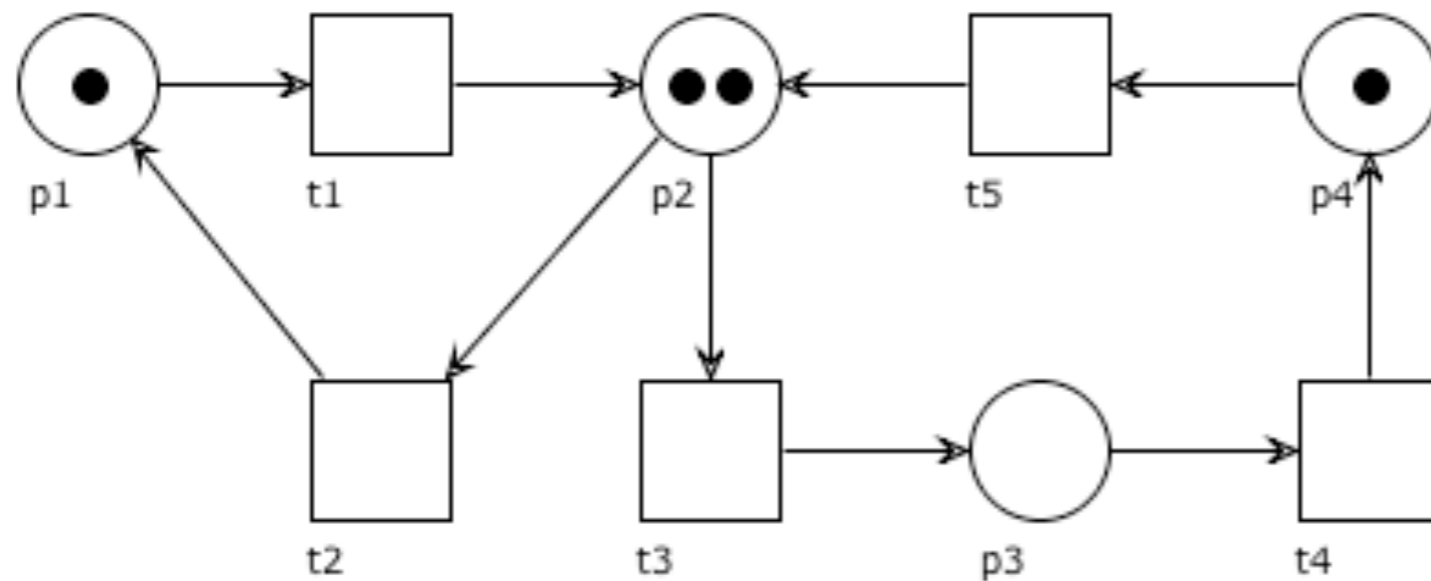
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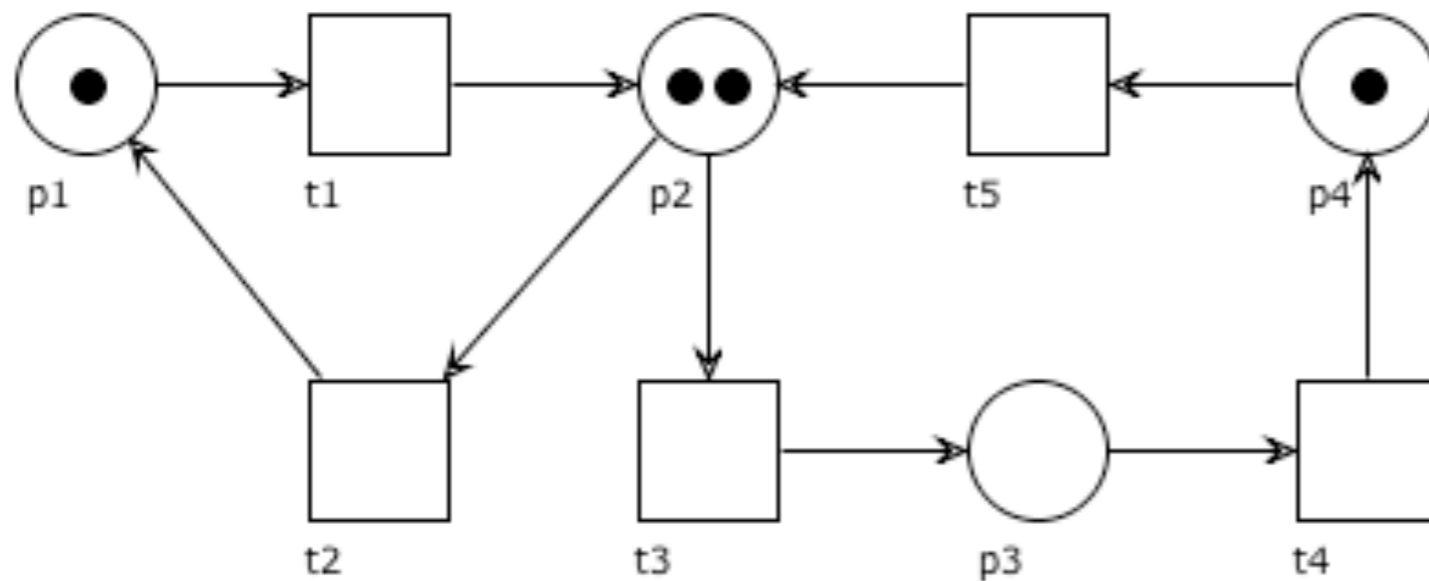
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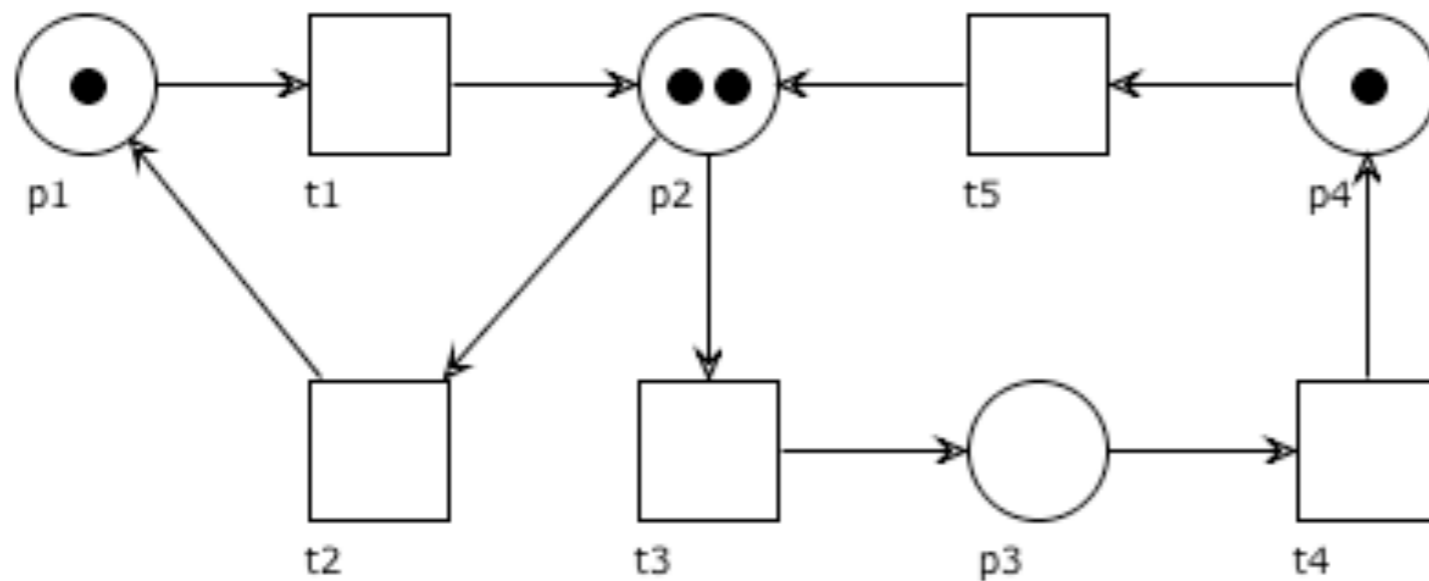
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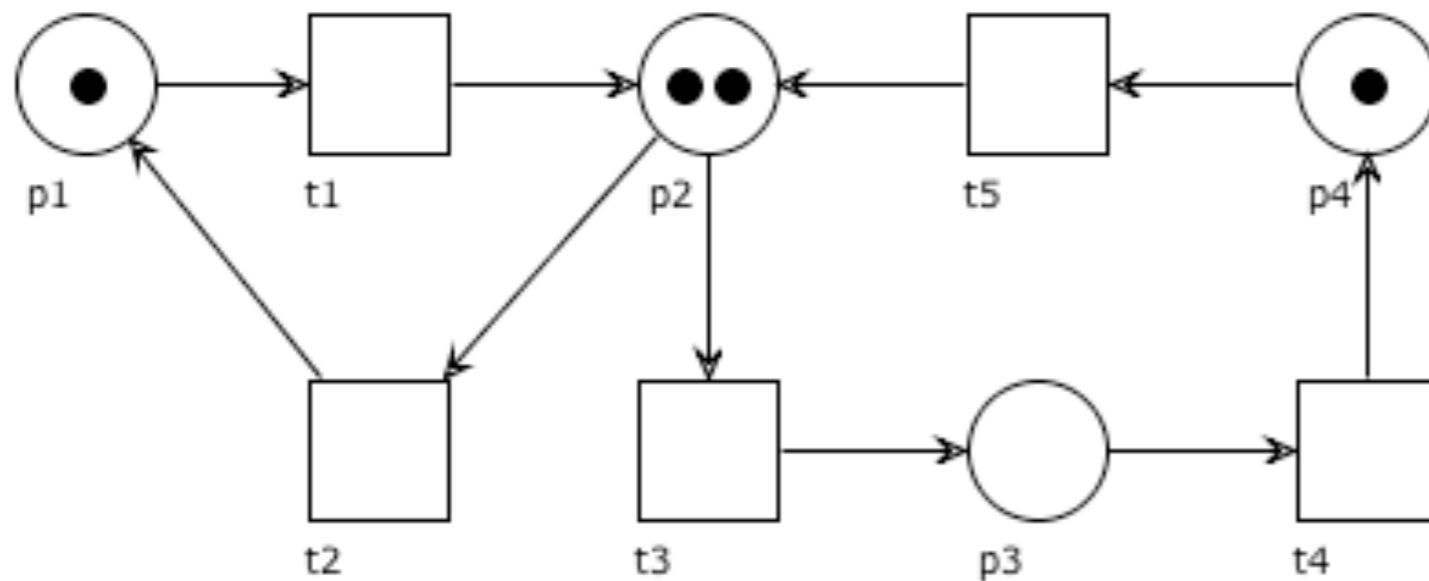
Question time

Which of the following are S-invariants? (why?)


$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

Question time

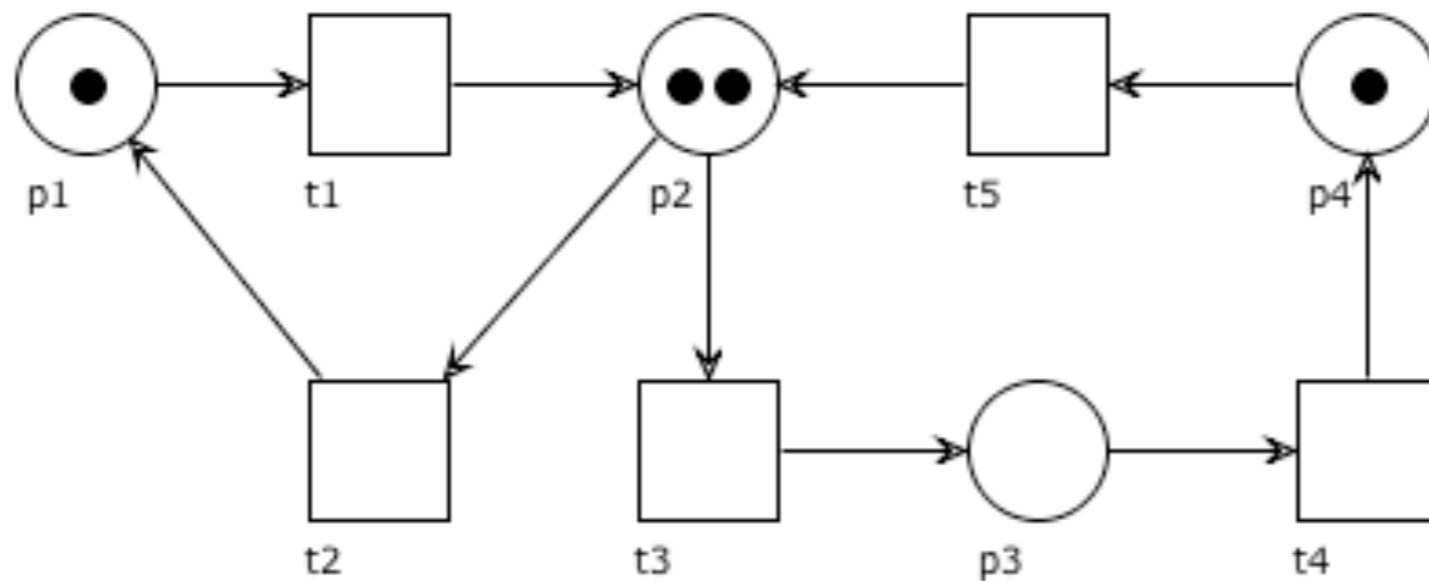
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[1	1	0	0]
[0	0	2	2]
[1	1	1	1]
[2	2	1	1]
[2	2	2	2]
[1	2	2	1]

Question time

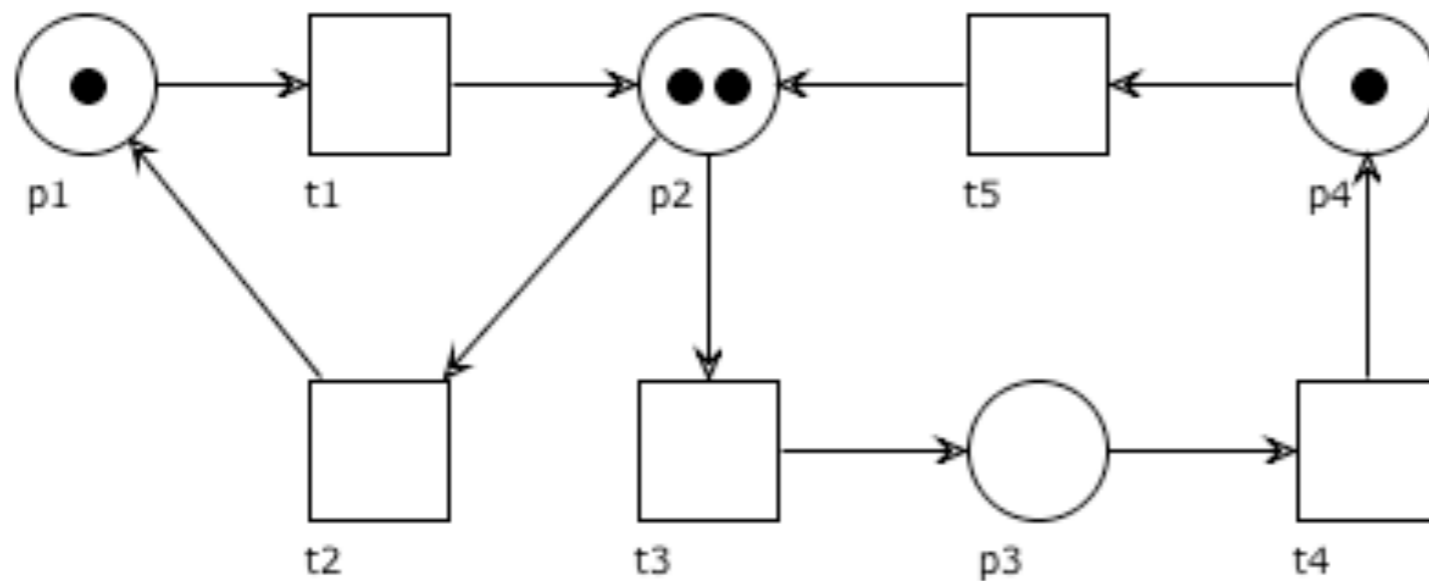
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$[1\ 1\ 0\ 0]$
 $[0\ 0\ 2\ 2]$
 $[1\ 1\ 1\ 1]$
 $[2\ 2\ 1\ 1]$
 $[2\ 2\ 2\ 2]$
 $[1\ 2\ 2\ 1]$

Question time

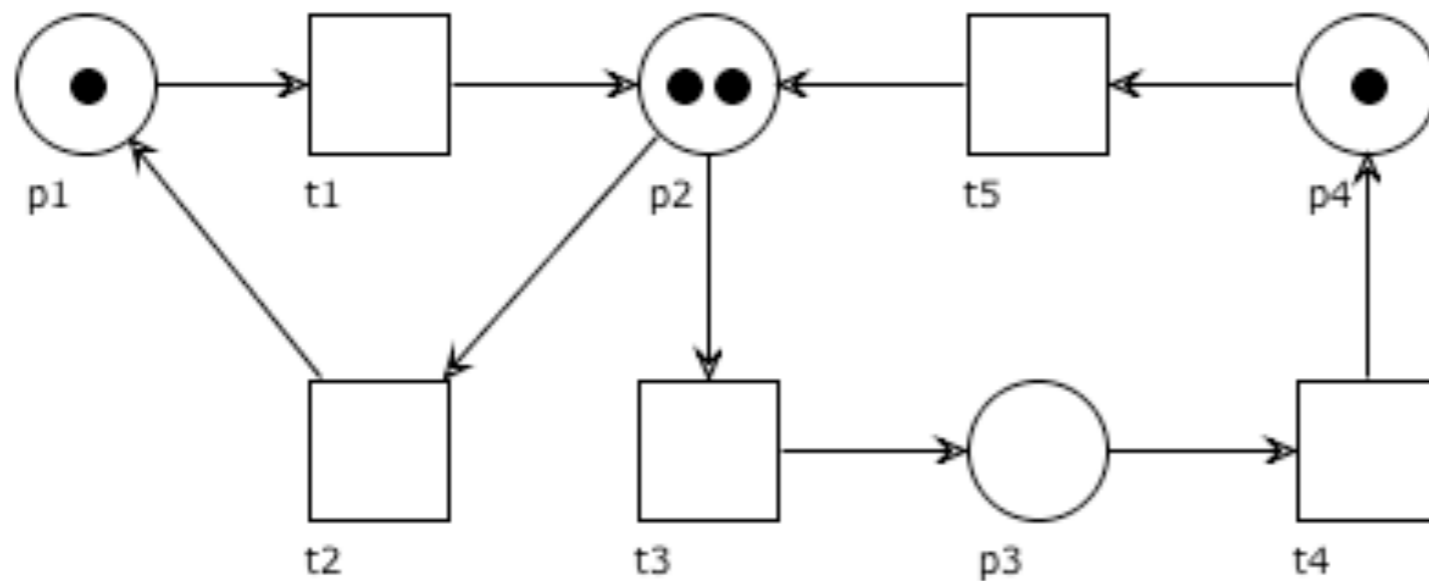
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$[1\ 1\ 0\ 0]$
 $[0\ 0\ 2\ 2]$
 $[1\ 1\ 1\ 1]$
 $[2\ 2\ 1\ 1]$
 $[2\ 2\ 2\ 2]$
 $[1\ 2\ 2\ 1]$

Question time

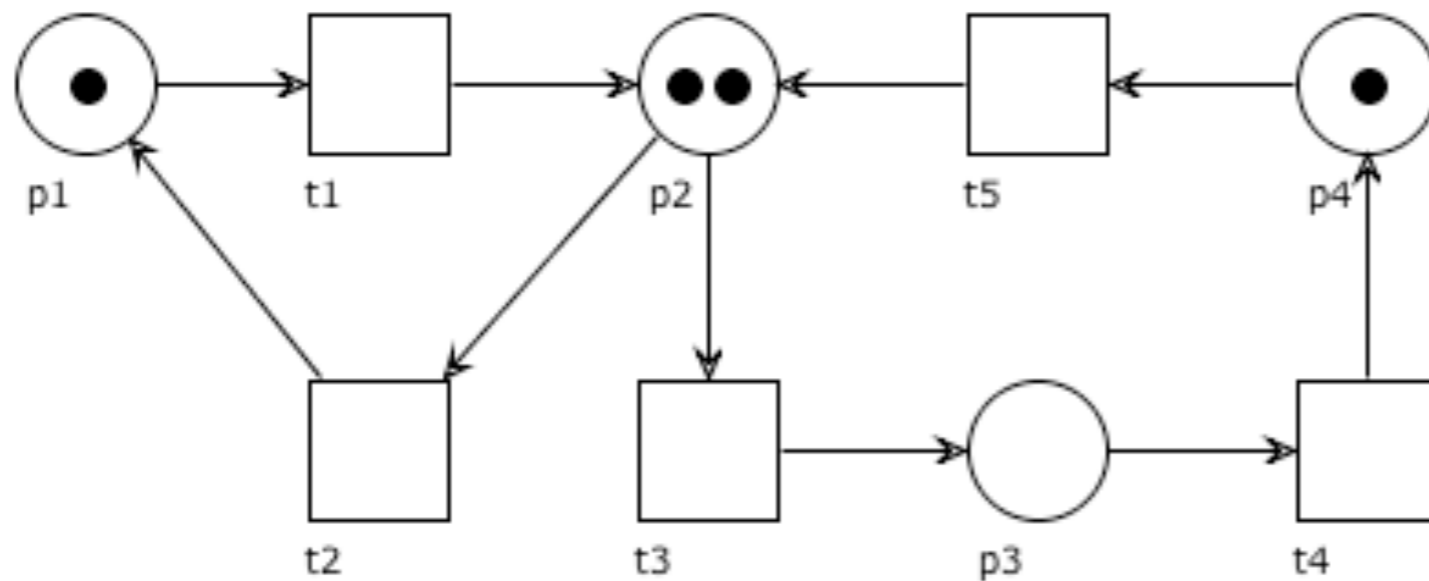
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 $[0\ 0\ 2\ 2]$
 $[1\ 1\ 1\ 1]$
 $[2\ 2\ 1\ 1]$
 $[2\ 2\ 2\ 2]$
 $[1\ 2\ 2\ 1]$

Question time

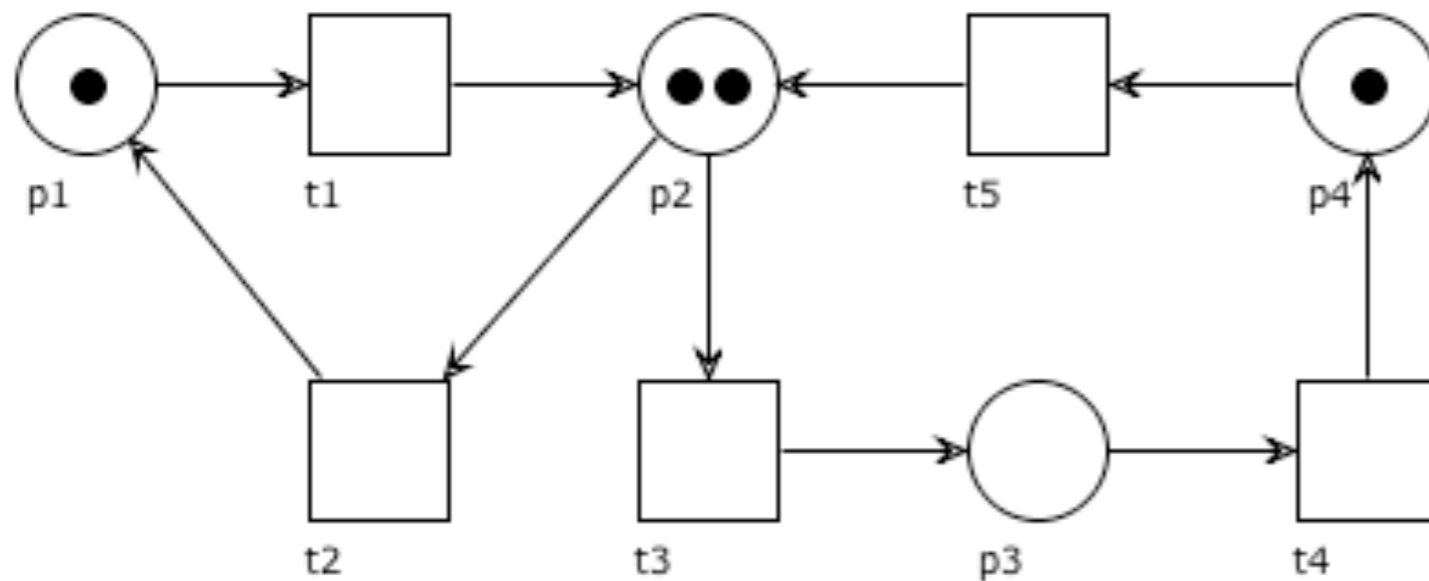
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 $[1\ 1\ 1\ 1]$
 $[2\ 2\ 1\ 1]$
 $[2\ 2\ 2\ 2]$
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Question time

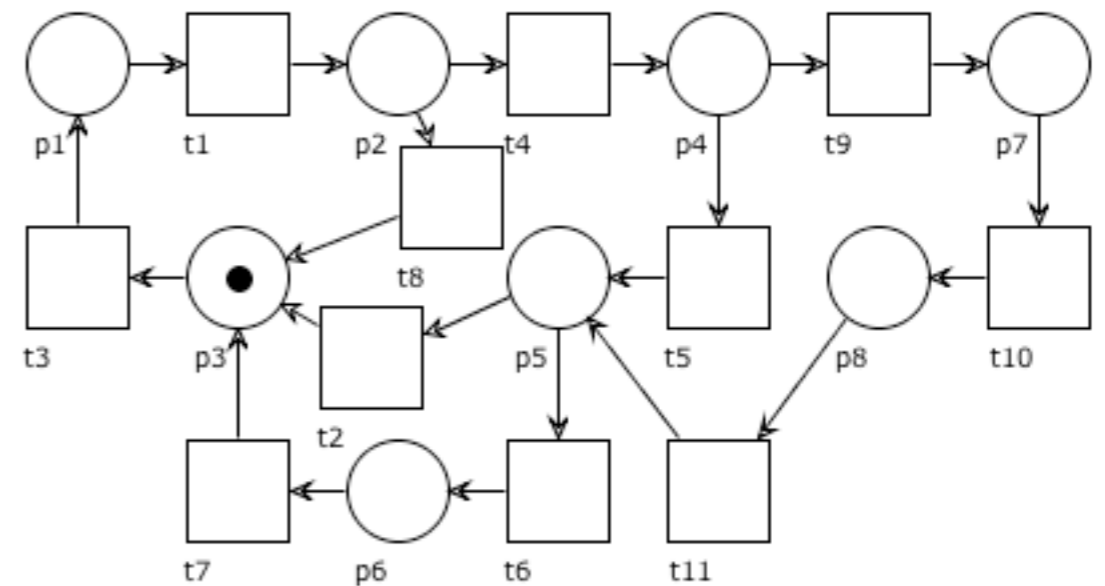
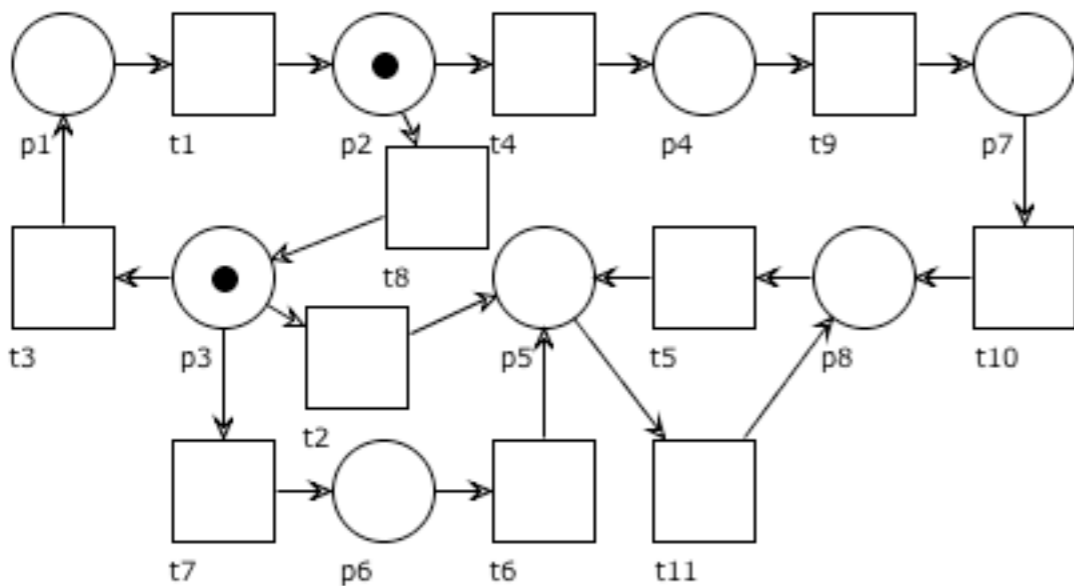
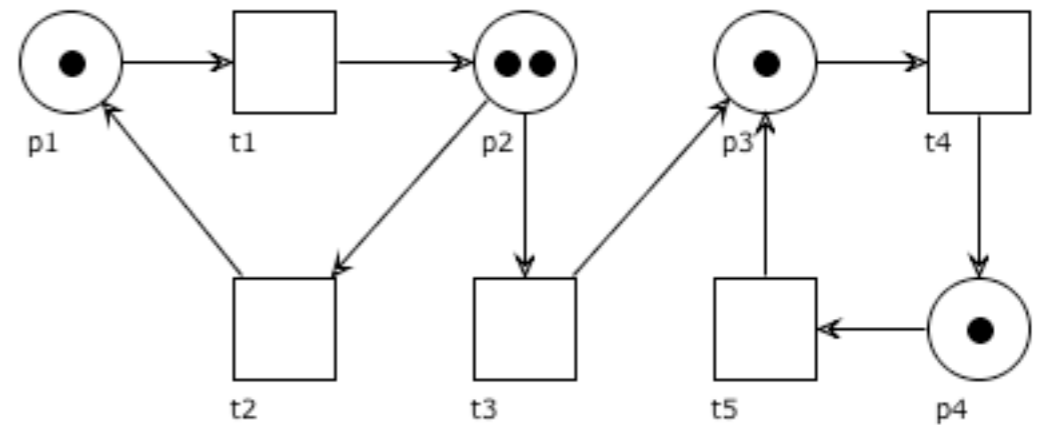
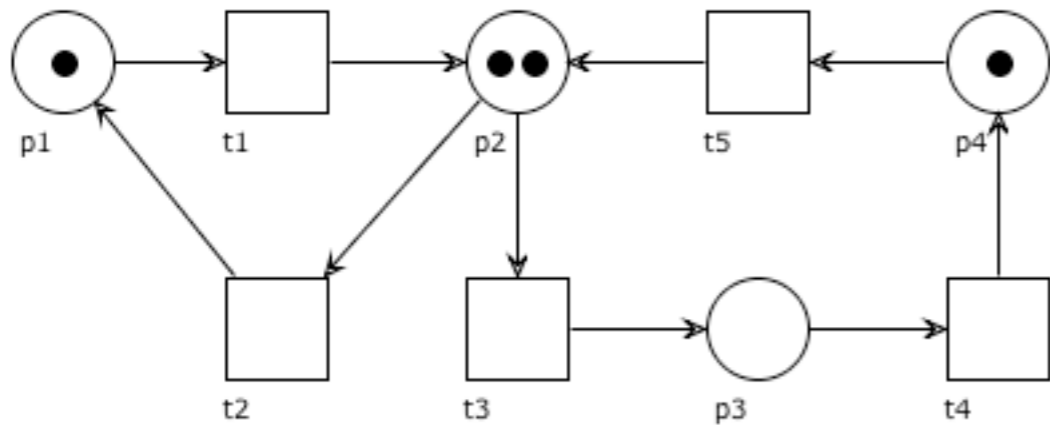
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$[1\ 1\ 0\ 0]$
 $[0\ 0\ 2\ 2]$
 $[1\ 1\ 1\ 1]$
 $[2\ 2\ 1\ 1]$
 $[2\ 2\ 2\ 2]$
 $[1\ 2\ 2\ 1]$

Exercises

Which of the following S-systems are live? (why?)



Exercise

Is the net below a workflow net?
Is it sound?

