Inventory Management Problems

(Ghiani, Laporte, Musmanno : Chap. 4)
(4.1, 4.2, 4.3)

Inventories: stockpiles of items (raw materials, components ... finished goods) waiting to be processed, transported or used

Motivations: improving service level, reducing overall logistics costs, coping with randomness in customer demands...

But: holding inventories can be very expensive

It is therefore crucial to solve inventory management problems: for each stocking point in the supply chain, to decide when to reorder (for a single or multiple products) and how much to order so that the expected (annual) cost is minimized while achieving a certain service level

Relevant involved costs:

1) Procurement costs: associated with the acquisition of goods; can be fixed costs or variable costs (i.e. dependent on the amount acquired): e.g. (fixed) reorder cost, (variable) purchasing or manufacturing cost, transportation cost ....

2) Inventory holding costs: incurred when materials are stocked for a period of time: e.g. opportunity (or capital) cost, i.e. the return on investment the firm would have realized if money had been invested in a profitable economic activity instead of inventory; warehousing cost (may includes space and equipment costs, maintance costs, taxes ... or fee to pay for storing)

3) Shortage costs: are paid when customer orders are not met:

- lost sales costs (lost profit and negative effect on future sales)
- back order costs (in case of a delayed sale, often a penalty has to be paied)

4) Obsolescence costs

Inventory Management models can be classified according to various criteria:

- Deterministic versus stochastic models: in a deterministic model, demands, costs, lead times ... are assumed to be known in advanced, while in a stochastic model some data are uncertain

- Fast-moving items versus slow-moving items: fast-moving items are manufactured and sold regularly, so we have to decide when to replenish stocking points and how much to order; slow-moving items (e.g. spare parts of complex machineries) have a very low demand (e.g. a few units every 20 years), and the main issue is to decide how many items to produce and store at the beginning of the planning horizon
- Number of stocking points: optimal inventory policies can be derived analytically only for single stocking point
- Number of commodities: single or multiple
- Instantaneous resupply versus noninstantaneous resupply: i.e. a stocking point can be replenished almost instantaneously or not (e.g. at a constant rate via a manufacturing process)

Single Stocking Point : Single Commodity under Constant Demand Rate

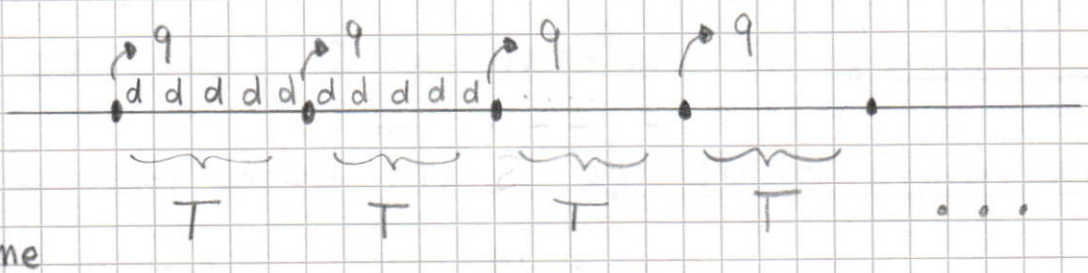
(Ghiani, Laporte, Musmanno: Chap. 4.4)

Assumptions:

- single stocking point for a single commodity constant rate demand (d)
- planning horizon: infinite

Since d is constant, a natural policy is to replenish the stocking point on a periodic basis : let

- T time lapse between two consecutive orders (period)
- q order size, i.e the amount of product to order each time



It is: $\boxed{q = d \cdot T}$

Notice: q (how much?) and T (when?) are variables to be determined

Moreover, let: (costs to be taken into account)

- c : value of an item
- K : fixed reorder cost
- h : holding cost per item and per time unit
- u : shortage cost per item (independent on the duration of the storage)
- v : shortage cost per item and per (shortage) time unit

Usually it is assumed:

$$\boxed{h = p \cdot c}$$ i.e. a fraction p of c

with p a banking interest rate

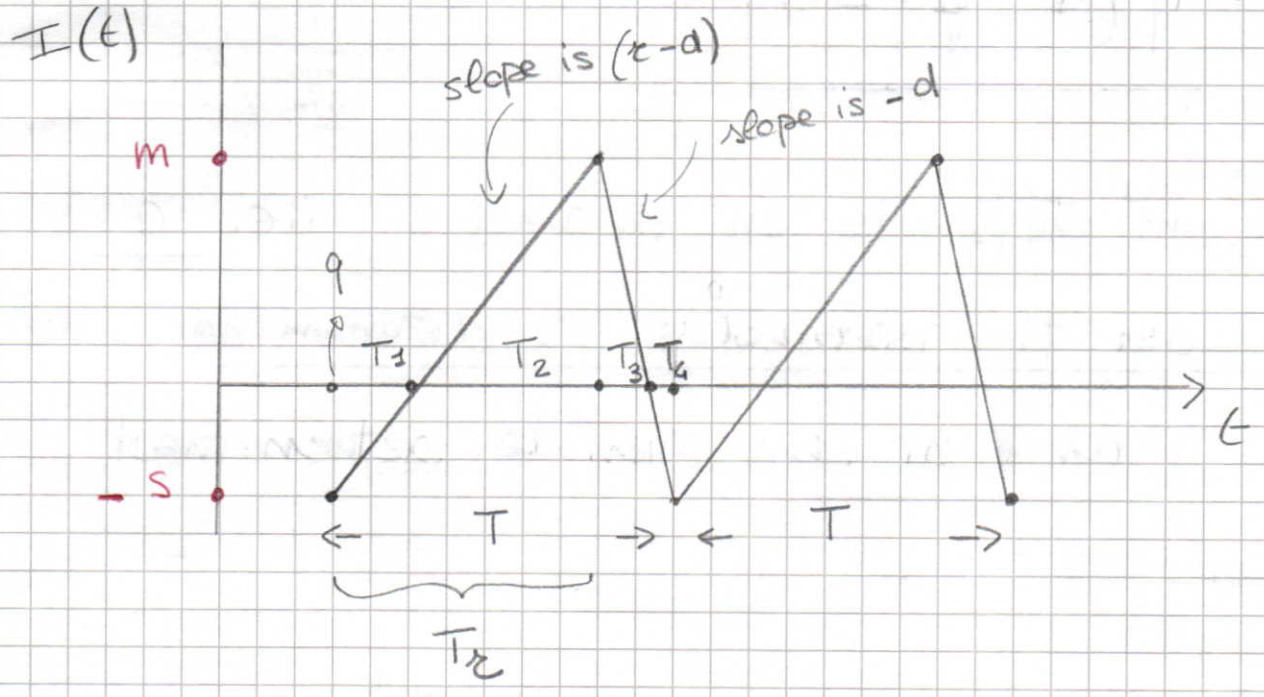
The problem: to determine q (and so T), the maximum shortage S (in each period) and the maximum inventory level m (in each period) in such a way as to minimize the overall average cost per time unit.

Let :

- T_r (≥ 0) <u>replenishment time</u>, i.e time needed for replenishment (of q)
- r <u>replenishment rate</u>, i.e. number of items per unite of time received during T_r

Then: $\boxed{q = r \cdot T_r}$

<u>Inventory level $I(t)$ as a function of time t:</u>



- $T = T_1 + T_2 + T_3 + T_4$ $\quad T_r = T_1 + T_2$

Note that:

- during the replenishment (i.e. in T_r), the net number of items stocked per unit of

time if $(r-d)$ (assuming $r > d$)

- after T_r, the stock level decreases at a rate equal to $-d$ (items are picked up at a rate d)

- $s + m = (r-d) \cdot T_r$, i.e.

$$\boxed{m} = (r-d)T_r - s = \underbrace{r\,T_r}_{=q} - d \cdot \underbrace{T_r}_{=\frac{q}{r}} - s =$$

$$= \boxed{q\left(1 - \frac{d}{r}\right) - s}$$

Therefore, m depends on q and s, i.e. q and s are the variables to determine, whereas T and m can then be determined by formulas:

$$T = \frac{q}{d} \quad \text{and} \quad m = q\left(1 - \frac{d}{r}\right) - s$$

The total average cost per time unit is:

$$u(q,s) = \frac{1}{T}\underbrace{\left(K + c\cdot q + h\cdot \overline{I}\cdot T + u\cdot s + v\cdot \overline{S}\cdot T\right)}_{\text{average cost per period}}$$

The holding cost $h\cdot \overline{I}\cdot T$ and the shortage costs (us and $v\cdot \overline{S}\cdot T$) depend on the average inventory level $\overline{I}$ and on the average shortage level $\overline{S}$, resp.:

- $\overline{I} = \frac{1}{T}\int_0^T I_+(t)\,dt = \frac{1}{T}\left(\frac{m\,(T_2+T_3)}{2}\right)$

  (I_+: "positive part" of $I(t)$)

- $\overline{S} = \frac{1}{T}\int_0^T I_-(t)\,dt = \frac{1}{T}\left(\frac{s\,(T_1+T_4)}{2}\right)$

  (I_-: "negative part" of $I(t)$)

but T_1, T_2, T_3 and T_4?

Since: $s = (r-d)\,T_1$

$m = (r-d)\,T_2$

$m = d\,T_3$

$s = d\,T_4$

we get: $T_1 = s/(r-d)$

$T_2 = m/(r-d)$

$T_3 = \frac{m}{d}$

$T_4 = \frac{s}{d}$

Therefore:

$$\bullet\ \boxed{\overline{I} = \frac{1}{T}\left(\frac{m(T_2+T_3)}{2}\right) = \frac{1}{T}\left(\frac{m^2\left(\frac{1}{(r-d)}+\frac{1}{d}\right)}{2}\right) =}$$

$$= \frac{d}{2q}\cdot m^2\left(\frac{d+r-d}{d(r-d)}\right) = \frac{m^2}{2q}\cdot\left(\frac{r}{r-d}\right) =$$

$$= \frac{m^2}{2q}\cdot\left(\frac{1}{\frac{(r-d)}{r}}\right) = \boxed{\frac{m^2}{2q\left(1-\frac{d}{r}\right)}}$$

as a function of q and m

$$= \frac{\left[q\left(1-\frac{d}{r}\right)-s\right]^2}{2q\left(1-\frac{d}{r}\right)}$$

as a function of q and s

Similarly: $\boxed{\overline{S} = \frac{s^2}{2q\left(1-\frac{d}{r}\right)}}$

By replacing in $\mu(q,s)$:

$$\mu(q,s) = K \cdot \frac{d}{q} + cd + \frac{h\left[q\left(1-\frac{d}{r}\right)-s\right]^2}{2q\left(1-\frac{d}{r}\right)} + \frac{u \cdot s \cdot d}{q} + \frac{v\, s^2}{2q\left(1-\frac{d}{r}\right)}$$

This is a nonlinear, but convex function. Therefore, a minimum (global) point (q^*, s^*) of $\mu(q,s)$ can be obtained by solving:

$$\begin{cases} \frac{\delta}{\delta q}\mu(q,s) = 0 \\ \frac{\delta}{\delta s}\mu(q,s) = 0 \end{cases}$$

stationary point

We get (no proof):

- $q^* = \sqrt{\frac{h+v}{v}} \cdot \sqrt{\frac{2Kd}{h\left(1-\frac{d}{r}\right)} - \frac{(ud)^2}{h(h+v)}}$
- $s^* = \frac{(hq^* - ud)\left(1-\frac{d}{r}\right)}{(h+v)}$

i.e.: in each period, the optimal amount to order to minimize the total average unit

cost is q^*, and the optimal time interval to order is $T^* = \frac{q^*}{d}$.

In each period, the optimal shortage level (to satisfy the demand by minimizing $\mathcal{U}(q,s)$) is s^*, while the optimal maximum inventory level is $m^* = q^*\left(1 - \frac{d}{r}\right) - s^*$.

Notice: in this (simple) case, we were able to determine the optimal inventory policy in an analytic way, i.e. by means of formulas for q^*, T^*, s^* and m^*

An example (page 127)

- Golden Food distributes tinned foodstuff in Great Britain
- In a warehouse in Birmingham:
  - demand rate d for tomato purée is 400 pallets a month (so, constant each month)
  - the value of a pallet is $c = £2500$

- the annual interest rate is $p = 14,5\%$ (including warehousing costs)
- issuing an order costs $K = £\,30$
- the replenishment rate is $r = 40$ pallets per day
- shortages are not allowed (i.e. $s^* = 0$) ($v = u = 0$)

Using the data, the holding cost is:

$$h = 0.145 \cdot 2500 = £\,362.5 \text{ per year per pallet}$$

$$= £\,30.2 \text{ per month per pallet}$$

Note that the unit of time is a month. By applying the optimal inventory policy:

$$q^* = \sqrt{\frac{2Kd}{h\left(1 - \frac{d}{r}\right)}} = \sqrt{\frac{2 \cdot 30 \cdot 400}{30.2\left(1 - \frac{400}{\underbrace{(40 \cdot 20)}_{\text{"monthly } r}}\right)}} \simeq 39.9 \simeq 40 \text{ pallets}$$

In the formula we assumed that the number of workdays in a month is 20, hence the monthly replenishment rate is $(40 \cdot 20)$

As a consequence:

$$T^* = \frac{40}{400} = \frac{1}{10} \text{ month} = 2 \text{ workdays}$$

$$T_r^* = \frac{40}{40} = 1 \text{ workday}$$

$$m^* = q^*\left(1 - \frac{d}{\text{"monthly" } r}\right) = \qquad (s=0)$$

$$= 40\left(1 - \frac{400}{(40 \cdot 20)}\right) = 40 \cdot \frac{1}{2} = 20 \text{ pallets}$$

Therefore: to minimize its total average unit cost the warehouse must order 40 ($=q^*$) pallets every 2 ($=T^*$) workdays. In each period, the replenishment will last 1 ($=T_r^*$) workday, and the level of inventory will vary between 0 (no shortage) and a maximum level equal to 20 ($=m^*$).

Special case of instantaneous resupply

the optimal inventory policy can be obtained from the previous, more general, policy by considering $\tau \to +\infty$ (and so, $T_\tau = \frac{q}{\tau} \to 0$):

- $q^* = \sqrt{\frac{h+v}{v}} \cdot \sqrt{\frac{2Kd}{h} - \frac{(ud)^2}{h(h+v)}}$
- $s^* = \frac{hq^* - ud}{h+v}$

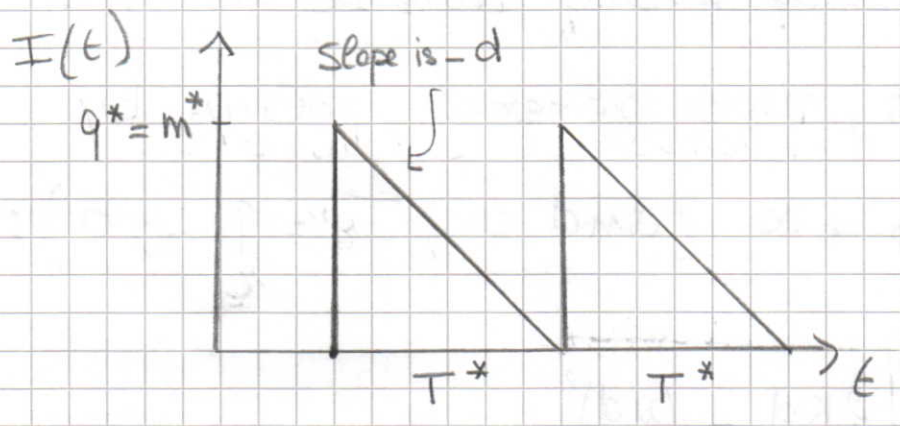
If shortages are not allowed (i.e. $s^* = 0$), then:

$$\boxed{q^* = \sqrt{\frac{2Kd}{h}}}$$

This is the classical Economic Order Quantity (EOQ) model introduced by F. W. Harris in 1913. The corresponding optimal cost is:

$$\mu(q^*) = Kd/q^* + c \cdot d + \frac{1}{2} h \cdot q^*$$

The inventory level, as a function of time, in EOQ becomes:



example

- Al-Bufaia Motors manufactures spare parts for aircraft engines in Saudi Arabia
- component YO2PN has:
  - demand $d = 220$ units per year
  - unit production cost $c = \$1200$
  - set-up cost (for producing) $K = \$800$
  - annual interest rate $p = 18\%$ (including warehousing costs)
  - shortages are not allowed (i.e. $s = 0$)

The holding cost is:

$$h = (0.18) \cdot 1200 = 216 \text{ dollars/year per unit}$$

Since replenishment is achieved via internal production, we can assume $r \to +\infty$, and so the EOQ model can be applied (d is constant per year):

$$q^* = \sqrt{\frac{2K \cdot d}{h}} = \sqrt{\frac{2 \cdot 800 \cdot 220}{216}} = 40,37 \text{ units}$$

$$T^* = \frac{40.37}{220} = 0.18 \text{ years} = 66.8 \text{ days}$$

Therefore, to minimize the cost Al-Bufeira Motors has to manufacture ~ 40 units of component Y02PX every ~ 67 days.

Notice: in practice, q^* and T^* must be rounded to apply the optimal policy

Single stocking point : a multicommodity model with capacity constraints

(Ghiani, Laporte, Musmanno : 4, 7, 1)

- more commodities in stock
- case where a limit is placed on the total investment in inventories, or on the common space

Assumption : for each product the EOQ model assumptions hold (i.e. constant demand rate d_j, $r \to +\infty$, no shortage)

Let :

- n : number of commodities in stock
- q_j : amount of commodity j to reorder at each replenishment, $j = 1, .. n$

↑ variable !

Under the EOQ assumption, the average unit cost for commodity j is:

$$\mu_j(q_j) = \frac{K_j d_j}{q_j} + c_j d_j + \frac{1}{2} h_j q_j \quad , j = 1, \ldots n$$

with:

K_j : fixed reorder cost

c_j : unit value of item j

$h_j = p_j \cdot c_j$: holding cost for item j

Therefore, the <u>average global unit cost</u> (for all commodities in stock) is:

$$\mu(q_1, q_2, \ldots q_n) = \sum_{j=1}^{n} \mu_j(q_j)$$

Assume that there is a side constraint (<u>capacity constraint</u>), usually representing a budget constraint or a warehouse constraint, of form:

$$\sum_{j=1}^{n} a_j \cdot q_j \leq b$$

(a_j, b: constants) — linear constraint

Then, the inventory management problem that we want to solve can be expressed as:

$$\text{Min} \sum_{j=1}^{n} \left(\frac{K_j \cdot d_j}{q_j} + c_j \cdot d_j + \frac{1}{2} h_j q_j \right)$$

subject to $\sum_{j=1}^{n} a_j \cdot q_j \leq b$

$$q_1, q_2, \ldots q_n \geq 0$$

Nonlinear (hard) model: it is possible to suggest a simple heuristic for the special case of identical interest rates, i.e. $P_j = P$, $j = 1, \ldots n$

Heuristic:

Step 1: compute the EOQ order sizes if the capacity constraint would be absent (or relaxed):

$$q_j' = \sqrt{\frac{2 K_j d_j}{P \cdot c_j}} \qquad j = 1, \ldots n$$

Note that $(q_1', \dots q_n')$ is the optimal solution of the relaxation, since by removing the capacity constraint the n commodities can be managed independently

If $\sum_{j=1}^{n} a_j \cdot q_j' \leq b$, then STOP:

$(q_1', \dots q_n')$ is optimal also for the original (i.e. not relaxed) model

Step 2: - consider to increase the common interest rate p of a quantity δ:

$$q_j(\delta) = \sqrt{\frac{2 K_j d_j}{(p+\delta) c_j}} \qquad j = 1, \dots n \;;$$

- determine the minimum value of δ, say δ^*, that allows to satisfy the capacity constraint, i.e.:

$$\sum_{j=1}^{n} a_j \, q_j(\delta^*) = b$$

Note: this was $>$ by considering $(q_1', \dots q_n')$

We get:

$$\sum_{j=1}^{n} \left(a_j \sqrt{\frac{2 K_j d_j}{(p+\delta) c_j}} \right) = b \quad \Leftrightarrow$$

$$\frac{1}{\sqrt{(p+\delta)}} \cdot \sum_{j=1}^{n} \left(a_j \cdot \sqrt{\frac{2 K_j d_j}{c_j}} \right) = b \quad \Leftrightarrow$$

$$\sqrt{(p+\delta)} = \frac{1}{b} \cdot \sum_{j=1}^{n} \left(a_j \cdot \sqrt{\frac{2 K_j d_j}{c_j}} \right) \quad \Leftrightarrow$$

$$\delta = \boxed{\delta^* = \left(\frac{1}{b} \cdot \sum_{j=1}^{n} \left(a_j \cdot \sqrt{\frac{2 K_j d_j}{c_j}} \right) \right)^2 - p}$$

- return the feasible solution

$(q_1(\delta^*), q_2(\delta^*), \ldots q_n(\delta^*))$: STOP

Example

- New Frontier distributes Knapsacks and suitcases
- The most successful models:
  - Preppie Knapsack:
    - yearly demand $d_1 = 150.000$ units
    - value $c_1 = \$ 30$
    - yearly $h_1 = 20\% c_1$

- Yuppie suitcase :
  - yearly demand $d_2 = 100.000$ units
  - value $c_2 = \$ 45$
  - yearly $h_2 = 20\% \, c_2$

so, identical interest rates

In addition : $K_1 = K_2 = \$ 250$

Company's constraint : the average capital invested in inventory can not exceed $\$ 75.000$, i.e.

$$\frac{30}{2} q_1 + \frac{45}{2} q_2 \leq 75.000$$

In fact, since

$$\bar{I} = \frac{\left[q\left(1 - \frac{d}{r}\right) - s \right]^2}{2q\left(1 - \frac{d}{r}\right)}$$

is the average inventory level in general (under the considered hypotheses), in EOQ we have :

$$\bar{I} = \frac{q}{2}$$

So $\frac{q_1}{2}$ is the average inventory level for Preppie Knapsack, $\frac{q_2}{2}$ is the average inventory level for Yuppie suitcase, and therefore $30\left(\frac{q_1}{2}\right) + 45\left(\frac{q_2}{2}\right)$ is the average capital in inventories

note: it is assumed, for precaution, that the overall average inventory level is the sum of the two average inventory levels

We can apply the heuristic:

Step 1:

$$q'_1 = \sqrt{\frac{2 \cdot 250 \cdot 150000}{(0.2)\,30}} = 3535,53 \text{ units}$$

$$q'_2 = \sqrt{\frac{2 \cdot 250 \cdot 100000}{(0.2)\,45}} = 2357,02 \text{ units}$$

It is $30 \cdot \frac{q'_1}{2} + 45 \cdot \frac{q'_2}{2} > 75.000$

Step 2: $\delta^* = 0.2$

$$q_1(0.2) = \sqrt{\frac{2 \cdot 250 \cdot 150000}{(0.2+0.2)\,30}} = 2500 \text{ units}$$

$$q_2(0.2) = \sqrt{\frac{2 \cdot 250 \cdot 100000}{(0.2+0.2)\,45}} \simeq 1666 \text{ units}$$

The reorder point policy

(Ghiani, Laporte, Musmanno: 4.8.3)

- Stochastic model (use data forecasts)

Main characteristic: the inventory level $I(t)$ is kept under observation in an almost continuous way; when it reaches a reorder point ℓ, then a constant quantity q is reordered:

The reorder size q may be computed as follows:

$$q = \sqrt{\frac{2K\bar{d}}{h}}, \qquad \text{EOQ formula}$$

where $\bar{d}$ is the expected demand rate in the time unit

Assumptions:

- the expected demand rate $\bar{d}$ and the standard deviation σ_d are constant in time
- the lead time t_e is constant

How to determine the reorder point ℓ: we want that $I(t)$ be nonnegative, during t_e, with a given probability α; this is equivalent to impose that, during the interval t_e, the total demand does not exceed ℓ with probability α.

Let d be a random variable denoting the total demand in the interval t_e; then

$$d = d_1 + d_2 + \dots + d_{t_e}$$

(a suff. large number!) (t_e denotes the width)

first time unit — last time unit

By assuming an identical demand distribution and assuming $d_1, d_2 \dots d_{t_e}$ independent, in each unit time, then from the Central limit theorem it is possible to assume that d is distributed according to a normal distribution with expected value $\bar{d} \cdot t_e$ and standard

deviation $\sigma_d \sqrt{T_e}$: $\boxed{d \text{ is } N(\bar{d} T_e, \sigma_d \sqrt{T_e})}$

standard normal variable

Property : if X is $N(0,1)$, then

$m + \sigma X$ is $N(m, \sigma)$.

Therefore, d can be represented as

$$\boxed{d = \bar{d} T_e + \sigma_d \cdot \sqrt{T_e} \cdot X}$$, with X being a $N(0,1)$

Given α, let z_α be the value under which X falls with probability α (e.g. $z_\alpha = 2$ for $\alpha = 0.9772$ and $z_\alpha = 3$ for $\alpha = 0.9987$):

Then : to guarantee ← our goal

$$\underbrace{d = \bar{d} T_e + \sigma_d \cdot \sqrt{T_e} \cdot X}_{\text{demand in } T_e} \leq e$$ with probability α

we can set $e = \bar{d} \cdot T_e + \sigma_d \cdot \sqrt{T_e} \cdot z_\alpha$;

in fact, $X \leq z_\alpha$ with probability α,

and so

$$d = \bar{d} \cdot T_e + \sigma_d \cdot \sqrt{T_e} \cdot X \leq \underbrace{\bar{d} \cdot T_e + \sigma_d \cdot \sqrt{T_e} \cdot z_\alpha}_{= e}$$

with probability α

This is why the policy sets the reorder point $\ell = \bar{d} \cdot t_\ell + \sigma_d \sqrt{t_\ell} \cdot z_\alpha$

- ℓ minus the average demand in the reorder period gives the <u>safety stock</u> I_s:

$$I_s = \underbrace{\bar{d} \cdot t_\ell + \sigma_d \sqrt{t_\ell} \cdot z_\alpha}_{\ell} - \underbrace{\bar{d} \cdot t_\ell}_{\text{average demand}} = \sigma_d \cdot \sqrt{t_\ell} \cdot z_\alpha$$

<u>An example</u>

- Papier is a French retail chain
- For the outlet in Lyon:
  - the expected demand $\bar{d}$ for mouse pads is 45 units <u>per month</u>
  - the value of an item is $c = €4$
  - the fixed reorder cost is $K = €30$
  - the annual interest rate is $p = 20\%$
  - the forecast for standard deviation is $\sigma_d = 5$
  - the lead time is $t_\ell = 1$ month

- the required service level must be 97.7%, i.e. it is imposed that the demand for mouse pads be satisfied with percentage 97.7% or, equivalently, in the reorder period $I(t) \geq 0$ with probability $\alpha = 0.977$

First compute the holding cost:

$$h = p \cdot c = (0.2) \cdot 4 = 0.8 \text{ €/year per item}$$
$$= 0.067 \text{ €/month per item}$$

< the time unit is month >

Then compute:

$$q = \sqrt{\frac{2 \cdot K \cdot \bar{d}}{h}} = \sqrt{\frac{2 \cdot 30 \cdot 45}{0.067}} \simeq 201 \text{ items}$$

To guarantee the required service level:

$$\ell = 45 \cdot 1 + 5 \cdot 1 \cdot \underbrace{2}_{= z_\alpha} = 45 + 10 = 55 \text{ units}$$

$$I_S = 55 - 45 = 10 \text{ units}$$

That is: we must reorder when the inventory level reaches $\ell = 55$ units; the corresponding safety stock is $I_S = 10$ units (to take into account variations around the average demand in the reorder period, i.e. 45 units)

Slow-moving items model

(Ghiani, Laporte, Musmanno: 4.11)

When demand is very low (e.g. space parts of complex machineries), the main issue is to determine how many items to purchase at the beginning of the planning horizon

Considered scenario:

- c : purchase cost of an item at the beginning of the planning horizon
- u : " " " during the planning horizon

shortage penalty ($u > c$)

On the other hand, holding costs are disregarded.

Let:

- n : number of items to be purchased at the beginning of the planning horizon (decision variable)
- m : number of items demanded in the planning horizon (unknown!)

Given m, the total cost is:

$$C(n, m) = \begin{cases} c \cdot n & \text{if } n \geq m \\ c \cdot n + u(m-n) & \text{if } n < m \end{cases}$$

However, m is unknown.

Let:

- $P(m)$: probability that m items will be demanded

Then, the expected cost if n items will be purchased is:

$$\overline{C}(n) = \sum_{m=0}^{+\infty} C(n, m) \cdot P(m) = c \cdot n + u \sum_{m=n+1}^{+\infty} (m-n) \cdot P(m)$$

The problem to be solved is:

$$\min \overline{C}(n)$$

$$n \geq 0, \text{ integer}$$

To solve the problem in an analytical way, let:

$F(m) = \text{Prob}(d \leq m)$, where

d denotes the demand (random variable). That is, $F(m)$ is the probability that m units (or less) will be demanded during the planning horizon.

Then, the following relationships hold:

$$1)\quad \overline{C}(n-1) = c\cdot(n-1) + u\cdot\sum_{m=n}^{+\infty}(m-n+1)P(m) =$$

$$= c\cdot n - c + u\sum_{m=n}^{+\infty}(m-n)\cdot P(m) + u\cdot\sum_{m=n}^{+\infty}P(m) =$$

$$= \overline{C}(n) - c + u\cdot\text{Prob}(d \geq n) =$$

$$= \overline{C}(n) - c + u\cdot\text{Prob}(d > n-1) =$$

$$= \overline{C}(n) - c + u\cdot(1 - F(n-1))$$

2) $\overline{C}(n+1) = c \cdot (n+1) + u \cdot \sum_{m=n+2}^{+\infty} (m-n-1) \cdot P(m) =$

$= c \cdot n + c + u \cdot \sum_{m=n+1}^{+\infty} (m-n-1) \cdot P(m) =$

↳ added a zero addendum for $m = n+1$

$= (cn) + c + \left(u \cdot \sum_{m=n+1}^{+\infty} (m-n) \cdot P(m) \right) - u \sum_{m=n+1}^{+\infty} P(m) =$

$= \overline{C}(n) + c - u \cdot \text{Prob}(d \geq n+1) =$

$= \overline{C}(n) + c - u \, \text{Prob}(d > n) =$

$= \overline{C}(n) + c - u \left(1 - F(n)\right)$

Now, the <u>minimum expected cost</u> is achieved if n^* items are purchased at the beginning of the planning horizon, where:

$$\begin{cases} \overline{C}(n^*) \leq \overline{C}(n^*-1) \\ \overline{C}(n^*) \leq \overline{C}(n^*+1) \end{cases}$$

By exploiting the previous relationships:

- $\bar{C}(n^*) \leq \bar{C}(n^*) - e + u\left(1 - F(n^* - 1)\right)$

  $\Leftrightarrow \quad u\,F(n^* - 1) \leq u - e$

  $\Leftrightarrow \quad F(n^* - 1) \leq \dfrac{u - e}{u}$

- $\bar{C}(n^*) \leq \bar{C}(n^*) + e - u\left(1 - F(n^*)\right)$

  $\Leftrightarrow \quad u\,F(n^*) \geq u - e$

  $\Leftrightarrow \quad F(n^*) \geq \dfrac{u - e}{u}$

That is, the optimal number n^* of items to be purchased is such that:

$$\boxed{F(n^* - 1) \leq \frac{u - e}{u} \leq F(n^*)}$$

Recall that F is a monotone nondecreasing function

example

Hydro Ltd uses five LIFO3 generators in its power plants in Nigeria:

- each piece of machinery (and so each generator) has an average life of 20 years
- in this period, the expected number of failures (i.e. the expected demand for spare parts of generators) is 1.4
- the cost of a spare part, purchased at the beginning, i.e. when a generator is manufactured, is $c = \$60.000$
- the production unit cost during the period is $u = \$300.000$
- the failure process during the planning horizon is assumed to be modelled according to a Poisson probability distribution with expected value $\boxed{\lambda = 5 \cdot (1.4) = 7}$ faults per life cycle

This is a discrete random variable (assuming values 0, 1, 2, ..) which is often used to model the random number of

"arrivals" (i.e. generator faults in our case) in a given time horizon. Therefore:

$$Prob(d = m) = \frac{e^{-\lambda} \cdot \lambda^m}{m!} \quad , m = 0, 1, \ldots$$

$$= \frac{e^{-7} \cdot 7^m}{m!}$$

while the cumulative probability can be computed according to:

$$F(0) = Prob(d = 0) = 0,009$$

$$F(m) = F(m-1) + Prob(d = m) \quad , m = 1, 2, \ldots$$

We get:

m	Prob(d = m)	F(m)	
0	0.0009	0.0009	
1	0.0064	0.0073	
2	0.0223	0.0296	
3	0.0521	0.0818	
4	0.0912	0.1730	
5	0.1277	0.3007	
6	0.1490	0.4497	
7	0.1490	0.5987	
8	0.1304	0.7291	← 0.8
n* 9	0.1014	0.8305	
10	0.0710	0.9015	

Since $\frac{\mu - e}{\mu} = \frac{300.000 - 60.000}{300.000} = 0.8$,

we have $F(\underset{\substack{\uparrow \\ n^*-1}}{8}) \leq 0.8 \leq F(\underset{\substack{\uparrow \\ n^*}}{9})$, and so

the number of spare parts to be purchased and stocked at the beginning of the planning horizon to minimize the expected cost is $n^* = 9$.

Note that, since $F(n)$ is monotone nondecreasing, we have a unique(•) n^* such that $F(n^*-1) \leq \frac{\mu - e}{\mu} \leq F(n^*)$, and therefore n^* is a local minimum but also a global minimum, i.e. it is the optimal solution of the presented inventory model.

(•) more precisely: in case of a sequence of equal probability values, select the minimum n^* such that $F(n^*-1) \leq \frac{\mu - e}{\mu} \leq F(n^*)$