Logistics

Lecture notes*

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Chapter 7

Routing problems

Routing problems are of fundamental importance in the management of the provision of goods and services in distribution systems. In particular, the use of optimization procedures allows substantial savings (from 5 % to 20 %) in the global transportation costs.

Here we address *Vehicle Routing Problems (VRP)*, concerning the distribution of goods from depots to final users (or customers). Typical applications are solid waste collection, school bus routing, dial-a-ride systems and, also, home care applications.

7.1 A general VRP formulation

Given a set of *customers*, a fleet of *vehicles* located in one or more *depots*, and given a *road network*, the generic Vehicle Routing Problem (VRP) consists in determining a set of *routes*, each performed by a single vehicle that starts ad ends at its own depot, in such a way that all the customer requirements are fulfilled, and all the operational constraints (if present) are satisfied, by minimizing the global transportation cost.

Let us describe some typical VRP characteristics by considering the main components, the different operation constraints that can be imposed, and the possible objectives to be achieved.

7.1.1 Typical VRP characteristics

The main components of a Vehicle Routing Problem are:

1. *Road network*: it is described by means of a graph, where nodes correspond to road junctions, depots and customer locations, and where arcs represent streets. Arcs can be directed (e.g. to model one-way streets) or undirected (to model traversal in both directions). Each arc is associated with a *cost*, which generally

represents its length, and a *travel time*, which may depend on the vehicle type or on the considered period.

2. *Customers*: typical characteristics of customers are:
   - the node of the road network where the customer is located;
   - the amount of goods (*demand*), possibly of different types, that must be delivered or collected;
   - the period of time (*time window*) during which the customer can be served;
   - the times required to deliver or collect goods at the customer location (*unloading* and *loading times*, respectively);
   - the subset of vehicles that can be used to serve the customer.

   If some customers can not be fully satisfied, then different *priorities*, or *penalties*, associated with (partial) lack of service, can be assigned to them.

3. *Routes*: each route starts and ends at a depot (usually the same). Each depot is characterized by the number and types of vehicles associated with it, and by the total amount of goods it can deal with.

4. *Fleet of vehicles*: for each available vehicle, we are given:
   - the home depot (vehicles may end the service at a depot other than their home depot);
   - the capacity, i.e. the maximum weight or volume or number of units the vehicle can load;
   - a possible subdivision into compartments;
   - some possible devices available for loading and unloading;
   - a subset of arcs which can be traversed by the vehicle;
   - the costs associated with vehicle utilization (per distance unit, per time unit...).

5. *Drivers*: they are equivalent to vehicles, in our context.

6. *Operational constraints*: typical constraints are:
   - the current load can not exceed the vehicle capacity;
   - customers served in a route can require only delivery, only collection, or both;
   - customers are served only within their time windows;
   - there can be precedence constraints on the order in which customers in a route are served:

Pickup and delivery: a route can perform both collection and delivery of goods, and goods collected by pickup customers must be carried to the corresponding delivery customers;

VRP with backhauls: all deliveries must be performed before the collections.

The evaluation of the global cost of the routes and the check of the operational constraints require to know the travel time and the travel cost between each pair of customers and between depots and customers. To this end, the original (sparse) road network is usually modelled as a complete graph G such that:

- the nodes in G correspond to customers and depots;
- for each edge (i, j) in G:
  - c_{ij} is the cost of the shortest path from i to j in the road network;
  - t_{ij} is the total time of such a shortest path in the road network.

G can be directed or undirected depending on the property of the corresponding cost and travel time matrices to be asymmetric or symmetric, respectively. The VRP models presented in the rest of the chapter will be defined on such a complete graph G.

7. *Optimization objectives* (they are often contrasting objectives):
   - minimization of the global transportation cost: this depends on the global distance travelled and on the fixed costs associated with the used vehicles;
   - balancing of the routes (for travel time, vehicle load...);
   - minimization of the penalties for partial service of some customers.

   Often a weighted combination of these objectives is addressed.

7.1.2 Relevant variants of VRP

Relevant variants of the generic VRP, previously introduced, are:

The Stochastic VRP: demands and/or travel times are random variables;

The Dynamic VRP: in this case, demands, costs and/or travel times are time dependent;

Arc Routing Problems: customers are located along the arcs of the road network (this is typical, for example, in postal delivery services).

There exists a particular case of VRP called the *Travelling Salesman Problem* (TSP): it is a VRP with a single depot, a single (uncapacitated) vehicle and no operational constraints.

Hereafter the Capacitated Vehicle Routing Problem (CVRP) will be introduced and formulated.

7.2 The Capacitated VRP (CVRP)

Let us make some assumptions on the problem:

- customers are of the same kind (e.g. delivery customers), with deterministic and unsplittable demand;
- the fleet of the vehicles is homogeneous, with vehicles located in a unique home depot;
- the only operational constraint is on vehicle capacity.

Let $G = (V, A)$ be a complete graph such that:

- $V = \{0, 1, \dots n\}$ is the set of the nodes, where 0 denotes the depot and $\{1, \dots n\}$ is the set denoting the customers;
- $c_{ij} \geq 0,\ \forall (i, j) \in A$, is the travelling cost from i to j.

If G is directed, then $c_{ij} \neq c_{ji}$ is possible: in this case we get the Asymmetric CVRP (ACVRP). If G is undirected, and so $c_{ij} = c_{ji}$, then we have the Symmetric CVRP (SCVRP).

Let us introduce the following additional input data:

- $d_i \geq 0$, the demand of customer $i,\ i = 1, \dots n$ ($d_0 = 0$);
- K identical vehicles available at the depot, with capacity C (assume $d_i \leq C,\ i = 1, \dots n$). Assume that $K \geq K_{\min}$, where $K_{\min}$ is the minimum number of vehicles to serve all customers ($K_{\min} \geq \left\lceil \frac{\sum_{i=1}^{n} d_i}{C} \right\rceil$).

The CVRP problem is to determine the tours of the K vehicles (i.e. K directed cycles in G) such that:

i) each tour includes the depot (i.e. node 0);

ii) each customer belongs to exactly one tour (i.e. he/she is visited by exactly one vehicle);

iii) the sum of the demands of the customers belonging to the same tour does not exceed C (the capacity of the vehicle);

by minimizing the total cost.

Some relevant variants to the Capacitated Vehicle Routing problem are:

- there can be unused vehicles (if $K > K_{\min}$); therefore it is possible to have fixed costs for using the vehicles;
- vehicles may have different capacities $C_k,\ k = 1, \dots K$;
- tours formed by a single customer can be forbidden.

Observe that CVRP generalizes TSP: TSP is the special case where $K = 1$ and $C \geq \sum_{i=1}^{n} d_i$. Therefore, CVRP is *NP*-hard (in a strong sense).

7.2.1 An example

Consider an instance where $n = 8, K = K_{\min} = \left\lceil \frac{\sum_{i=1}^{8} d_i}{8} \right\rceil = \left\lceil \frac{13}{8} \right\rceil = 2, C = 8$ and $d = (1, 3, 2, 1, 2, 1, 1, 2)$.

If all arcs in Figure 7.1 cost 1, then the cost of the feasible solution to CVRP shown in the figure is 10. Note that the numbers associated with the nodes represent the customer demands.

7.3 Basic models to CVRP

Hereafter we shall present some basic ILP models for the asymmetric version (ACVRP), which can be easily adapted to the symmetric one.

7.3.1 Basic two-index model (VRP1)

Let us introduce the following decision variables:

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ belongs to a tour} \\ 0 & \text{otherwise} \end{cases}, \quad \forall (i,j) \in A.$$

Overall, there are $\mathcal{O}(n^2)$ variables. Using these variables, CVRP can e formulated as follows:

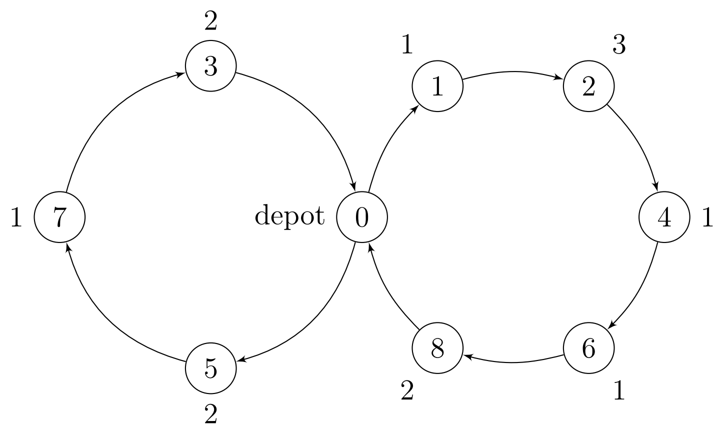


Figure 7.1: A feasible CVRP solution

$$
\begin{aligned}
&\min \sum_{(i,j)\in A} c_{ij}\, x_{ij} && && \\
&\sum_{(i,j)\in \mathrm{BS}(j)} x_{ij} = 1 && \forall j \in V\setminus\{0\} && ① \\
&\sum_{(j,i)\in \mathrm{FS}(j)} x_{ji} = 1 && \forall j \in V\setminus\{0\} && ② \\
&\sum_{(i,0)\in A} x_{i0} = K && && ③ \\
&\sum_{(0,i)\in A} x_{0i} = K && && ④ \\
&\sum_{i\notin S}\sum_{j\in S} x_{ij} \geq r(S) && \forall S\subseteq V\setminus\{0\}, S\neq\varnothing && ⑤ \\
&x_{ij}\in\{0,1\} && \forall (i,j)\in A && ⑥
\end{aligned}
\qquad \text{(VRP1)}
$$

Observe that equations ①, ②, ③ and ④ are linearly dependent, and thus one constraint could be removed.

Are ①, ②, ③, ④ sufficient to model ACVRP solutions? To answer this question, let us present a counterexample. Consider the example introduced in Section 7.2.1. The tour in Figure 7.2 satisfies ①, ②, ③ and ④, but it is not a feasible solution to ACVRP, since it contains a subtour which does not include the depot. Additional constraints, such as ⑤, are therefore necessary to correctly formulate the problem.

The *Cut Capacity Constraints (CCC)* (⑤) are in fact introduced to guarantee the connection of each tour to the depot, i.e. to avoid the situation above. In ⑤, $r(s)$ is the minimum number of vehicles to serve all the customers in S ($r(S)$ are assumed to be

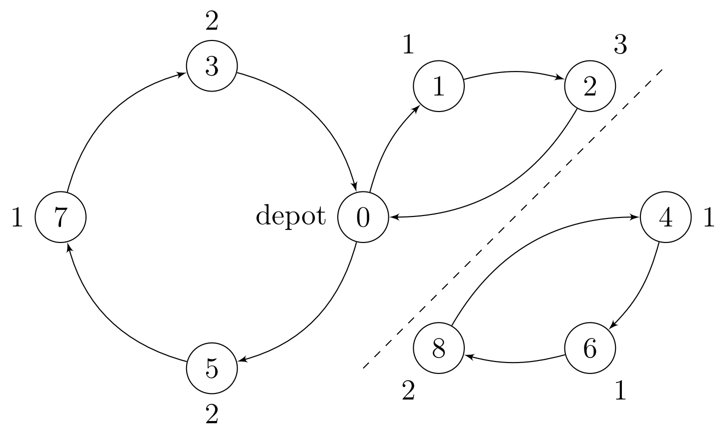


Figure 7.2: An unfeasible solution to CVRP

input data). For every S, $r(S)$ can be computed by solving a *Bin Packing Problem* related to S: find the minimum number of bins (which are vehicles in our context), each having a capacity C, to load all the elements (which are customers in our context) in S. This problem is *NP*-hard, but solvable in an efficient way.

The meaning of constraints ⑤ is that, for each *cut* $(V \setminus S, S)$ in the logistics network, which separates the customers in S from the depot, at least $r(S)$ arcs going from $V \setminus S$ to S must be present in any feasible solution, i.e. a sufficient number of vehicles must travel from $V \setminus S$ (where the depot is located) to S to serve the customers in S.

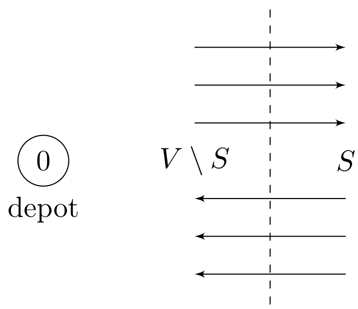
Let us resume the last example to see how these constraints work. In Figure 7.2, $V \setminus S = \{0, 1, 2, 3, 5, 7\}$ and $S = \{4, 6, 8\}$. S, and so the cut $(V \setminus S, S)$, violates the CCC constraints: in fact, since $C = 8$, then $r(S) = 1$, and so the CCC constraint related to $S = \{4, 6, 8\}$ is

$$\sum_{i \in \{0,1,2,3,5,7\}} \sum_{j \in \{4,6,8\}} x_{ij} \geq 1 = r(S),$$

which is not satisfied by the considered solution. Therefore, such a solution is not feasible to model (VRP1) (in fact, it is not feasible for CVRP).

Note that constraints ① – ④ imply:

$$\sum_{i \in V \setminus S} \sum_{j \in S} x_{ij} = \sum_{i \in S} \sum_{j \in V \setminus S} x_{ij}, \ \forall S \neq \varnothing, S \subseteq V \setminus \{0\}$$



i.e., at least $r(S)$ vehicles must travel from $V \setminus S$ (where the depot is located) to S, and the same number of vehicles come back from S to $V \setminus S$, as depicted by the figure above.

The CCC constraints guarantee not only the connection of each tour to the depot, but also the vehicle capacity satisfaction. By referring to the same example as before, the solution in Figure 7.3 is composed of two tours, but it is unfeasible: in fact, for the subset $S = \{1, 2, 4, 6, 7, 8\}$, it is $\sum_{i \in S} d_i = 9 > 8 = C$. The CCC constraint related to $S = \{1, 2, 4, 6, 7, 8\}$ is:

$$\sum_{i \in V \setminus S} \sum_{j \in S} x_{ij} \geq r(S) = 2 \ (\leftarrow \text{this number takes into account the vehicle capacity } C).$$

Notice that this constraint is violated by the solution in Figure 7.3, since only one arc crosses the cut. Therefore, the solution is not feasible to the ILP model VRP1.

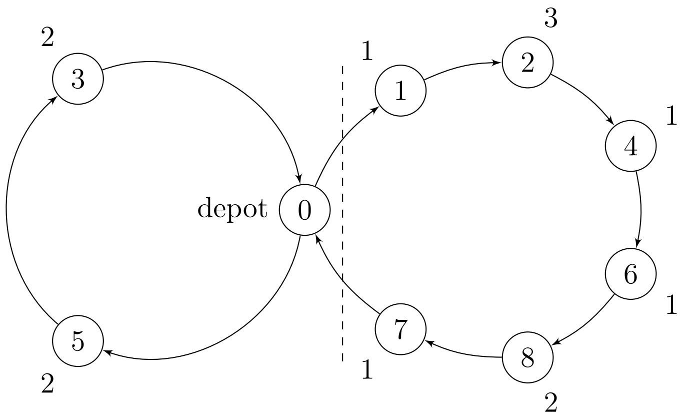


Figure 7.3: Another unfeasible solution to CVRP

The CCC constraints ⑤ can be replaced by the following *Generalized Subtour Elimination Constraints (GSEC)*:

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - r(S), \ \forall S \subseteq V \setminus \{0\}, S \neq \varnothing. \quad ⑦$$

The meaning is that at least $r(S)$ arcs (i.e. vehicles) must go out of S (and so, enter S from outside).

By considering the previous unfeasible solutions:

1. Infeasibility due to a subtour (see Figure 7.2): by considering $S = \{4, 6, 8\}$, the related GSEC constraint is:

$$\overbrace{\sum_{i \in S} \sum_{j \in S} x_{ij}}^{\text{internal selected arcs}} = 3 \leq |S| - r(S) = 3 - 1 = 2;$$

   it is violated, so the solution is not feasible to the ILP model (as it should be).

2. Infeasibility due to capacity violation (see Figure 7.3): by considering $S = \{1, 2, 4, 6, 7, 8\}$, the related GSEC constraint is:

$$\sum_{i \in S} \sum_{j \in S} x_{ij} = 5 \leq |S| - r(S) = 6 - 2 = 4;$$

   it is violated (due to $r(S) = 2$, caused by capacity considerations); so, also this solution is not feasible to our ILP model (as it must be).

Indeed, CCC (constraints ⑤) and GSEC (constraints ⑦) are equivalent, so:

① – ④ CCC ⑥	and	① – ④ GSEC ⑥

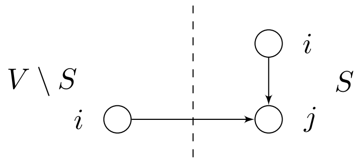
are equivalent formulations to CVRP. More formally, for each subset of customers S,

$$\sum_{j \in S} \overbrace{\sum_{(i,j) \in \mathrm{BS}(j)} x_{ij}}^{1} = |S| \tag{7.1}$$

is equivalent to

$$\sum_{j \in S} \left(\sum_{i \in S} x_{ij} + \sum_{i \in V \setminus S} x_{ij} \right) = |S|, \tag{7.2}$$

as illustrated by the figure below:



That is:

$$\sum_{i \in S} \sum_{j \in S} x_{ij} + \sum_{i \in V \setminus S} \sum_{j \in S} x_{ij} = |S|, \text{ for each } S.$$

Therefore:

$$\sum_{i \in V \setminus S} \sum_{j \in S} x_{ij} \geq r(S) \text{ (CCC constraints)}$$

if and only if

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - r(S) \text{ (GSEC constraints).}$$

Observe that there is an inconvenience for both the CCC and the GSEC constraints: their cardinality can be exponential with respect to the input size (i.e. n). Possible approaches to deal with this issue are:

1. Cutting Plane approaches (not discussed here);
2. alternative constraints, of polynomial cardinality; they will be presented in the next section.

7.3.2 Miller, Tucker and Zemlin (MTZ) constraints

These constraints are alternative to CCC and to GSEC. Let us introduce some additional auxiliary variables:

$$u_i \geq 0, \ \forall i \in V \setminus \{0\}.$$

Using these variables, the Miller, Tucker and Zemlin (MTZ) constraints can be stated as follows:

(i) $u_i - u_j + C\, x_{ij} \leq C - d_j, \ \forall i, j \in V \setminus \{0\}$ such that $d_i + d_j \leq C$,

(ii) $d_i \leq u_i \leq C, \ \forall i \in V \setminus \{0\}$.

In the MTZ constraints, u_i represents the load of the vehicles serving customer i, after the loading operation at i, $\forall i \in V \setminus \{0\}$. Therefore, constraints (ii) $d_i \leq u_i \leq C$ must be satisfied, since the load must be at least d_i but no more than C, due to the capacity of the vehicle.

Concerning constraints (i), two cases are possible $\forall i, j \in V \setminus \{0\}$:

- $x_{ij} = 0$, i.e. no vehicle moves from i to j along (i, j); in this case $u_i - u_j \leq C - d_j$ is always true since $u_i \leq C$ and $u_j \geq d_j$ due to constraints (ii). That is, if $x_{ij} = 0$ the constraint is redundant.
- $x_{ij} = 1$, i.e. a vehicle moves from i to j along (i, j); in this case:

$$u_i - u_j + \cancel{C} \leq \cancel{C} - d_{ij}, \ \text{ i.e. } \ u_j \geq u_i + d_j,$$

  in fact, there is a vehicle serving j after i, and so its load after visiting j, i.e. u_j, must be $\geq u_i + d_j$.

Observe that the MTZ constraints do exclude solutions containing subtours, such as the one in Figure 7.4. In fact, since $x_{46} = x_{68} = x_{84} = 1$, and the MTZ impose:

$$\begin{aligned} u_4 - u_6 + \cancel{C\, x_{46}} &\leq \cancel{C} - d_6 \\ u_6 - u_8 + \cancel{C\, x_{68}} &\leq \cancel{C} - d_8 \\ u_8 - u_4 + \cancel{C\, x_{84}} &\leq \cancel{C} - d_4, \end{aligned}$$

by summing up the inequalities, they imply:

$$0 \leq -d_4 - d_6 - d_8, \quad \text{i.e.} \quad d_4 + d_6 + d_8 \leq 0$$

which is true if and only if $d_4 = d_6 = d_8 = 0$ (but, in such a case, we can disregard customers 4, 6 and 8).

References P. Toth and D. Vigo (2002): Chapter 1

An example of (VRP1)

Consider the following instance to CVRP:

$$\begin{aligned} K = 2, &\quad V = \{0, 1, 2, 3, 4, 5\} \\ C = 5, &\quad d = (2, 1, 4, 1, 1) \end{aligned}$$

The corresponding model VRP1 is presented in Model 7.1.

In case the CCC constraints are used, some examples are:

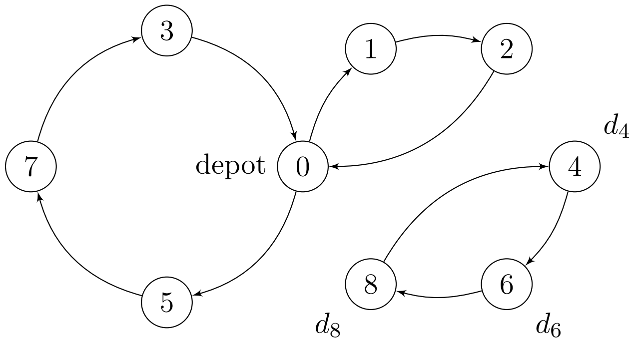


Figure 7.4: Infeasible solution due to a subtour

$$
\begin{aligned}
&\min \sum_{(i,j)} c_{ij}\, x_{ij} \\
&x_{01} + x_{02} + x_{03} + x_{04} + x_{05} = 2 \quad 0 \\
&x_{10} + x_{20} + x_{30} + x_{40} + x_{50} = 2 \quad 0 \\
&x_{10} + x_{12} + x_{13} + x_{14} + x_{15} = 1 \quad 1 \\
&x_{01} + x_{21} + x_{31} + x_{41} + x_{51} = 1 \quad 1
\end{aligned}
$$

(the same for customers 2, 3, 4 and 5)

$$\vdots$$

(CCC or GSEC or MTZ constraints)

$$x_{01}, x_{02}, \ldots \in \{0, 1\}$$

Model 7.1

- for $S = \{1, 2, 5\}$: since $d_1 + d_2 + d_5 = 4 < C = 5$, then $r(S) = 1$, i.e. one vehicle is sufficient to serve all customers in S (see Figure 7.5); the corresponding CCC constraint is therefore:

$$x_{01} + x_{02} + x_{05} + x_{31} + x_{32} + x_{35} + x_{41} + x_{42} + x_{45} \geq 1.$$

  Notice that we are imposing that this vehicle comes from $V \setminus S$, where there is the depot (i.e. node 0).

- for $S = \{2, 3, 4\}$: since $d_2 + d_3 + d_4 = 6 > C = 5$, then $r(S) = 2$, i.e. at least 2 vehicles are needed (for their capacity) to serve all customers in S (see Figure 7.6); the corresponding CCC constraint is therefore:

$$x_{02} + x_{03} + x_{04} + x_{12} + x_{13} + x_{14} + x_{52} + x_{53} + x_{54} \geq 2.$$

Notice that, if we consider the case $K = 1$, i.e. we address the *Travelling Salesman Problem*, then $r(S) = 1$, $\forall S$, and therefore:

$$\sum_{i \in V \setminus S} \sum_{j \in S} x_{ij} \geq 1, \qquad \forall S \subseteq V \setminus \{0\}, S \neq \varnothing \qquad \text{(CCC)}$$

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1, \quad \forall S \subseteq V \setminus \{0\}, S \neq \varnothing \qquad \text{(GSEC)}$$

whereas MTZ do not change (C can be set equal to n, i.e. the cardinality of the node set).

7.3.3 A three-index model

The main drawback of the (VRP1) model is that we have no information about the assignment of vehicles to routes. If this information is required, we need to adopt

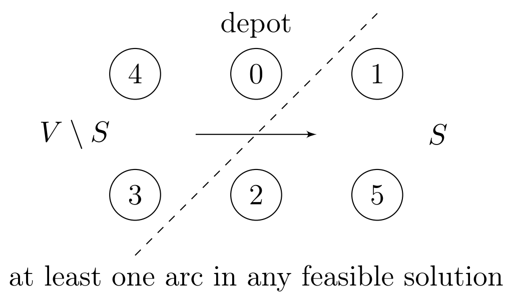


Figure 7.5

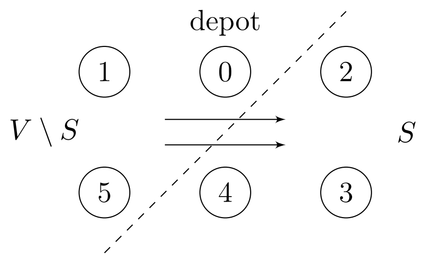


Figure 7.6

an alternative *three-index model*. In order to present this model, let us introduce the following decision variables:

$$x_{ijk} = \begin{cases} 1 & \text{if the arc } (i,j) \text{ belongs to the tour of vehicle } k \\ 0 & \text{otherwise} \end{cases}, \ \forall (i,j) \in A, k = 1, \dots K.$$

These are $\mathcal{O}(n^2\,K)$ design variables. We need also:

$$y_{ik} = \begin{cases} 1 & \text{if customer } i \text{ is served by vehicle } k \\ 0 & \text{otherwise} \end{cases}, \ \forall i \in V \setminus \{0\}, k = 1, \dots K$$

which are additional $\mathcal{O}(n\,K)$ variables.

The ILP three-index model is the following:

$$\begin{array}{lll}
\min \displaystyle\sum_{(i,j)\in A} c_{ij} \sum_{k=1}^{K} x_{ijk} & & \\
\displaystyle\sum_{k=1}^{K} y_{ik} = 1, & \forall i \in V \setminus \{0\} & ① \\
\displaystyle\sum_{k=1}^{K} y_{0k} = K & & ② \\
\displaystyle\sum_{(i,j)\in \mathrm{FS}(i)} x_{ijk} = \sum_{(j,i)\in \mathrm{BS}(i)} x_{jik} = y_{ik}, & \forall i \in V, k = 1, \dots K & ③ \\
\displaystyle\sum_{i\in V} d_i\, y_{ik} \le C, & k = 1, \dots K & ④ \\
\displaystyle\sum_{i\in V\setminus S} \sum_{j\in S} x_{ijk} \ge y_{hk}, & \forall S \subseteq V \setminus \{0\}, \forall h \in S, k = 1, \dots K & ⑤ \\
x_{ijk} \in \{0,1\} & \forall (i,j) \in A, k = 1, \dots K & \\
y_{ik} \in \{0,1\} & \forall i \in V, k = 1, \dots K &
\end{array} \qquad \text{(VRP2)}$$

where the numbered constraints have the following meaning:

① each customer is assigned to exactly one vehicle;

② the depot is traversed by the K vehicles;

③ if node i is assigned to vehicle k, then vehicle k enters and leaves i exactly once;

④ these are the vehicle capacity constraints (which are now explicit thanks to variables $\{y_{ik}\}$);

⑤ these constraints are a reformulation of the CCC constraints using variables $\{y_{hk}\}$: if customer h belongs to S and h is served by vehicle k (i.e. $y_{hk} = 1$), then vehicle k must traverse the cut $(V \setminus S, S)$, since the depot is in $V \setminus S$ (to avoid subtours). See also Figure 7.7. Observe that it is also possible to reformulate the GSEC and the MTZ constraints for the case of three-index variables, but such reformulations are not provided in these notes.

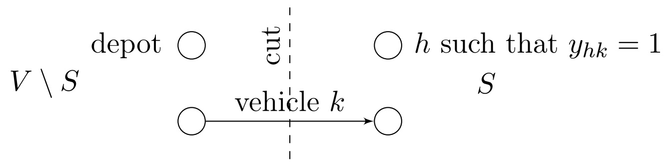


Figure 7.7

7.3.4 Possible extensions of (VRP1) and (VRP2)

Some possible extensions of models (VRP1) and (VRP2) are listed below:

1. *Unused vehicles*, i.e. let some of the K vehicles stay unused. In (VRP1), this can be achieved by substituting constraints ③ – ④ with the following:

$$\sum_{(i,0)\in A} x_{i0} \leq K \qquad ③$$

$$\sum_{(i,0)\in A} x_{i0} = \sum_{(0,i)\in A} x_{0i} \qquad ④$$

   On the other hand, in (VRP2), constraints ② must be replaced by:

$$\sum_{k=1}^{K} y_{0k} \leq K. \qquad ②$$

2. *Fixed costs for using vehicles* (only for (VRP2)): if f_k denotes the fixed cost for using vehicle k, $k = 1, \dots K$, then the objective function of model (VRP2)

becomes:

$$\min \sum_{(i,j)\in A} c_{ij} \sum_{k=1}^{K} x_{ijk} + \sum_{k=1}^{K} f_k\, y_{0k}$$

where a vehicle-specific cost c_{ij}^k may be used instead of c_{ij}, if it depends on k.

3. *Heterogeneous fleet of vehicles* (only for (VRP2)): if C_k is the capacity of vehicle k, $k = 1, \dots K$, then constraints ④ can be replaced by:

$$\sum_{i\in V} d_i\, y_{ik} \le C_k,\ k = 1, \dots K. \qquad ④$$

4. *Forbidden routes composed of a single customer*: in (VRP1), add the following constraint:

$$x_{0j} + x_{j0}, \le 1\ \forall j \in V \setminus \{0\}.$$

An example of (VRP2)

Consider an instance of CVRP where $K = 2$ and $C = 5$. Furthermore, let $V = \{0, 1, 2, 3, 4, 5\}$ and $d = (2, 1, 4, 1, 1)$.

The objective function of the corresponding model (VRP2) is:

$$\min \sum_{(i,j)\in A} c_{ij} \sum_{k=1}^{2} x_{ijk} = \min \sum_{(i,j)\in A} c_{ij}(x_{ij1} + x_{ij2}).$$

Let us build the model, constraint by constraint:

① Either vehicle 1 or vehicle 2 must visit customer 1:

$$y_{11} + y_{12} = 1.$$

The analogous constraint must be fixed for customers 2, 3, 4 and 5.

② Both vehicles must visit the depot:

$$y_{01} + y_{02} = 2.$$

③ If $y_{11} = 1$, i.e. customer 1 is assigned to vehicle 1, then exactly one arc entering 1 and exactly one arc leaving 1 must be visited by vehicle 1, since customer 1 belongs to the tour of vehicle 1. On the other hand, if $y_{11} = 0$, then no arcs entering 1 and no arcs leaving 1 must be traversed by vehicle 1 (see Figure 7.8):

$$\underbrace{x_{101} + x_{121} + x_{131} + x_{141} + x_{151}}_{\text{FS}(1)} = \underbrace{x_{011} + x_{211} + x_{311} + x_{411} + x_{511}}_{\text{BS}(1)} = y_{11}.$$

Considering now customer 1 and vehicle 2:

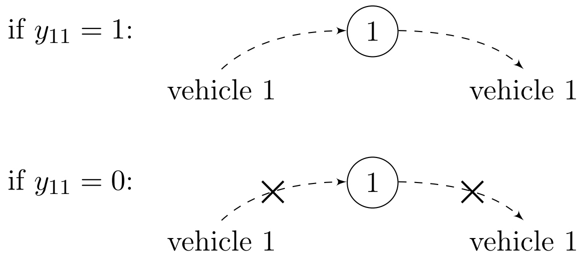


Figure 7.8

$$x_{102} + x_{122} + x_{132} + x_{142} + x_{152} = x_{012} + x_{212} + x_{312} + x_{412} + x_{512} = y_{12}.$$

The analogous constraints must be fixed for customers 2, 3, 4 and 5. Also, the depot must be visited by both vehicles:

$$x_{011} + x_{021} + x_{031} + x_{041} + x_{051} = x_{101} + x_{201} + x_{301} + x_{401} + x_{501} = y_{01}$$
$$x_{012} + x_{022} + x_{032} + x_{042} + x_{052} = x_{102} + x_{202} + x_{302} + x_{402} + x_{502} = y_{02},$$

where both $y_{01} = y_{02} = 1$ for constraint ②.

④ The vehicle capacity must not be exceeded:

$$d_1\, y_{11} + d_2\, y_{21} + d_3\, y_{31} + d_4\, y_{41} + d_5\, y_{51} \leq 5 = C$$
$$d_1\, y_{12} + d_2\, y_{22} + d_3\, y_{32} + d_4\, y_{42} + d_5\, y_{52} \leq 5$$

⑤ $\sum_{i \in V \setminus S} \sum_{j \in S} x_{ijk} \geq y_{hk}, \ \forall S \subseteq V \setminus \{0\}, \forall h \in S, k = 1, 2$

Let us explicit some of these CCC constraints for $S = \{1, 2, 3\}$:

- if $y_{11} = 1$, then vehicle 1 must traverse the cut (see Figure 7.9):

$$\sum_{i \in \{0,4,5\}} \sum_{j \in \{1,2,3\}} x_{ij\mathbf{1}} \geq y_{\mathbf{11}};$$

- if $y_{21} = 1$, then vehicle 1 must traverse the cut (see Figure 7.10):

$$\sum_{i \in \{0,4,5\}} \sum_{j \in \{1,2,3\}} x_{ij\mathbf{1}} \geq y_{\mathbf{21}};$$

- if $y_{31} = 1$, then vehicle 1 must traverse the cut (see Figure 7.11):

$$\sum_{i \in \{0,4,5\}} \sum_{j \in \{1,2,3\}} x_{ij\mathbf{1}} \geq y_{\mathbf{31}};$$

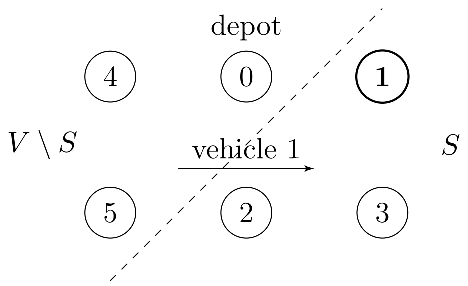


Figure 7.9: If $y_{11} = 1$, then vehicle 1 must traverse the cut

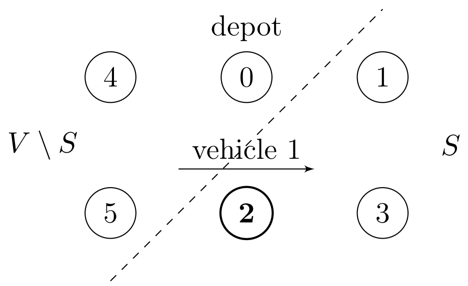


Figure 7.10: If $y_{21} = 1$, then vehicle 1 must traverse the cut

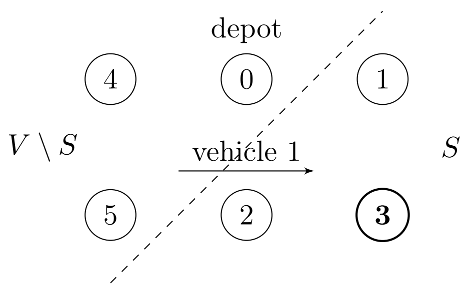


Figure 7.11: If $y_{31} = 1$, then vehicle 1 must traverse the cut

$$\left.\begin{array}{l} \bullet \displaystyle\sum_{i\in\{0,4,5\}}\sum_{j\in\{1,2,3\}} x_{ij\mathbf{2}} \geq y_{1\mathbf{2}}; \\[2ex] \bullet \displaystyle\sum_{i\in\{0,4,5\}}\sum_{j\in\{1,2,3\}} x_{ij\mathbf{2}} \geq y_{2\mathbf{2}}; \\[2ex] \bullet \displaystyle\sum_{i\in\{0,4,5\}}\sum_{j\in\{1,2,3\}} x_{ij\mathbf{2}} \geq y_{3\mathbf{2}}. \end{array}\right\} \text{analogous constraints for vehicle 2}$$

⑥ $x_{011}, x_{012}, x_{021}, x_{022} \ldots \in \{0,1\}$;
$y_{01}, y_{02}, y_{11}, y_{12}, y_{31}, y_{32} \ldots \in \{0,1\}$.

The overall model is presented in Model 7.2.

$$\min \sum_{(i,j)\in A} c_{ij}(x_{ij1} + x_{ij2})$$

$y_{11} + y_{12} = 1$ ①

(analogous constraints for customers 2, 3, 4 and 5) ①

$\vdots$ ①

$y_{01} + y_{02} = 2$ ②

$x_{101} + x_{121} + x_{131} + x_{141} + x_{151} = x_{011} + x_{211} + x_{311} + x_{411} + x_{511} = y_{11}$ ③

$x_{102} + x_{122} + x_{132} + x_{142} + x_{152} = x_{012} + x_{212} + x_{312} + x_{412} + x_{512} = y_{12}$ ③

(analogous constraints for customers 2, 3, 4 and 5) ③

$\vdots$ ③

$x_{011} + x_{021} + x_{031} + x_{041} + x_{051} = x_{101} + x_{201} + x_{301} + x_{401} + x_{501} = y_{01}(=1)$ ③

$x_{012} + x_{022} + x_{032} + x_{042} + x_{052} = x_{102} + x_{202} + x_{302} + x_{402} + x_{502} = y_{02}(=1)$ ③

$d_1\, y_{11} + d_2\, y_{21} + d_3\, y_{31} + d_4\, y_{41} + d_5\, y_{51} \leq 5\,(= C)$ ④

$d_1\, y_{12} + d_2\, y_{22} + d_3\, y_{32} + d_4\, y_{42} + d_5\, y_{52} \leq 5$ ④

$\displaystyle\sum_{i\in V\setminus S}\sum_{j\in S} x_{ijk} \geq y_{hk},\ \forall S \subseteq V \setminus \{0\}, \forall h \in S, k = 1,2$ ⑤

$x_{ijk} \in \{0,1\},\ \forall (i,j) \in A, k = 1,2$ ⑥

$y_{ik} \in \{0,1\},\ \forall i \in V, k = 1,2$ ⑥

Model 7.2

7.4 A flow based model for CVRP

The idea behind the flow based model for CVRP is to use "flow constraints" in place of the CCC, GSEC or MTZ constraints in model (VRP1). The goal of these flow constraints is the same, i.e. to avoid subtours and impose the vehicle capacity satisfaction.

In order to present the model, let us introduce the flow variables y_{ij}, where y_{ij} represents the amount of good to push along (i,j), $\forall (i,j) \in A$.

The ILP model is the following:

$$
\begin{array}{llr}
\min \sum_{(i,j)\in A} c_{ij}\, x_{ij} & & \\
\sum_{(i,j)\in \mathrm{BS}(j)} x_{ij} = 1 & \forall j \in V \setminus \{0\} & ① \\
\sum_{(j,i)\in \mathrm{FS}(j)} x_{ji} = 1 & \forall j \in V \setminus \{0\} & ② \\
\sum_{(i,0)\in A} x_{i0} = K & & ③ \\
\sum_{(0,i)\in A} x_{0i} = K & & ④ \\
\sum_{(j,i)\in \mathrm{BS}(i)} y_{ji} - \sum_{(i,j)\in \mathrm{FS}(i)} y_{ij} = d_i & \forall i \in V \setminus \{0\} & ⑤ \\
\sum_{(j,0)\in \mathrm{BS}(0)} y_{j0} - \sum_{(0,j)\in \mathrm{FS}(0)} y_{0j} = -\sum_{i\in V\setminus\{0\}} d_i & & ⑥ \\
0 \le y_{ij} \le C\, x_{ij} & \forall (i,j) \in A & ⑦ \\
x_{ij} \in \{0,1\} & \forall (i,j) \in A & ⑧
\end{array}
\qquad \text{(VRP3)}
$$

Observe that constraints ⑤, ⑥ and ⑦ replace the CCC, GSEC or MTZ constraints, yet their cardinality is polynomial with respect to the input size.

An example

Let us resume the example presented in Section 7.2.1. Let $n = 8, K = K_{\min} = 2, C = 8$, and $d = (1, 3, 2, 1, 2, 1, 1, 2)$.

Do constraints ⑤, ⑥ and ⑦ allow to avoid subtours? In order to show this, let us refer to the solution proposed in Figure 7.12. It is not a feasible solution to (VRP3), since 0 is the only source of flow, and so the destinations 4, 6 and 8 must be necessarily connected to 0, i.e. the depot. Therefore, no solution of model (VRP3) can include subtours.

Do constraints ⑤, ⑥ and ⑦ impose the vehicle capacity satisfaction? Let us now refer to Figure 7.13. It is not a feasible solution to (VRP3), since the link $(0, 1)$ violates the related constraint $0 \leq y_{01} \leq 8x_{01}$. Therefore, no solution of model (VRP3) can violate the vehicle capacities (due to constraints ⑦).

In addition, thanks to the flow variables $\{y_{ij}\}$, we can generalize (VRP3) by also taking into account transportation costs. To achieve this, replace the previously stated objective function by:

$$\min \underbrace{\sum_{(i,j)\in A} c_{ij}\, x_{ij}}_{\text{design cost}} + \underbrace{\sum_{(i,j)\in A} \alpha_{ij}\, y_{ij}}_{\text{transportation cost}}$$

where α_{ij} is the unit transportation cost along (i, j), $\forall (i, j) \in A$.

A feasible solution to the previous instance of CVRP is proposed in Figure 7.14. It takes into account both design and routing decisions. In particular, the decision variables x_{ij} indicate the arcs to be used, i.e. the ones traversed by the vehicles, while the routing variables y_{ij} indicate the load of the vehicles along the traversed arcs. Note that $y_{ij} \leq 8 = C$, for each traversed arc.

In terms of such design and routing variables, the solution in Figure 7.14 is modelled as:

$$\begin{gathered} x_{01} = x_{12} = x_{24} = x_{46} = x_{60} = x_{08} = x_{85} = x_{57} = x_{73} = x_{30} = 1, \\ x_{ij} = 0 \text{ otherwise}, \\ y_{01} = 6, y_{12} = 5 \ldots \end{gathered}$$

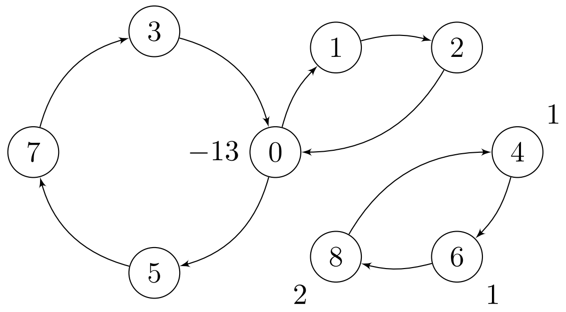


Figure 7.12: Infeasible solution due to a subtour

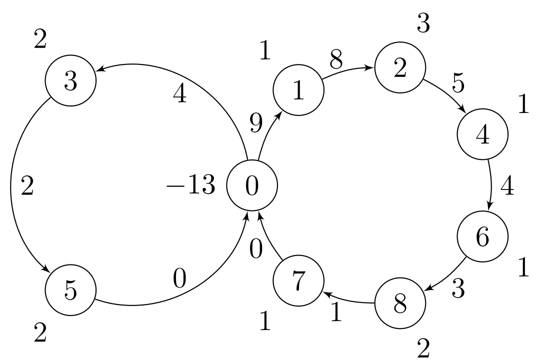


Figure 7.13: Infeasible solution due to capacity violation

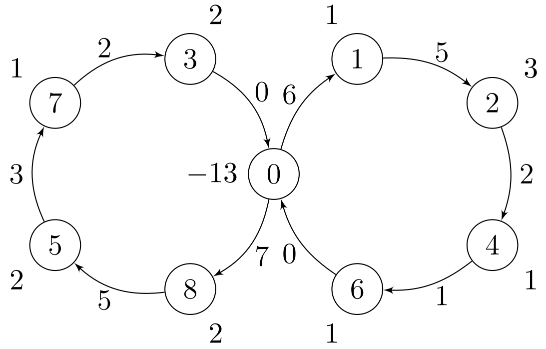


Figure 7.14: A feasible solution to (VRP3). Labels on arcs denote vehicle loads.

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