Logistics

Lecture notes*

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Chapter 6

Multicommodity flows

The problems studied in the previous chapters are *single commodity flow problems*, i.e. they address the transportation along a network, from some origins to some destinations, of a unique type of product or *commodity* (e.g. cars in BMC and ACA, fruit in Tropicsun). However, in several applications many products must be sent along the same network, leading to a multicommodity scenario: if the products are independent, then we can decompose the problem into several independent single commodity min-cost flow problems, one per commodity. Otherwise, if the products share the network resources (e.g. link capacities), then we need to model and solve a *multicommodity flow problem*, which generalizes the minimum cost flow problem (which is a single commodity one).

For instance, in airline scheduling the commodities are the different flights, each one having its own departure node (origin) and destination node, in the logistics network associated with the airline company.

6.1 The minimum cost multicommodity flow problem

Consider the following input data:

- $G = (N, A)$ is a directed graph (logistics network);
- K is the number of commodities;
- b_i^k is the balance of node i for commodity k, $\forall i \in N, k = 1, \dots K$;
- u_{ij}^k is the capacity of link (i, j) for commodity k, $\forall (i, j) \in A, k = 1, \dots K$;
- u_{ij} is the global capacity of link (i, j), $\forall (i, j) \in A$;
- c_{ij}^k is the unit transportation cost along (i, j) for commodity k, $\forall (i, j) \in A, k = 1, \dots K$.

Let us define the multicommodity flow variables x_{ij}^k, where x_{ij}^k is the amount of commodity k to be pushed along (i,j), $\forall (i,j) \in A, k = 1, \dots K$.

The minimum cost multicommodity flow problem can then be modelled as follows:

$$
\begin{aligned}
&\min \sum_{(i,j)\in A} \sum_{k=1}^{K} c_{ij}^k \, x_{ij}^k \\
&\sum_{(j,i)\in \mathrm{BS}(i)} x_{ji}^k - \sum_{(i,j)\in \mathrm{FS}(i)} x_{ij}^k = b_i^k \quad \forall i \in N, \qquad k = 1, \dots K \\
&0 \le x_{ij}^k \le u_{ij}^k \qquad \forall (i,j) \in A, \quad k = 1, \dots K \\
&\sum_{k=1}^{K} x_{ij}^k \le u_{ij} \qquad \forall (i,j) \in A
\end{aligned}
\tag{6.1}
$$

The last constraints within the model are the ones linking the K commodities (without these linking constraints, the model could be solved by solving K independent minimum cost flow problems). An important point is that now the integrality property does not hold.

6.2 The fixed-charge network design problem

Assume now that, in addition to organize the transportation of the commodities along the network, you have also to design the network: you can send some products along (i,j) only if you first decide to build the link (i,j), with a related fixed cost f_{ij}. We then need the additional input data f_{ij}: the fixed cost for activating (i,j), $\forall (i,j) \in A$.

Let us introduce some additional decision variables (design variables):

$$
y_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is activated} \\ 0 & \text{otherwise} \end{cases}, \ \forall (i,j) \in A.
$$

The ILP model is therefore

$$
\begin{aligned}
&\min \overbrace{\sum_{(i,j)\in A} f_{ij}\, y_{ij}}^{\text{fixed cost}} + \overbrace{\sum_{(i,j)\in A} \sum_{k=1}^{K} c_{ij}^{k}\, x_{ij}^{k}}^{\text{transportation cost}} \\
&\sum_{(j,i)\in \mathrm{BS}(i)} x_{ji}^{k} - \sum_{(i,j)\in \mathrm{FS}(i)} x_{ij}^{k} = b_i^k \quad && \forall i \in N, \quad && k = 1, \dots K \\
&0 \le x_{ij}^{k} \le u_{ij}^{k} && \forall (i,j) \in A, && k = 1, \dots K \\
&\sum_{k=1}^{K} x_{ij}^{k} \le u_{ij}\, y_{ij} && \forall (i,j) \in A && \circledast \\
&y_{ij} \in \{0,1\} && \forall (i,j) \in A
\end{aligned}
\tag{6.2}
$$

The starred constraints ($\circledast$) are capacity constraints and also link the transportation decisions ($\{x_{ij}^k\}$) with the design decisions ($\{y_{ij}\}$). Concerning the time complexity issue, the fixed-charge network design problem is *NP*-hard.

Textbooks

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