Logistics

Lecture notes*

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Chapter 10

Inventory management problems

Inventories are stockpiles of items (raw materials, components and finished goods) waiting to be processed, transported or used. The motivations for stockpiling such items are for improving the service level, reducing the overall logistics costs, coping with randomness in customer demands, and so on. However, keeping inventories can be very expensive. It is therefore crucial to solve *inventory management problems*: that is, for each stocking point in the supply chain, to decide when to reorder (for a single or multiple products) and how much to order, so that the expected (annual) cost is minimized while achieving a certain service level.

Among the relevant involved costs, there are:

1. *Procurement costs* associated with the acquisition of goods; they can be fixed costs or variable costs (i.e. dependent on the amount acquired): e.g. (fixed) reorder costs, (variable) purchasing or manufacturing costs, transportation costs.

2. *Inventory holding costs* incurred when materials are stocked for a period of time: e.g. *opportunity (or capital) costs*, i.e. the return on investment the firm would have realized if money had been invested in a profitable economic activity instead of inventory; *warehousing costs*, which may include space and equipment costs, maintenance costs, taxes, or fees to pay for storing the goods.

3. *Shortage costs* are paid when customer orders are not met; for example:

   - *lost sales costs*: they are related to lost profit and to a negative effect on future sales;

   - *back order costs*: they are due to the fact that, in case of a delayed sale, often a penalty has to be paid.

4. *Obsolescence costs.*

Inventory management models can be classified according to various criteria:

- *Deterministic versus stochastic models*: in a deterministic model, demands, costs and lead times are assumed to be known in advance, while in a stochastic model

some data are uncertain.

- *Fast-moving items versus slow-moving items*: fast-moving items are manufactured and sold regularly, so we have to decide when to replenish stocking points and how much to order each time; slow-moving items (e.g. spare parts of complex machineries) have a very low demand (e.g. few units every 20 years), and the main issue is to decide how many items to produce and store at the beginning of the planning horizon.

- *Number of stocking points*: usually optimal inventory policies can be derived analytically only for the case of a single stocking point.

- *Number of commodities*: inventory management models are different depending on the fact that they deal with single or with multiple commodities.

- *Instantaneous resupply versus noninstantaneous resupply*: there are differences depending on the fact that a stocking point can be replenished almost instantaneously or not (e.g. at a constant rate via a manufacturing process).

Hereafter we shall present some basic inventory management models, with different characteristics according to the criteria that have been listed before.

10.1 Single stocking point: single commodity under constant demand rate

The model is based on the following assumptions:

- we are dealing with a single stocking point for a single commodity, under a constant demand rate, say d;

- the planning horizon is infinite.

Since d is a constant, a natural policy is to replenish the stocking point on a periodic basis. To this end, let:

- T be the time lapse between two consecutive orders (*period*),

- q be the *order size*, i.e. the amount of product to order each time.

The figure below provides a graphical representation of the relationship among demand, period and order size.
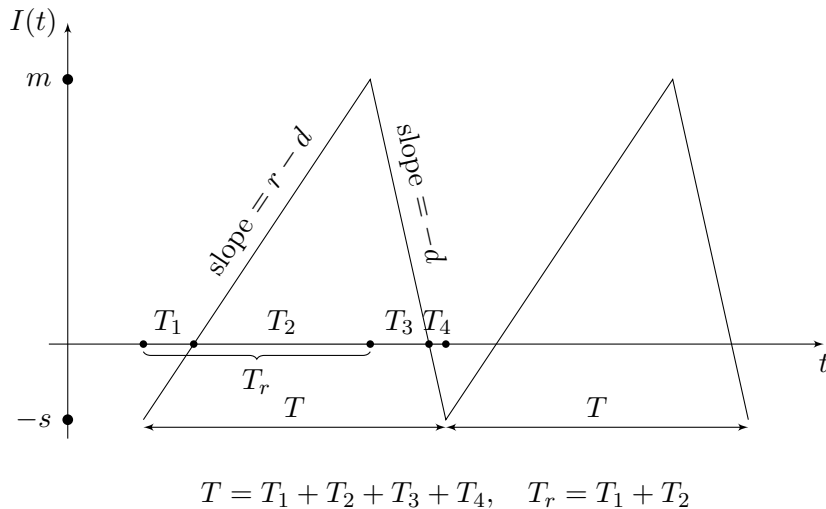

Clearly it is: $q = d \cdot T$.

$$T = T_1 + T_2 + T_3 + T_4, \quad T_r = T_1 + T_2$$
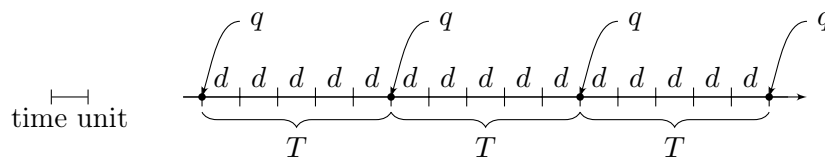
Figure 10.1: Inventory level $I(t)$ as a function of time t

Notice that q (how much to reorder?) and T (when to reorder?) are variables to be determined. In order to determine q and T, the following costs are taken into account by the considered model:

- c: the value of an item;

- K: the fixed reorder cost;

- h: the holding cost per item and per time unit;

- u: the shortage cost per item (this is independent on the duration of the storage);

- v: the shortage cost per item and per (shortage) time unit.

Usually it is assumed $h = p \cdot c$, i.e. h is a fraction p of c, with p a banking interest rate.

The inventory management problem under presentation is to determine q (and so T), the maximum shortage s (in each period) and the maximum inventory level m (in each period) in such a way as to minimize the overall average cost per time unit. See Figure 10.1 for a graphical representation of the maximum inventory level and of the maximum shortage in each period.

In order to solve the problem, let:

- $T_r \geq 0$ be the *replenishment time*, i.e. the time needed to replenish the stocking point with q items;

- r be the *replenishment rate*, i.e. the number of items per unit of time received by the stocking point during T_r.

Therefore, $q = r \cdot T_r$.

Note that during the replenishment (i.e. in T_r), the net number of items stocked per unit of time is $r - d$ (assuming $r > d$), whereas, after T_r, the stock level decreases at a

rate equal to $-d$, since items are picked up at a rate d. Also, $s + m = (r - d) T_r$, i.e.:

$$m = (r - d)T_r - s = \underbrace{r\,T_r}_{q} - d\,\underbrace{T_r}_{\frac{q}{r}} - s = q\left(1 - \frac{d}{r}\right) - s$$

Therefore, m depends on q and s, i.e. q and s are the variables to be determined, whereas T and m can then be computed by means of the formulas:

$$T = \frac{q}{d} \qquad \text{and} \qquad m = q\left(1 - \frac{d}{r}\right) - s$$

Recall that the optimal values of variables q and s are the ones that minimize the total average cost per unit of time, which is given by the following function:

$$\mu(q, s) = \frac{1}{T}\underbrace{(K + c\,q + h\,\bar{I}\,T + u\,s + v\,\bar{S}\,T)}_{\text{average cost per period}}.$$

In this function, the *holding cost* $h \cdot \bar{I} \cdot T$ and the *shortage costs* ($u \cdot s$ and $v \cdot \bar{S} \cdot T$) depend on the average inventory level $\bar{I}$ and on the average shortage level $\bar{S}$, given respectively by:

- $\bar{I} = \dfrac{1}{T}\int_0^T \underbrace{I_+(t)}_{\text{``positive part'' of } I(t)}\mathrm{d}t = \dfrac{1}{T}\dfrac{m(T_2 + T_3)}{2}$;

- $\bar{S} = \dfrac{1}{T}\int_0^T \underbrace{I_-(t)}_{\text{``negative part'' if } I(t)}\mathrm{d}t = \dfrac{1}{T}\dfrac{s(T_1 + T_4)}{2}$.

See Figure 10.1 for a graphical illustration of the subintervals T_1, T_2, T_3 and T_4. In order to express T_1, T_2, T_3 and T_4 as functions of variables q and s, observe that:

$$\begin{cases} s = (r - d)T_1 \\ m = (r - d)T_2 \\ m = d\,T_3 \\ s = d\,T_4 \end{cases} \qquad \text{and so:} \qquad \begin{cases} T_1 = s/(r - d) \\ T_2 = m/(r - d) \\ T_3 = m/d \\ T_4 = s/d. \end{cases}$$

Therefore:

$$\bar{I} = \frac{1}{T}\frac{m(T_2 + T_3)}{2} = \frac{1}{T}\frac{m^2(\frac{1}{r-d} + \frac{1}{d})}{2} = \frac{\not{d}}{2q}m^2\frac{\not{d} + r - \not{d}}{\not{d}\,(r - d)} =$$

$$= \frac{m^2}{2q}\frac{r}{r - d} = \frac{m^2}{2q}\frac{1}{\frac{r-d}{r}} = \underbrace{\frac{m^2}{2q(1 - \frac{d}{r})}}_{\text{as a function of } q \text{ and } m} = \underbrace{\frac{[q(1 - \frac{d}{r}) - s]^2}{2q(1 - \frac{d}{r})}}_{\text{as a function of } q \text{ and } s}. \qquad (10.1)$$

In a similar way we can derive:

$$\bar{S} = \frac{s^2}{2q(1 - \frac{d}{r})}.$$ (10.2)

By replacing (10.1) and (10.2) in $\mu(q, s)$ we get:

$$\mu(q, s) = K\frac{d}{q} + c\,d + \frac{h[q(1 - \frac{d}{r}) - s]^2}{2q(1 - \frac{d}{r})} + \frac{u\,s\,d}{q} + \frac{v\,s^2}{2q(1 - \frac{d}{r})}.$$

This is a nonlinear, yet convex, function. Therefore, a (global) minimum point (q^*, s^*) of $\mu(q, s)$ exists and can be obtained by solving:

$$\begin{cases} \dfrac{\partial}{\partial q}\mu(q, s) = 0 \\[2mm] \dfrac{\partial}{\partial s}\mu(q, s) = 0 \end{cases} \qquad \text{stationary point.}$$

We get (no proof provided):

$$\begin{cases} q^* = \sqrt{\dfrac{h + v}{v}} \cdot \sqrt{\dfrac{2Kd}{h(1 - \frac{d}{r})} - \dfrac{(u\,d)^2}{h(h + v)}} \\[4mm] s^* = \dfrac{(h\,q^* - u\,d)(1 - \frac{d}{r})}{h + v} \end{cases}$$

i.e., in each period, the optimal amount to order to minimize the total average unit cost is q^*, and the optimal time interval to order is $T^* = q^*/d$.

Furthermore, in each period the optimal shortage level (to satisfy the demand by minimizing $\mu(q, s)$) is s^*, while the optimal maximum inventory level is $m^* = q^*(1 - \frac{d}{r}) - s^*$.

Notice that, in the simple inventory scenario under consideration, we have been able to determine the optimal inventory policy in an analytical way, i.e. by means of the formulas that have been derived for q^*, T^*, s^* and m^*.

An example

Golden Food distributes tinned foodstuff in Great Britain. Consider a warehouse in Birmingham where:

- the demand rate d for tomato purée is 400 pallets a month (so, it is constant each month);

- the value of a pallet is $c = $ £2,500;

- the annual interest rate is $p = 14.5\,\%$, including warehousing costs;

- issuing an order costs $K = $ £30;

- the replenishment rate is $r = 40$ pallets per day;

- shortages are not allowed, i.e. $s^* = 0$ (and so the shortage costs u and v are zero).

Using the data, the holding cost is:

$$h = 0.145 \cdot \text{£2,500} = \text{£362.5 per year per pallet}$$
$$= \text{£30.2 per month per pallet.}$$

Note that the unit of time is a month. By applying the optimal inventory policy previously presented, we get:

$$q^* = \sqrt{\frac{2Kd}{h(1 - \frac{d}{r})}} = \sqrt{\frac{2 \cdot 30 \cdot 400}{30.2 \cdot (1 - \frac{400}{40 \cdot 20})}} \approx 39.9 \approx 40 \text{ pallets.}$$

In the formulas we assumed that the number of workdays in a month is 20, hence the monthly replenishment rate is $40 \cdot 20$. As a consequence:

$$T^* = \frac{40}{400} = \frac{1}{10} \text{ month} = 2 \text{ workdays}$$

$$T_r^* = \frac{40}{40} = 1 \text{ workday}$$

$$m^* = q^* \left(1 - \frac{d}{40 \cdot 20}\right) = 40 \cdot \left(1 - \frac{400}{40 \cdot 20}\right) = 40 \cdot \frac{1}{2} = 20 \text{ pallets.}$$

Therefore, to minimize its total average unit cost the warehouse must order 40 ($= q^*$) pallets every 2 ($= T^*$) workdays. In each period, the replenishment will last 1 ($= T_r^*$) workday, and the level of inventory will vary between 0 (because shortage is not allowed) and a maximum level equal to 20 ($= m^*$).

10.1.1   The special case of instantaneous resupply

In case the resupply is instantaneous, the optimal inventory policy can be obtained from the previous, more general, policy by considering $r \to +\infty$ and, so, $T_r = \frac{q}{r} \to 0$. Then we obtain:

$$\begin{cases} q^* = \sqrt{\dfrac{h + v}{v}} \cdot \sqrt{\dfrac{2Kd}{h} - \dfrac{(u\,d)^2}{h(h + v)}} \\ s^* = \dfrac{h\,q^* - u\,d}{h + v} \end{cases}$$

If shortages are not allowed (i.e. $s^* = 0$), then:

$$q^* = \sqrt{\frac{2Kd}{h}}.$$

This is the classical *Economic Order Quantity (EOQ) model* introduces by F.W. Harris in 1913. The corresponding optimal cost is:

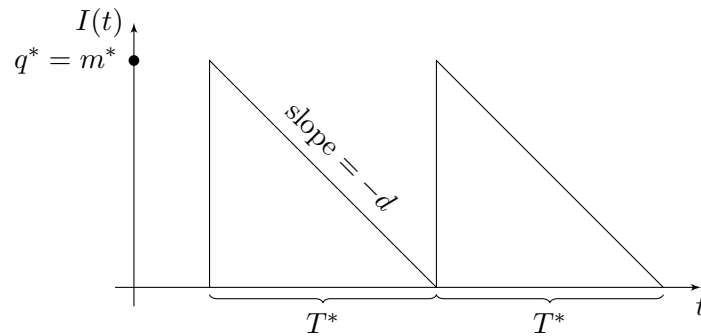$$\mu(q^*) = K\frac{d}{q^*} + c\,d + \frac{1}{2}h\,q^*.$$

Figure 10.2: Inventory level as a function of time in the EOQ model

The inventory level, as a function of time, in EOQ becomes the one depicted in Figure 10.2.

An example

Al-Bufeira motors manufactures spare parts for aircraft engines in Saudi Arabia. A certain component Y02PN has the following supply requirements:

- the demand is $d = 220$ units per year;

- the unit production cost is $c = \$1200$;

- the set-up cost (for producing the component) is $K = \$800$;

- the annual interest rate is $p = 18\%$ (including warehousing costs);

- no shortages are allowed (i.e. $s = 0$).

Therefore the holding cost is:

$$h = 0.18 \cdot \$1200 = \$216 \text{ per year per unit.}$$

Since replenishment is achieved via internal production, we can assume $r \to +\infty$, and so the EOQ model can be applied (observe that d is a constant per year). According to this model we can therefore derive:

$$\begin{cases} q^* = \sqrt{\dfrac{2Kd}{h}} = \sqrt{\dfrac{2 \cdot 800 \cdot 220}{216}} = 40.37 \text{ units} \\ T^* = \dfrac{40.37}{220} = 0.18 \text{ years} = 66.8 \text{ days} \end{cases}$$

Therefore, to minimize its cost Al-Bufeira Motors has to manufacture ~ 40 units of the component Y02PN every ~ 67 days.

Notice that, in practice, q^* and T^* must be suitably rounded in order to apply the optimal policy.

10.2   Single stocking point: multicommodity with capacity constraints

Let us consider now inventory problems where different types of commodity must be stocked and a limit is placed on the total investment in inventories, or on the common space. Assume that, for each product j, the assumptions made for the EOQ model can be applied (see Section 10.1.1). In other words assume that, for each product j, the demand rate d_j is constant, the replenishment rate r_j goes to infinity, and that shortage is not allowed.

Let:

- n be the number of commodities in stock;

- q_j be the amount of commodity j to reorder at each replenishment, $j = 1, \ldots n$.

Under the EOQ assumption, the average unit cost for commodity j is:

$$\mu_j(q_j) = \frac{K_j \, d_j}{q_j} + c_j \, d_j + \frac{1}{2} h_j \, q_j, \ j = 1, \ldots n,$$

where K_j is the fixed reorder cost, c_j is the unit value of item j and $h_j = p_j \, c_j$ is the holding cost for item j.

Therefore, the average global unit cost (for all commodities in stock) is:

$$\mu(q_1, q_2, \ldots q_n) = \sum_{j=1}^{n} \mu_j(q_j).$$

Assume now that there is a side, linear, constraint (*capacity constraint*), usually representing a budget constraint or a warehouse constraint, of form:

$$\sum_{j=1}^{n} a_j \, q_j \leq b$$

where $a_j, \ j = 1, \ldots n$, and b are constants. Then, the inventory management problem that we want to solve can be expressed as:

$$
\begin{aligned}
\min \sum_{j=1}^{n} & \left(\frac{K_j \, d_j}{q_j} + c_j \, d_j + \frac{1}{2} h_j \, q_j \right) \\
\sum_{j=1}^{n} & a_j \, q_j \leq b \\
& q_1, q_2, \ldots q_n \geq 0
\end{aligned}
\tag{10.3}
$$

This is a nonlinear (hard) model, yet it is possible to suggest a simple heuristic for the special case of identical interest rates, i.e. $p_j = p, \ j = 1, \ldots n$.

10.2.1 A heuristic for the case of identical interest rates

The heuristic algorithm computes the sizes of the orders, i.e. $q_1, q_2, \ldots q_n$ by performing the following steps:

Step 1:   Compute the EOQ order sizes as if the capacity constraint was absent:

$$q'_j = \sqrt{\frac{2K_j d_j}{p\, c_j}}, \quad j = 1, \ldots n.$$

Note that $(q'_1, \ldots q'_n)$ is the optimal solution of a *relaxation* of the nonlinear problem (10.3), since by removing the capacity constraint the n commodities can be managed independently.

If $\sum_{j=1}^{n} a_j\, q'_j \le b$, then STOP: $(q'_1, \ldots q'_n)$ is optimal also for the original (i.e. not relaxed) problem.

Step 2:

- Increase the common interest rate p of a quantity δ:

$$q_j(\delta) = \sqrt{\frac{2K_j d_j}{(p+\delta)c_j}}, \quad j = 1, \ldots n.$$

- Determine the minimum value of δ, say δ^*, that allows to satisfy the capacity constraint, i.e.:

$$\sum_{j=1}^{n} a_j\, q_j(\delta^*) = b.$$

We get:

$$\sum_{j=1}^{n} \left(a_j \sqrt{\frac{2K_j d_j}{(p+\delta)c_j}} \right) = b$$

$$\frac{1}{\sqrt{p+\delta}} \sum_{j=1}^{n} \left(a_j \sqrt{\frac{2K_j d_j}{c_j}} \right) = b$$

$$\sqrt{p+\delta} = \frac{1}{b} \sum_{j=1}^{n} \left(a_j \sqrt{\frac{2K_j d_j}{c_j}} \right)$$

$$\delta = \delta^* = \left[\frac{1}{b} \sum_{j=1}^{n} \left(a_j \sqrt{\frac{2K_j d_j}{c_j}} \right) \right]^2 - p.$$

- Return the feasible solution $(q_1(\delta^*), q_2(\delta^*), \ldots q_n(\delta^*))$, and STOP.

model	yearly demand	value	yearly interest price
Preppie knapsack	$d_1 = 150,000$ units	$c_1 = \$30$	$h_1 = 20\,\% \, c_1$
Yuppie suitcase	$d_2 = 100,000$ units	$c_2 = \$45$	$h_2 = 20\,\% \, c_2$

<div align="center">Table 10.1</div>

An example

New Frontier distributes Knapsacks and suitcases, whose most successful models are the Preppie knapsack and the Yuppie suitcase. Table 10.1 summarizes the relevant information about demand, value and interest rate of the two products. Notice that the interest rate is at $20\,\%$ for both products. In addition, the reorder cost is the same, too: $K_1 = K_2 = \$250$.

The company imposes the following additional constraint: the average capital invested in inventory can not exceed \$75,000, i.e. $\frac{30}{2}q_1 + \frac{45}{2}q_2 \leq 75,000$. In fact, since

$$\bar{I} = \frac{[q(1 - \frac{d}{r}) - s]^2}{2q(1 - \frac{d}{r})}$$

is the average inventory level under the inventory policy presented in Section 10.1, in the special case of EOQ the average inventory level is $\bar{I} = q/2$.

So, $q_1/2$ is the average inventory level for Preppie knapsacks, $q_2/2$ is the average inventory level for Yuppie suitcases, and therefore $30\frac{q_1}{2} + 45\frac{q_2}{2}$ is the average capital invested in inventories. It is assumed, for precaution, that the overall average inventory level is the sum of the average inventory levels for the two products.

Therefore, we can apply the previously presented heuristic:

Step 1:

$$q'_1 = \sqrt{\frac{2 \cdot 250 \cdot 150,000}{0.2 \cdot 30}} = 3,535.53 \text{ units}$$

$$q'_2 = \sqrt{\frac{2 \cdot 250 \cdot 100,000}{0.2 \cdot 45}} = 2,357.02 \text{ units}$$

It is $30\frac{q'_1}{2} + 45\frac{q'_2}{2} > 75,000$.

Step 2: $\delta^* = 0.2$

$$q_1(0.2) = \sqrt{\frac{2 \cdot 250 \cdot 150,000}{(0.2 + 0.2) \cdot 30}} = 2,500 \text{ units}$$

$$q_2(0.2) = \sqrt{\frac{2 \cdot 250 \cdot 100,000}{(0.2 + 0.2) \cdot 45}} \approx 1,666 \text{ units}$$
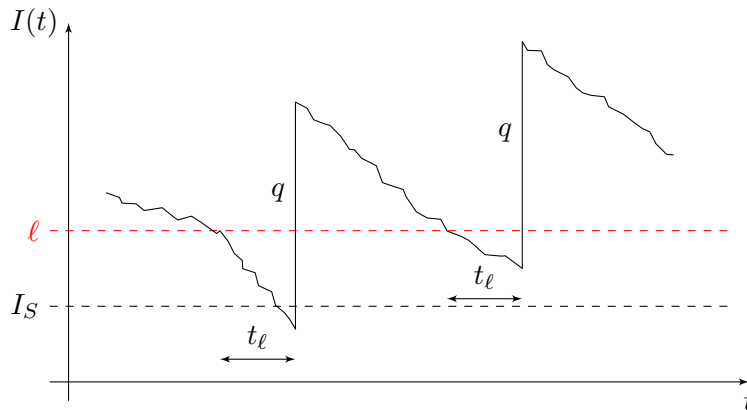
Figure 10.3: Example of utilization of the reorder point policy

Therefore, according to the heuristic we have to maintain in stock 2500 Preppie knapsacks and about 1666 Yuppie suitcases.

10.3   The reorder point policy

The reorder point policy is a stochastic model,, since it assumes that the demand is uncertain. Precisely, it is assumed that the demand to the stocking point is a random variable. The inventory level $I(t)$ is kept under observation in an almost continuous way; when it reaches a *reorder point* ℓ, then a constant quantity q is reordered. Such a quantity q will be delivered after a reorder time, or lead time, t_ℓ, which is assumed to be constant. Figure 10.3 shows an example of the policy at work.

The reorder size q may be computed as follows:

$$q = \sqrt{\frac{2K\bar{d}}{h}},$$

which is the EOQ formula, where $\bar{d}$ is the expected demand rate in the time unit. In order to develop the model, we will make the assumption that the expected demand rate $\bar{d}$ and the standard deviation σ_d are constant in each unit of time.

In order to determine the reorder point ℓ, we want that the inventory level $I(t)$ be nonnegative, during t_ℓ, with a given probability α. This is equivalent to impose that, during the interval t_ℓ, the total demand does not exceed ℓ with probability α.

Let d be the random variable denoting the total demand in the lead time; then

$$d = \underbrace{d_1}_{\text{first time unit}} + d_2 + \ldots + \underbrace{d_{t_\ell}}_{\text{last time unit}}.$$

By assuming an identical demand distribution in each unit of time, and assuming d_1, d_2, $\ldots d_{t_\ell}$ independent, if t_ℓ is sufficiently large then from the *Central limit theorem*

it is possible to assume that d is distributed according to a normal distribution with expected value $\bar{d} \cdot t_\ell$, and standard deviation $\sigma_d \sqrt{t_\ell}$, i.e. d is $\mathcal{N}(\bar{d}\, t_\ell, \sigma_d \sqrt{t_\ell})$.

Since the following property holds:

Property: if X is $\mathcal{N}(0,1)$ (standard normal variable), then $m + \sigma X$ is $\mathcal{N}(m, \sigma)$, i.e. a normal variable with expected value m and standard deviation σ.

then d can be represented as

$$d = \bar{d}\, t_\ell + \sigma_d \sqrt{t_\ell} X,$$

with X being a $\mathcal{N}(0,1)$.

Given α, let z_α be the value under which X falls with probability α (e.g. $z_\alpha = 2$ for $\alpha = 0.9772$ and $z_\alpha = 3$ for $\alpha = 0.9987$). Then, to guarantee

$$d = \underbrace{\bar{d}\, t_\ell + \sigma_d \sqrt{t_\ell}\, X}_{\text{demand in } t_\ell} \leq \ell \quad \text{with probability } \alpha,$$

we can set $\ell = \bar{d}\, t_\ell + \sigma_d \sqrt{t_\ell}\, z_\alpha$; in fact, $X \leq z_\alpha$ with probability α, and so

$$d = \bar{d}\, t_\ell + \sigma_d \sqrt{t_\ell} X \leq \bar{d}\, t_\ell + \sigma_d \sqrt{t_\ell}\, z_\alpha \quad \text{with probability } \alpha.$$

Due to this, the reorder point policy sets the reorder point $\ell = \bar{d}\, t_\ell + \sigma_d \sqrt{t_\ell}\, z_\alpha$.

Moreover, ℓ minus the average demand in the reorder period gives the *safety stock* I_S:

$$I_S = \underbrace{\bar{d}\, t_\ell + \sigma_d \sqrt{t_\ell}\, z_\alpha}_{\ell} - \underbrace{\bar{d}\, t_\ell}_{\text{average demand}} = \sigma_d \sqrt{t_\ell}\, z_\alpha.$$

10.3.1   An example

Papier is a French retail chain which owns an outlet in Lyon. For this outlet:

- the expected demand $\bar{d}$ for mouse pads is 45 units per month;

- the value of an item is $c = €4$;

- the fixed reorder cost is $K = €30$;

- the annual interest rate is $p = 20\,\%$;

- the forecast for standard deviation is $\sigma_d = 5$;

- the lead time is $t_\ell = 1$ month;

- the required service level must be $97.7\,\%$, i.e. it is imposed that the demand for mouse pads be satisfied with percentage $97.7\,\%$ or, equivalently, in the reorder period $I(t) \geq 0$ with probability $\alpha = 0.977$.

First, compute the holding cost (the time unit is a month):

$$h = p\,c = 0.2 \cdot \text{€}4 = \text{€}0.8/\text{year per item} = \text{€}0.067/\text{month per item}.$$

Then, compute the reorder size:

$$q = \sqrt{\frac{2K\bar{d}}{h}} = \sqrt{\frac{2 \cdot 30 \cdot 45}{0.067}} \approx 201 \text{ items}.$$

Since $z_\alpha = 2$, to guarantee the required service level we have to set

$$\ell = 45 \cdot 1 + 5 \cdot 1 \cdot 2 = 45 + 10 = 55 \text{ units, and so}$$
$$I_S = 55 - 45 = 10 \text{ units}.$$

That is, we must reorder when the inventory level reaches $\ell = 55$ units; the corresponding safety stock is $I_S = 10$ units (to take into account variations around the average demand in the reorder period).

10.4   The slow-moving item model

When the demand is very low (e.g. spare parts of complex machineries), the main issue is to determine how many items to purchase at the beginning of the planning horizon. Consider a scenario in which:

- c is the purchase cost of an item at the beginning of the planning horizon;

- u is the purchase cost of an item during the planning horizon ($u > c$ due to storage penalty);

- holding costs are disregarded;

- n is the number of items to be purchased at the beginning of the planning horizon (decision variable);

- m is the number of items demanded in the planning horizon (unknown).

Given m and n, the total purchase cost is:

$$C(n, m) = \begin{cases} c\,n & \text{if } n \geq m \\ c\,n + \mu(m - n) & \text{if } n < m. \end{cases}$$

However, m is unknown. Let $P(m)$ be the probability that m items will be demanded during the planning horizon. Then, the expected cost if n items will be purchased can be expressed as:

$$\overline{C}(n) = \sum_{m=0}^{+\infty} C(n, m)\, P(m) = c\,n + u \sum_{m=n+1}^{+\infty} (m - n)\, P(m).$$

The problem to be solved is:

$$\min \overline{C}(n)$$
$$n \geq 0, \text{ integer} \tag{10.4}$$

To solve the problem in an analytical way, let $F(m) = \text{Prob}(d \leq m)$, where d denotes the demand in the planning horizon, which is a discrete random variable. That is, $F(m)$ is the probability that m units (or less) will be demanded during the planning horizon. Then, the following two relationships hold:

$$\overline{C}(n-1) = c\,(n-1) + u\sum_{m=n}^{+\infty}(m-n+1)\,P(m) =$$
$$= c\,n - c + u\sum_{m=n}^{+\infty}(m-n)\,P(m) + u\sum_{m=n}^{+\infty}P(m) =$$
$$= \overline{C}(n) - c + u\,\text{Prob}(d \geq n) = \tag{10.5}$$
$$= \overline{C}(n) - c + u\,\text{Prob}(d > n-1) =$$
$$= \overline{C}(n) - c + u\,[1 - F(n-1)]$$

$$\overline{C}(n+1) = c\,(n+1) + u\sum_{m=n+2}^{+\infty}(m-n-1)\,P(m) =$$
$$= c\,n + c + u\sum_{m=n+1}^{+\infty}(m-n-1)\,P(m) =$$
$$= c\,n + c + u\sum_{m=n+1}^{+\infty}(m-n)\,P(m) - u\sum_{m=n+1}^{+\infty}P(m) = \tag{10.6}$$
$$= \overline{C}(n) + c - u\,\text{Prob}(d \geq n+1) =$$
$$= \overline{C}(n) + c - u\,\text{Prob}(d > n) =$$
$$= \overline{C}(n) + c - u\,[1 - F(n)]$$

Now, the minimum expected cost is achieved if n^* items are purchased at the beginning of the planning horizon, where:

$$\begin{cases} \overline{C}(n^*) \leq \overline{C}(n^* - 1) \\ \overline{C}(n^*) \leq \overline{C}(n^* + 1) \end{cases}$$

By exploiting (10.5) and (10.6), we derive:

$$\overline{C}(n^*) \leq \overline{c}(n^*) - c + u\,[1 - F(n^* - 1)] \quad \Longleftrightarrow \quad F(n^* - 1) \leq \frac{u - c}{u} \tag{10.7}$$

$$\overline{C}(n^*) \leq \overline{c}(n^*) + c - u\,[1 - F(n^*)] \quad \Longleftrightarrow \quad F(n^*) \geq \frac{u - c}{u} \tag{10.8}$$

That is, the optimal number n^* of items to be purchased is such that:

$$F(n^* - 1) \leq \frac{u - c}{c} \leq F(n^*).^1$$

Recall that F is a monotone decreasing function.

10.4.1   An example

Hydro Ltd uses five LIF03 generators in its power plants in Nigeria:

- each piece of machinery (and so each generator) has an average life of 20 years;

- in this period, the expected number of failures (i.e. the expected demand for spare parts of generators) is 1.4;

- the cost of a spare part, purchased at the beginning, i.e. when a generator is manufactured, is $c = \$60,000$;

- the production unit cost of spare parts during the average period of life of the generators is $u = \$300,000$;

- the failure process during the planning horizon, described by a random variable d, is assumed to be modelled according to a Poisson probability distribution with expected value $\lambda = 5 \cdot 1.4 = 7$ faults per cycle of life.

Recall that the Poisson random variable is a discrete random variable (assuming values $0, 1, 2\ldots$), which is often used to model the random number of "arrivals" (i.e. generator faults, in our case) in a given time horizon. According to the Poisson probability distribution:

$$\text{Prob}(d = m) = \frac{e^{-\lambda}\,\lambda^m}{m!} = \frac{e^{-7}\,7^m}{m!}, \quad m = 0, 1\ldots$$

while the cumulative probability can be computed according to:

$$\begin{cases} F(0) = \text{Prob}(d = 0) = 0.009 \\ F(m) = F(m - 1) + \text{Prob}(d = m), \ m = 1, 2, \ldots \end{cases}$$

By applying this recursive formula, we get the values in Table 10.2.

Since $\frac{u-c}{u} = (300{,}000 - 60{,}000)/300{,}000 = 0.8$, according to the table we have $F(8) \leq 0.8 \leq F(9)$ and so the number of spare parts to be purchased and stocked at the beginning of the planning horizon to minimize the expected cost is $n^* = 9$ (circled in Table 10.2).

Note that, since $F(m)$ is a monotone nondecreasing function, we have a unique[2] n^* such that $F(n^* - 1) \leq \frac{u-c}{u} \leq F(n^*)$, and therefore n^* is a local minimum but also a global minimum, i.e. it is the optimal solution of the presented inventory model.

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[1] Recall that F is a monotone nondecreasing function.

[2] More precisely: in case of a sequence of equal probability values, we select the minimum n^* such that $F(n^* - 1) \leq \frac{u-c}{u} \leq F(n^*)$.

m	$\text{Prob}(d = m)$	$F(m)$
0	0.0009	0.0009
1	0.0064	0.0073
2	0.0223	0.0296
3	0.0521	0.0818
4	0.0912	0.1730
5	0.1277	0.3007
6	0.1490	0.4497
7	0.1490	0.5987
8	0.1304	0.7291
⑨	0.1014	0.8305
10	0.0710	0.9015

Table 10.2: The cumulative probability distribution of d

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