[Ex. 1] Let us extend the syntax of arithmetic expressions with the term \(a^x\), whose operational semantics is defined by the rules

\[
\begin{align*}
\langle a, \sigma \rangle &\rightarrow n & \langle a \times, \sigma \rangle &\rightarrow n \\
\langle a^x, \sigma \rangle &\rightarrow n & \langle a^x, \sigma \rangle &\rightarrow m \\
\langle a^x, \sigma \rangle &\rightarrow n \times m
\end{align*}
\]

1. Prove termination of extended expressions by structural induction.
2. Prove by rule induction that \(\forall \sigma, n. P((1^x, \sigma) \rightarrow n)\), where

\[P((1^x, \sigma) \rightarrow n) \overset{\text{def}}{=} n = 1\]

[Ex. 2] Let \(w\) be the command:

\[w \overset{\text{def}}{=} \text{while } x \times x = y \text{ do } (x := x \times x ; y := x \times y)\]

Find the set of memories \(S\) such that \(\forall \sigma \in S. \langle w, \sigma \rangle \not\rightarrow\).

[Ex. 3] Let \((D, \preceq)\) be the CPO with bottom such that \(D = \mathbb{N} \cup \{\infty_1, \infty_2\}\) and \(\preceq \cap (\mathbb{N} \times \mathbb{N}) = \preceq\), \(\infty_2\) is the top element and \(x \preceq \infty_1\) iff \(x \neq \infty_2\).

1. Consider the function \(\text{succ} : D \rightarrow D\) such that \(\forall n \in \mathbb{N}. \text{succ}(n) = n + 1\) and \(\text{succ}(\infty_1) = \text{succ}(\infty_2) = \infty_2\).
   Prove that the function \(\text{succ}\) is monotone but not continuous.
2. [Optional] Let \(\{d_i\}_{i \in \mathbb{N}}\) be a chain.
   Prove that if \(\bigsqcup_{i \in \mathbb{N}} d_i = \infty_2\) then the chain is finite.
   \textit{Hint:} Note that if \(\infty_1\) or \(\infty_2\) belong to the chain then it is finite.

[Ex. 4] Let \(\mathbf{Pf}\) be the domain of partial functions over positive natural numbers (ordered as usual and whose bottom element \(\bot_{\mathbf{Pf}}\) is the always undefined function). Let \(\Gamma : \mathbf{Pf} \rightarrow \mathbf{Pf}\) the continuous function defined by

\[\Gamma \overset{\text{def}}{=} \lambda \varphi. \lambda m. (m = 1) \rightarrow 2, 2m + \varphi(m - 1)\]

Take \(\varphi_n \overset{\text{def}}{=} \Gamma^n(\bot_{\mathbf{Pf}})\) and \(f \overset{\text{def}}{=} \text{fix } \Gamma = \bigcup_{n \in \mathbb{N}} \varphi_n\).
Prove that \(\forall n > 0. \varphi_n(n) = n(n+1)\) to conclude that \(\forall n > 0. f(n) = n(n+1)\).

[Ex. 5] Consider the Haskell types \(\text{Arc } a = (a, a)\), \(\text{Graph } a = \text{[Arc } a\]\) and \(\text{Nodes } a = [a]\). Implement a function

\[\text{nodes :: (Eq } a\) \Rightarrow \text{Graph } a \rightarrow \text{Nodes } a\]

that returns the list of nodes of a graph, without repeated elements.