



PSC 2023/24 (375AA, 9CFU)

Principles for Software Composition

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<http://didawiki.di.unipi.it/doku.php/magistraleinformatica/psc/start>

08b - Kleene's fixed point theorem

Partial functions

Comparing functions

given two functions $f, g : A \rightarrow B$, when can we say $f = g$?

$$\forall a \in A . f(a) = g(a)$$

if we see functions as relations

$$\{ (a, f(a)) \mid a \in A \} \subseteq A \times B$$

we can use set equality

Example

$$f(n) = n!$$

$$f = \left\{ \begin{array}{l} \overset{n}{\quad} \overset{f(n)}{\quad} \\ (0, 1), \\ (1, 1), \\ (2, 2), \\ (3, 6), \\ \dots \\ (k, k!), \\ \dots \end{array} \right\}$$

$$f(n) = n(n+1)/2$$

$$f = \left\{ \begin{array}{l} (0, 0), \\ (1, 1), \\ (2, 3), \\ (3, 6), \\ \dots \\ (k, T_k), \\ \dots \end{array} \right\}$$

Partial functions

let $f: A \rightarrow B$, or equivalently $f: A \rightarrow B \cup \{ \perp \}$

the function f can be undefined on some inputs

we can still see partial functions as relations

$$\{ (a, f(a)) \mid a \in A, f(a) \neq \perp \} \subseteq A \times B$$

omit pairs where $f(a)$ is undefined

Partial functions

$D = (A \rightarrow B) = \mathbf{Pf}(A, B) = \{f : A \rightarrow B\}$ partial functions

$f \sqsubseteq g$ if $f(a)$ is defined, $g(a)$ is defined and $g(a) = f(a)$

but $g(a)$ can be defined when $f(a)$ is not

if we see partial functions as relations

$$\{(x, f(x)) \mid f(x) \neq \perp\} \subseteq A \times B$$

$f \sqsubseteq g$ means essentially $f \subseteq g$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \perp & \text{otherwise} \end{cases}$$

$$f = \{ \begin{array}{l} \begin{array}{cc} n & f(n) \end{array} \\ (0, 0), \\ (2, 1), \\ (4, 2), \\ (6, 3), \\ \dots \\ (2k, k), \\ \dots \end{array} \}$$

Example

Pf(\mathbb{N}, \mathbb{N})

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ 2 \cdot n & \text{otherwise} \end{cases}$$

$$g = \left\{ \begin{array}{l} (0, 0), (1, 2), \\ (2, 1), (3, 6), \\ (4, 2), (5, 10), \\ (6, 3), (7, 14), \\ \dots \\ (2k, k), (1 + 2k, 2 + 4k), \\ \dots \end{array} \right\}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ 2 \cdot n & \text{otherwise} \end{cases}$$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \perp & \text{otherwise} \end{cases}$$

$$g = \{ \begin{array}{l} (0, 0), (1, 2), \\ (2, 1), (3, 6), \\ (4, 2), (5, 10), \\ (6, 3), (7, 14), \\ \dots \\ (2k, k), (1 + 2k, 2 + 4k), \\ \dots \end{array} \}$$

$$f = \{ \begin{array}{l} (0, 0), \\ (2, 1), \\ (4, 2), \\ (6, 3), \\ \dots \\ (2k, k), \\ \dots \end{array} \}$$

$$\begin{array}{l} f \sqsubseteq g? \quad \checkmark \\ g \sqsubseteq f? \quad \times \end{array}$$

Example

Pf(\mathbb{N}, \mathbb{N})

$$\emptyset \sqsubseteq \{ (0,0) \} \sqsubseteq \{ (0,0), (1,1) \} \sqsubseteq \dots$$

which function(s) are we approximating?

Example

Pf(\mathbb{N}, \mathbb{N})

$$\emptyset \sqsubseteq \{ (0,0) \} \sqsubseteq \{ (0,0), (1,1) \} \sqsubseteq \{ (0,0), (1,1), (2,2) \} \sqsubseteq \dots$$

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which function(s) are we approximating?

Example

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$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (1,1) \} \sqsubseteq \{ (0,1), (1,1), (2,2) \} \sqsubseteq \dots$$

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which function(s) are we approximating?

Example

Pf(\mathbb{N}, \mathbb{N})

$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (1,1) \} \sqsubseteq \{ (0,1), (1,1), (2,2) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6), (4,24) \} \sqsubseteq \dots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (4,24) \} \sqsubseteq \{ (0,1), (1,1), (4,24) \} \sqsubseteq \{ (0,1), (1,1), (3,6), (4,24) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6), (4,24) \} \sqsubseteq \dots$$

which function(s) are we approximating?

Example

Pf(\mathbb{N}, \mathbb{N})

$$\emptyset \sqsubseteq \{ (1,1) \} \sqsubseteq \{ (1,1), (2,4) \} \sqsubseteq \{ (1,1), (2,4), (3,81) \} \sqsubseteq \{ (1,1), (2,4), (3,81), (4,256) \} \sqsubseteq \dots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (4,2) \} \sqsubseteq \{ (4,2), (6,3) \} \sqsubseteq \{ (4,2), (6,3), (8,4) \} \sqsubseteq \{ (4,2), (6,3), (8,4), (9,3) \} \sqsubseteq \{ (4,2), (6,3), (8,4), (9,3), (10,5) \} \sqsubseteq \dots$$

which function(s) are we approximating?

Example

Pf(\mathbb{N}, \mathbb{N})

$$\emptyset \sqsubseteq \{ (1,6) \} \sqsubseteq \{ (1,6), (2,28) \} \sqsubseteq \{ (1,6), (2,28), (3,496) \} \sqsubseteq \{ (1,6), (2,28), (3,496), (4,8128) \} \sqsubseteq \dots$$

which function(s) are we approximating?

Functional property

$\mathbf{Pf}(A, B) = \{f : A \rightarrow B\}$ partial functions

$\mathbf{Pf}(A, B) = \{f \subseteq A \times B \mid \forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2\}$

functional property

$f(a) \downarrow \triangleq \exists b \in B. (a, b) \in f$ function f is defined on a

$f \sqsubseteq g \Leftrightarrow (\forall a \in A. f(a) \downarrow \Rightarrow (g(a) \downarrow \wedge f(a) = g(a)))$
 $\Leftrightarrow f \subseteq g$

$(\mathbf{Pf}(A, B), \sqsubseteq)$ is a PO with bottom
what is bottom?
is it complete?

the empty relation
(the function always undefined)

Is Pf complete?

$(\mathbf{Pf}(A, B), \sqsubseteq)$

complete?

Given a chain $\{f_i\}_{i \in \mathbb{N}}$ let us consider $\bigcup_{i \in \mathbb{N}} f_i \subseteq A \times B$

we want to prove that $\bigcup_{i \in \mathbb{N}} f_i \in \mathbf{Pf}(A, B)$

i.e. that $f = \bigcup_{i \in \mathbb{N}} f_i$ satisfies the functional property

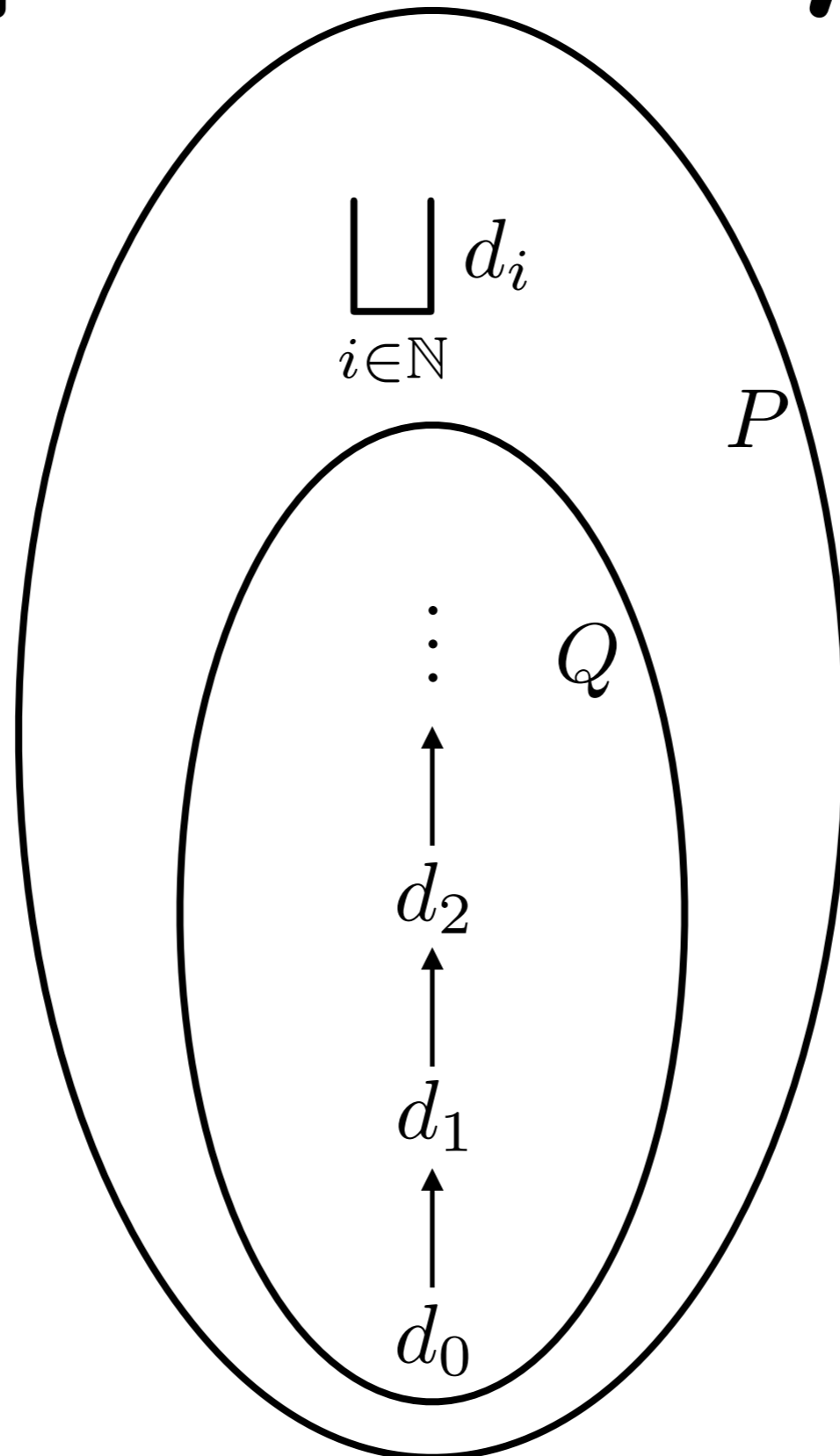
we know that each f_i is functional

$$\forall i \in \mathbb{N}. \forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f_i \wedge (a, b_2) \in f_i \Rightarrow b_1 = b_2$$

we need to prove f is functional

$$\forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2$$

pictorially



is the limit in Q ?

Pf is complete

we need to prove f is functional

$$\forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2$$

Take $a \in A, b_1, b_2 \in B$ such that $(a, b_1) \in f \wedge (a, b_2) \in f$

we need to prove $b_1 = b_2$

$$(a, b_1) \in f = \bigcup_{i \in \mathbb{N}} f_i \Leftrightarrow \exists k \in \mathbb{N}. (a, b_1) \in f_k$$

$$m \triangleq \max\{k, h\}$$

$$(a, b_2) \in f = \bigcup_{i \in \mathbb{N}} f_i \Leftrightarrow \exists h \in \mathbb{N}. (a, b_2) \in f_h$$

Clearly $f_k \subseteq f_m$ $f_h \subseteq f_m$ f_m is functional

$$(a, b_1) \in f_m \quad (a, b_2) \in f_m \quad \Rightarrow \quad b_1 = b_2$$

Example

$$\begin{array}{l} \mathbf{Pf}(\mathbb{N}, \mathbb{N}) \\ f_0 \quad \emptyset \\ \subseteq \\ \subseteq \\ \subseteq \\ \subseteq \\ \subseteq \\ \subseteq \\ \subseteq \end{array} \begin{array}{l} \{(0, 1)\} \\ \{(0, 1), (1, 1)\} \\ \{(0, 1), (1, 1), (2, 2)\} \\ \{(0, 1), (1, 1), (2, 2), (3, 6)\} \\ \{(0, 1), (1, 1), (2, 2), (3, 6), (4, 24)\} \\ \dots \end{array} \begin{array}{l} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array}$$

$\bigcup_{i \in \mathbb{N}} f_i$ is (maybe) the factorial function

note: the limit of partial functions can be a total function

Total functions

$\mathbf{Tf}(A, B) = (A \rightarrow B)$ total functions

$\mathbf{Pf}(A, B) \equiv \mathbf{Tf}(A, B_{\perp})$ $B_{\perp} \triangleq B \uplus \{\perp\}$

$\sqsubseteq_{B_{\perp}} \triangleq$ flat order

$$f \sqsubseteq g \Leftrightarrow \forall a \in A. f(a) \sqsubseteq_{B_{\perp}} g(a)$$

PO? immediate to check

bottom? $f_{\perp}(a) = \perp$ for any $a \in A$

complete? we will prove it later

(as an instance of a more general result)

$$\left(\bigsqcup_{i \in \mathbb{N}} f_i \right)(a) \triangleq \bigsqcup_{i \in \mathbb{N}} f_i(a) \quad (\text{flat order, limit exists})$$

Monotone functions

Monotone function

(D, \sqsubseteq_D) PO (E, \sqsubseteq_E) PO $f : D \rightarrow E$

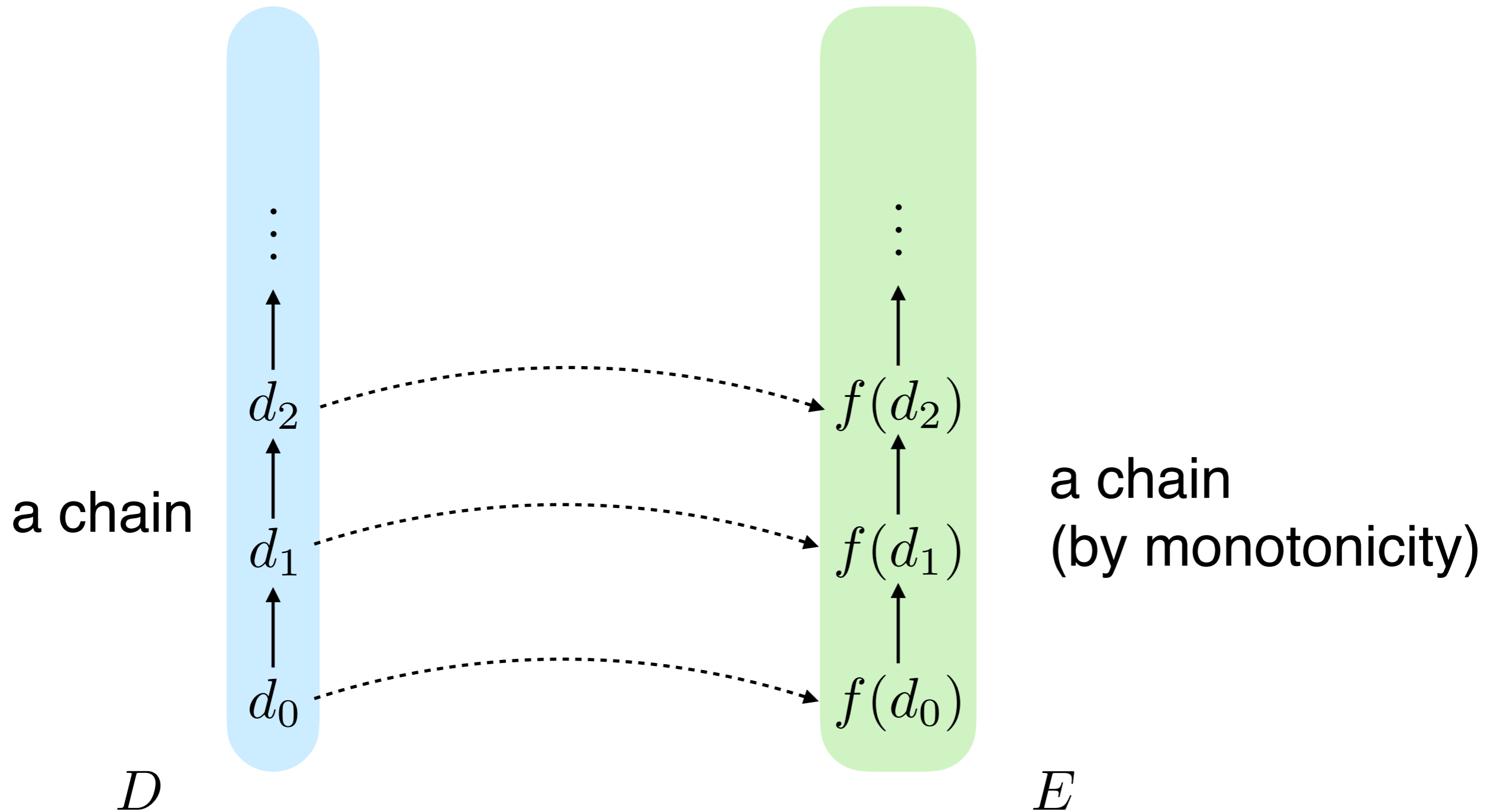
f is **monotone** if $\forall d_1, d_2 \in D. d_1 \sqsubseteq_D d_2 \Rightarrow f(d_1) \sqsubseteq_E f(d_2)$

Monotone = Order preserving

$\left. \begin{array}{l} \{d_i\}_{i \in \mathbb{N}} \text{ a chain in } D \\ f \text{ monotone} \end{array} \right\} \Rightarrow \{f(d_i)\}_{i \in \mathbb{N}} \text{ a chain in } E$

When $D = E$ we say $f : D \rightarrow D$ is a function on D

Monotonicity illustrated

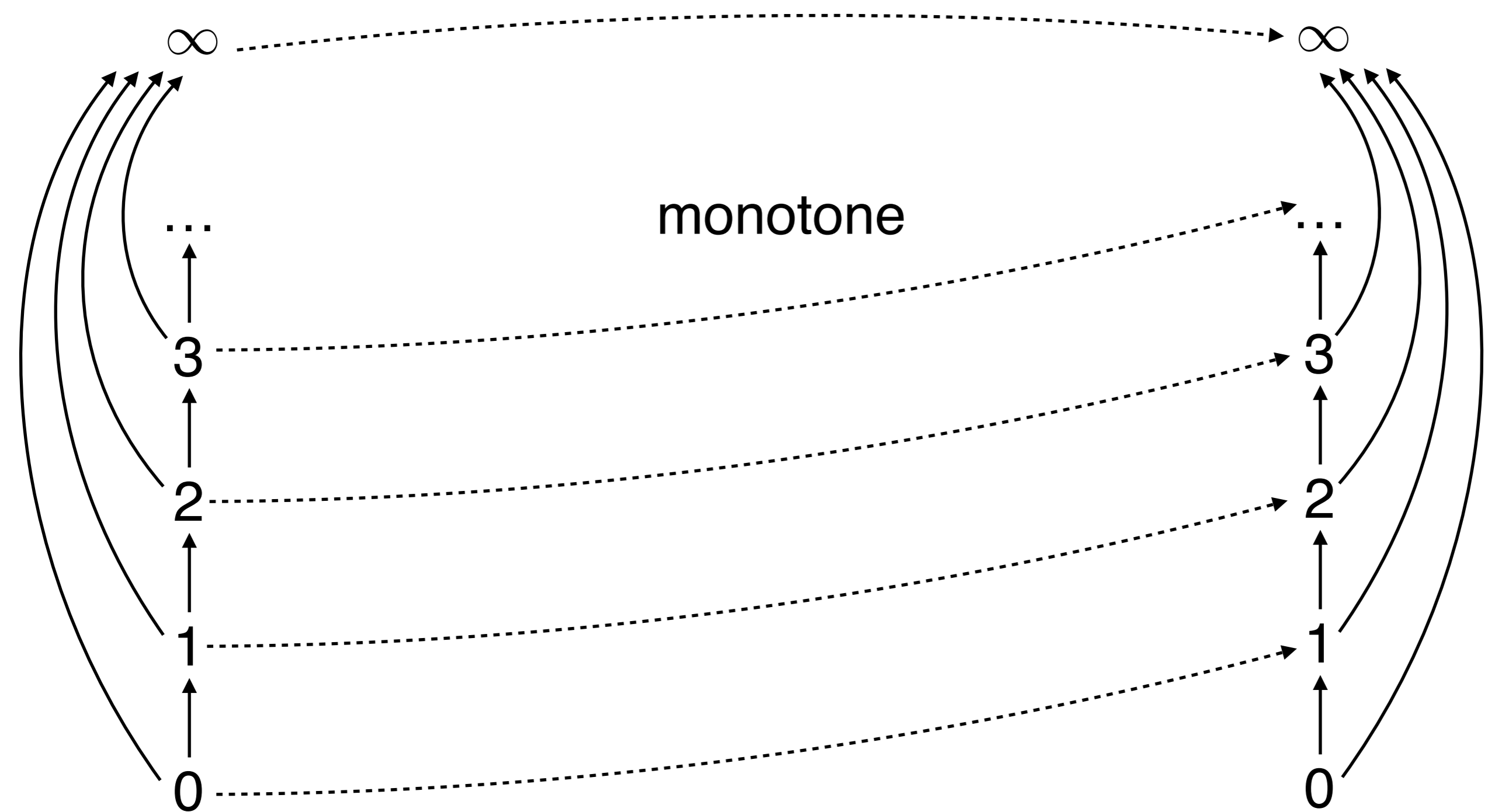


Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$



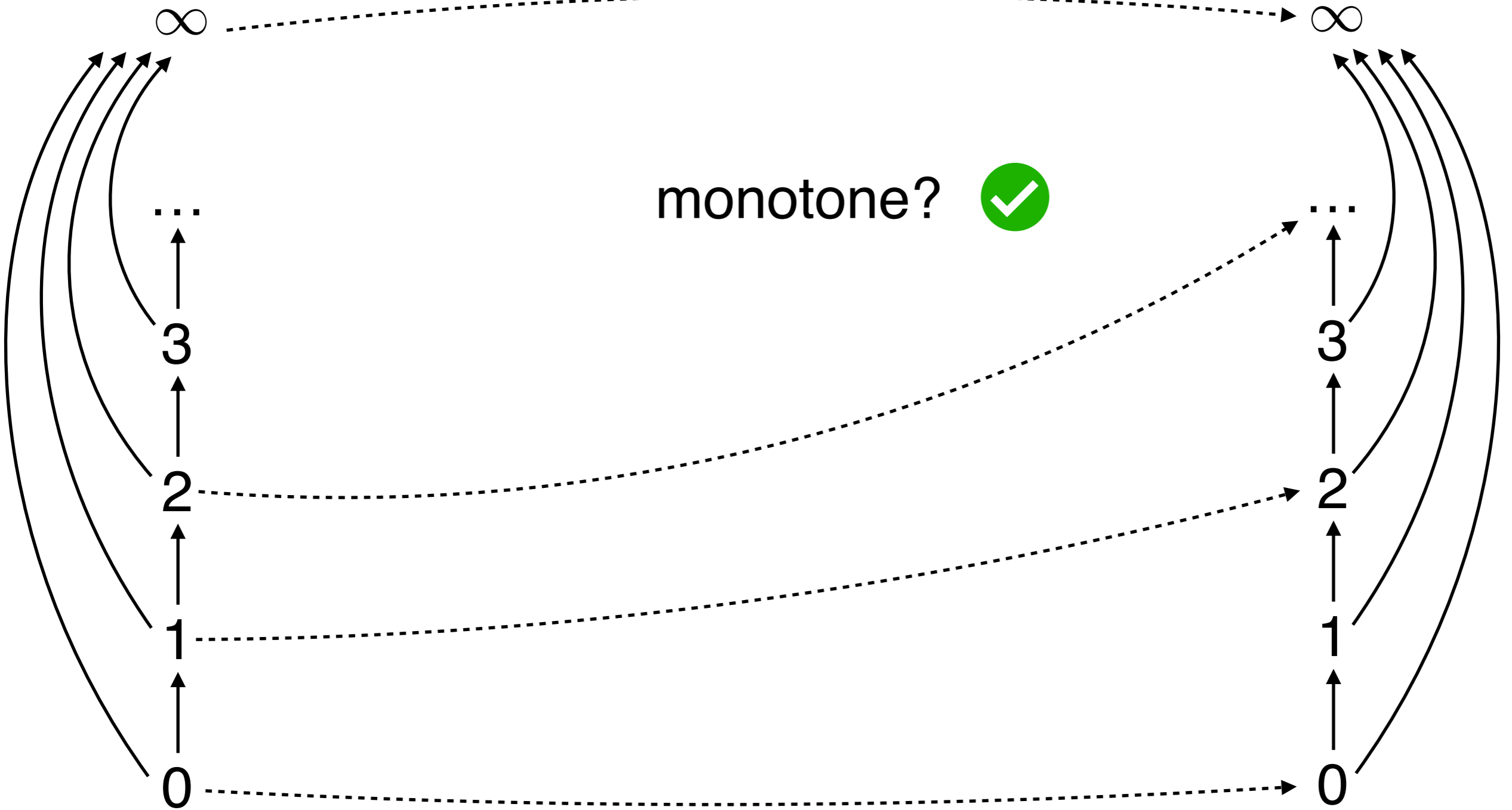


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$



monotone?



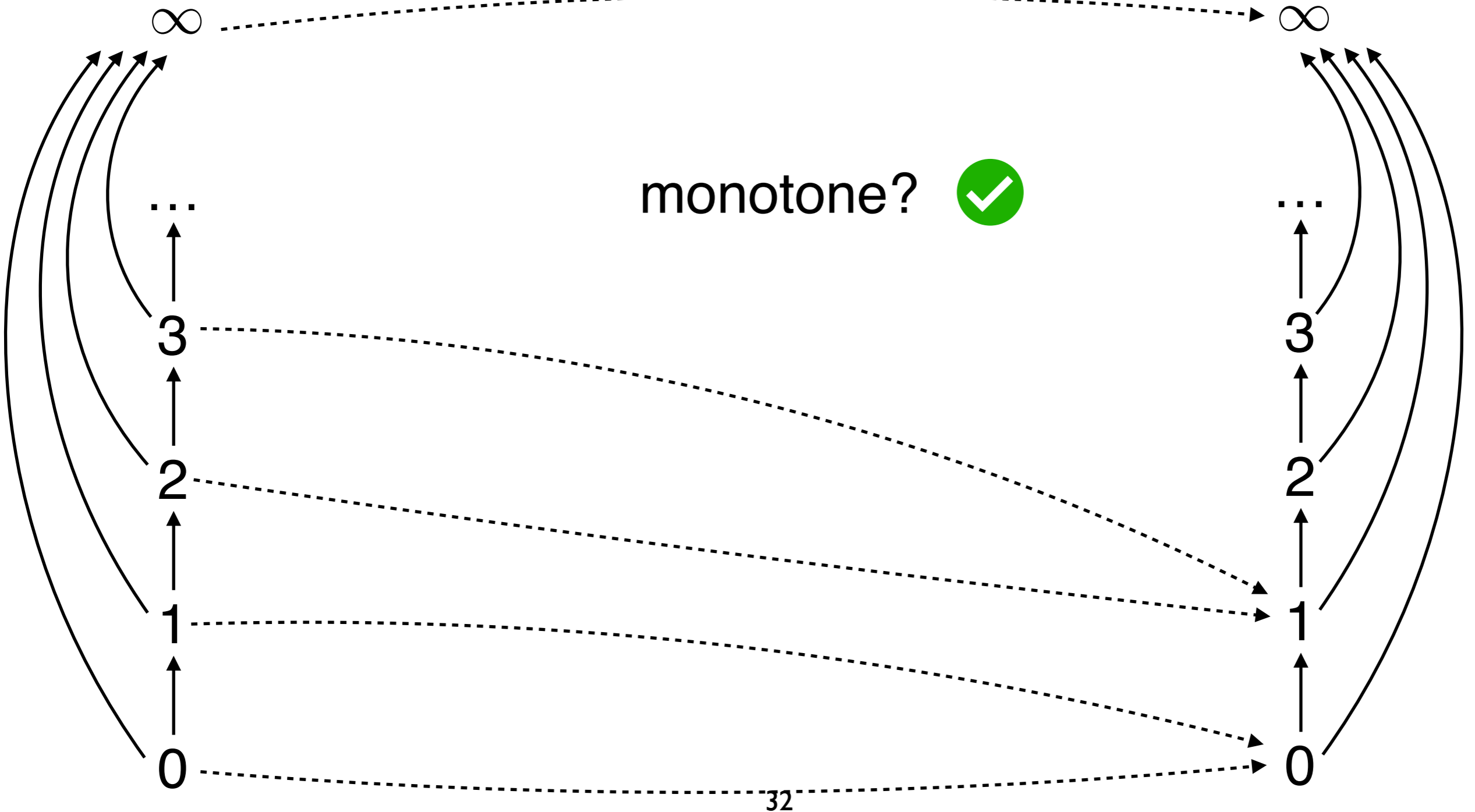


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n/2$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$



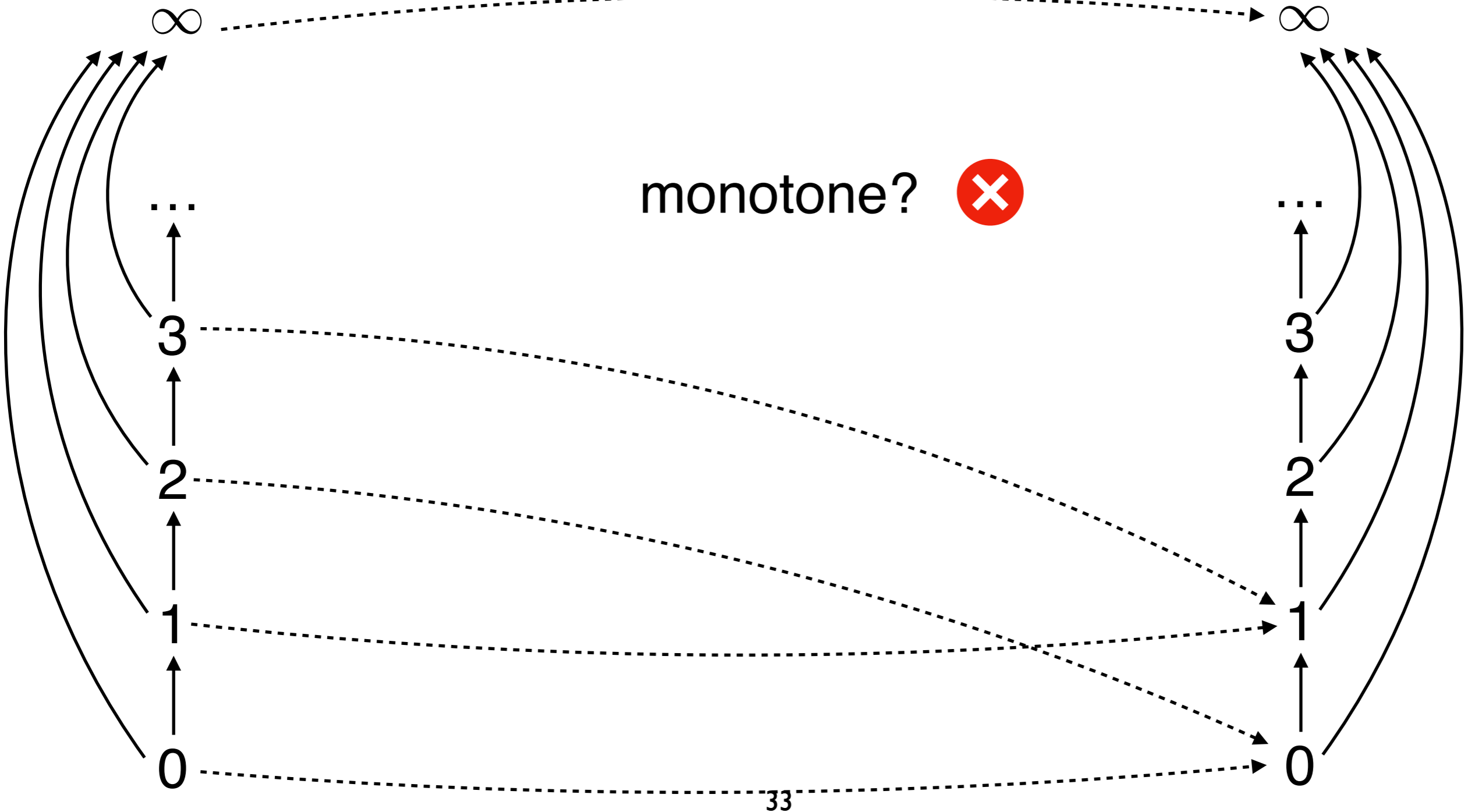


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n \% 2$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$



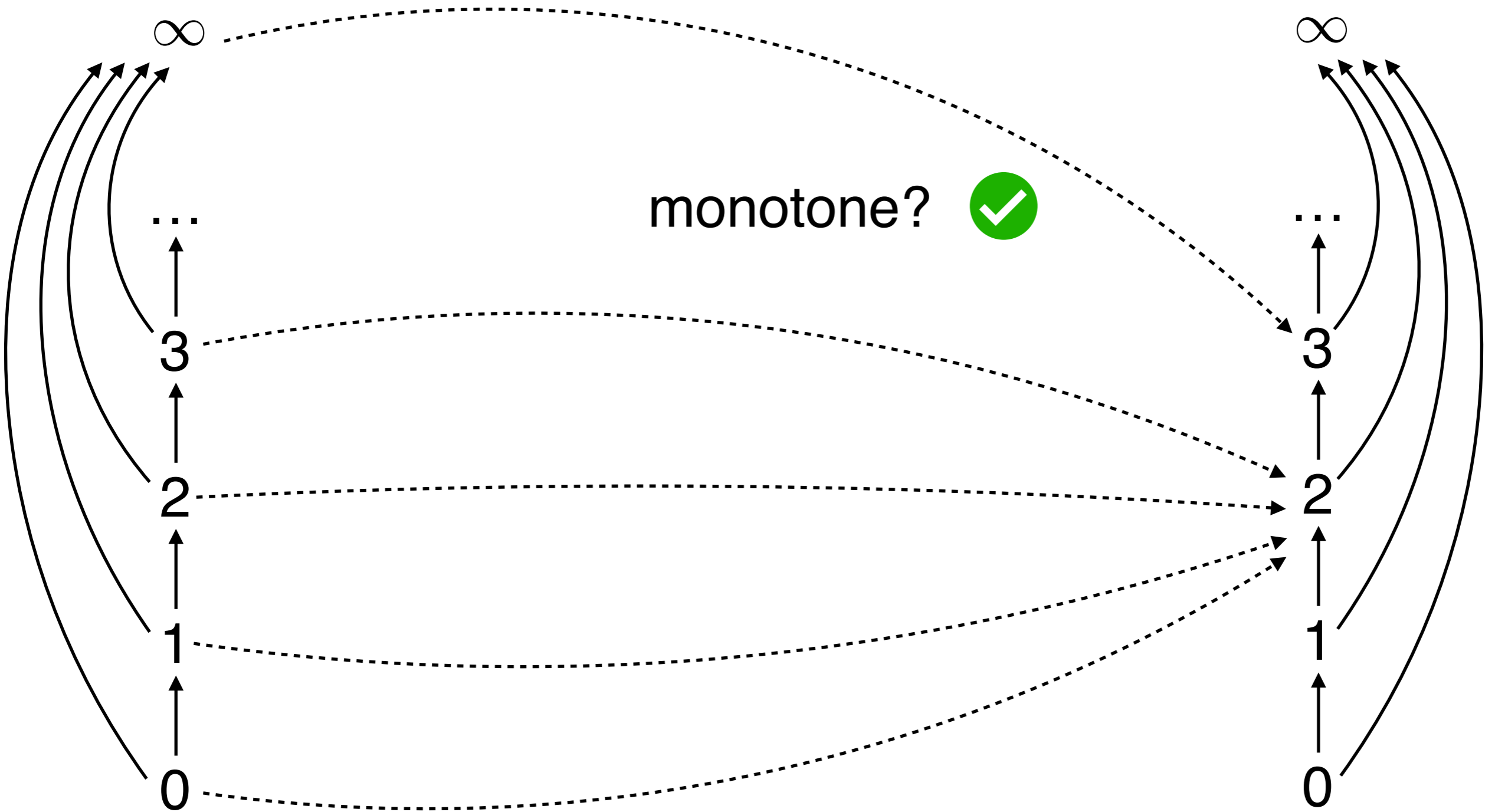


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

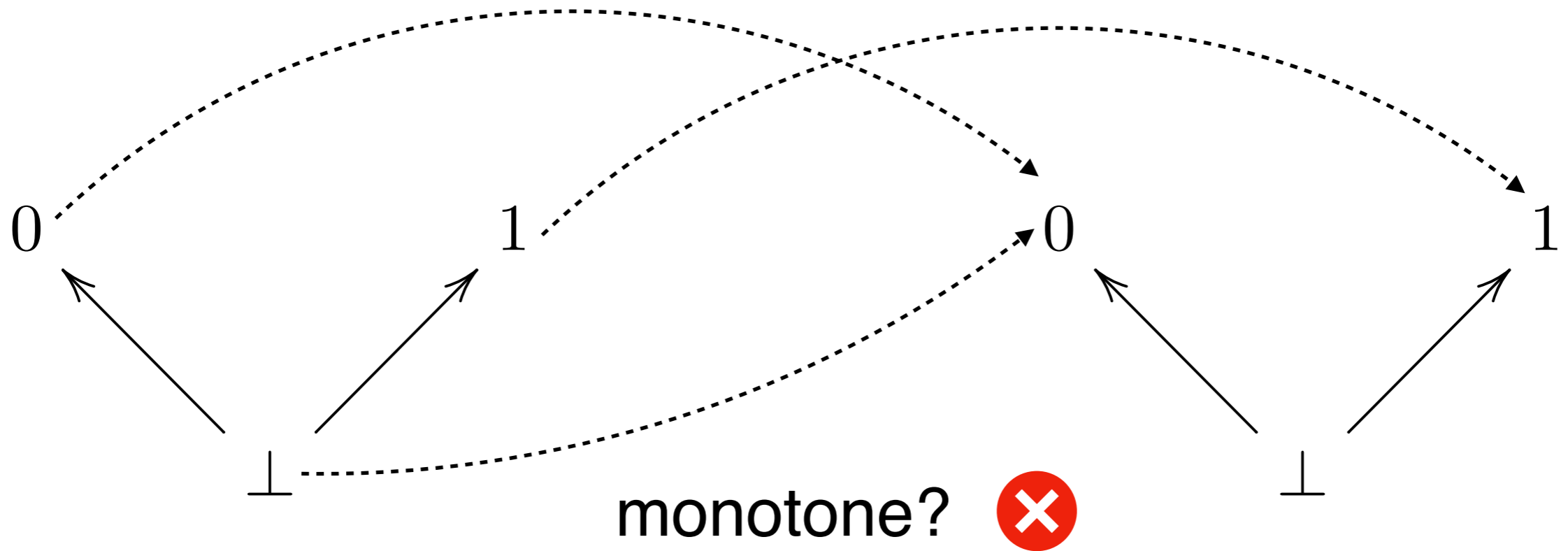
$$f(n) = 2$$
$$f(\infty) = 3$$

$(\mathbb{N} \cup \{\infty\}, \leq)$





Exercise



D

$f : D \rightarrow D$

D

$$f(\perp) = f(0) = 0$$

$$f(1) = 1$$

$$\perp \sqsubseteq 1$$

$$f(\perp) = 0 \not\sqsubseteq 1 = f(1)$$



Exercise

$$(\wp(\mathbb{N}), \subseteq) \quad f(S) = \{ m \in \mathbb{N} \mid \exists n \in S, m \leq n \} \quad (\wp(\mathbb{N}), \subseteq)$$

monotone?





Exercise

$$(\wp(\mathbb{N}), \subseteq) \quad f(S) = \{ m \in \mathbb{N} \mid \forall n \in S, n < m \} \quad (\wp(\mathbb{N}), \subseteq)$$

monotone?



Composition

TH. Any composition of monotone functions is monotone

$$\begin{array}{llll} (D, \sqsubseteq_D) & \text{PO} & f : D \rightarrow E \text{ monotone} & \\ (E, \sqsubseteq_E) & \text{PO} & g : E \rightarrow F \text{ monotone} & \Rightarrow h = g \circ f : D \rightarrow F \\ (F, \sqsubseteq_F) & \text{PO} & & \text{monotone} \end{array}$$

proof. we need to prove $\forall x, y \in D. x \sqsubseteq_D y \Rightarrow h(x) \sqsubseteq_F h(y)$

take $x \sqsubseteq_D y$

we want to prove $h(x) \sqsubseteq_F h(y)$

then $f(x) \sqsubseteq_E f(y)$ because f is monotone

then $g(f(x)) \sqsubseteq_F g(f(y))$ because g is monotone

$$\begin{array}{ccc} = & & = \\ h(x) & & h(y) \end{array}$$

Continuous functions

Continuous function

(D, \sqsubseteq_D) CPO (E, \sqsubseteq_E) CPO $f : D \rightarrow E$ monotone

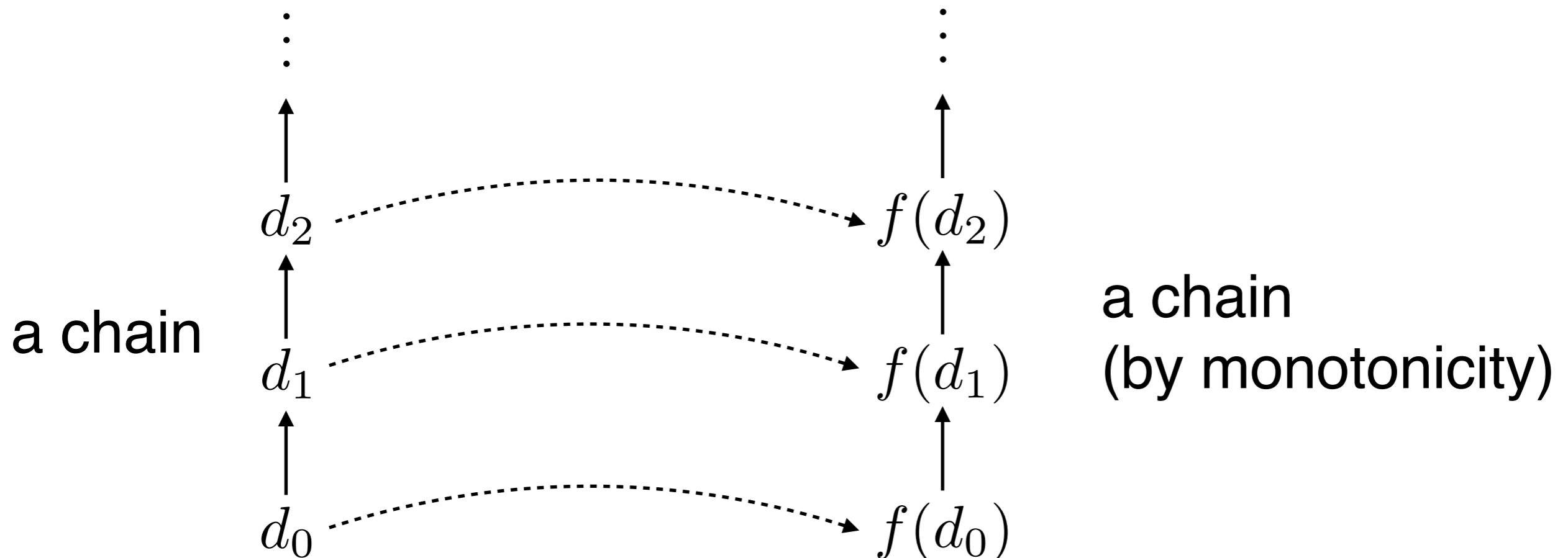
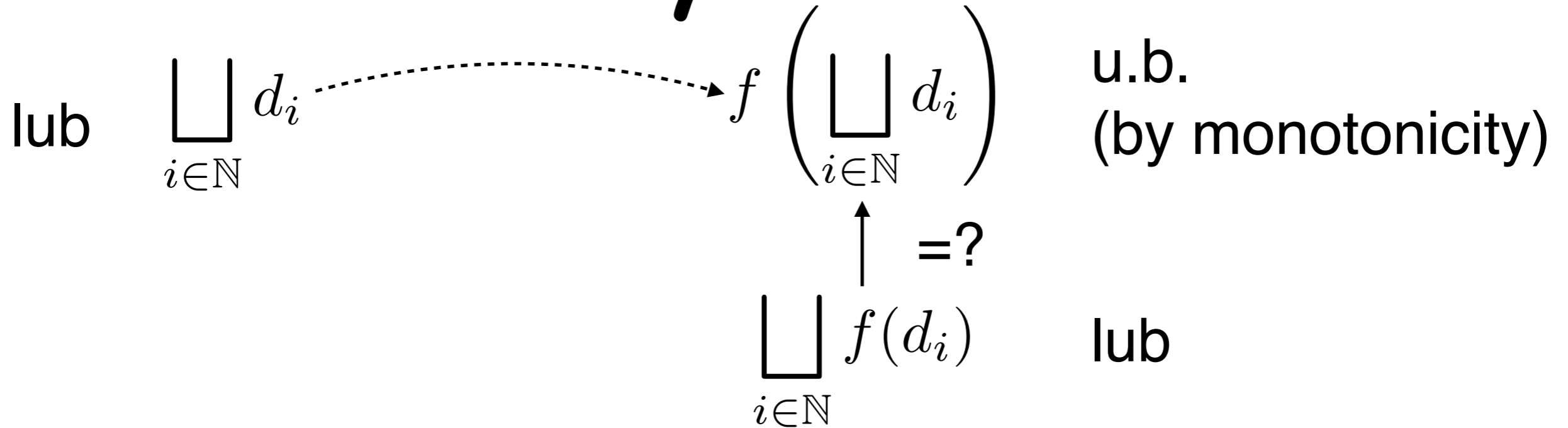
f is **continuous** if $\forall \{d_i\}_{i \in \mathbb{N}}$ chain

$$f \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)$$

limit in D limit in E

Continuous = limit preserving

Continuity illustrated



Continuity illustrated

lub $\bigsqcup_{i \in \mathbb{N}} d_i \xrightarrow{\text{dotted arrow}} f \left(\bigsqcup_{i \in \mathbb{N}} d_i \right)$

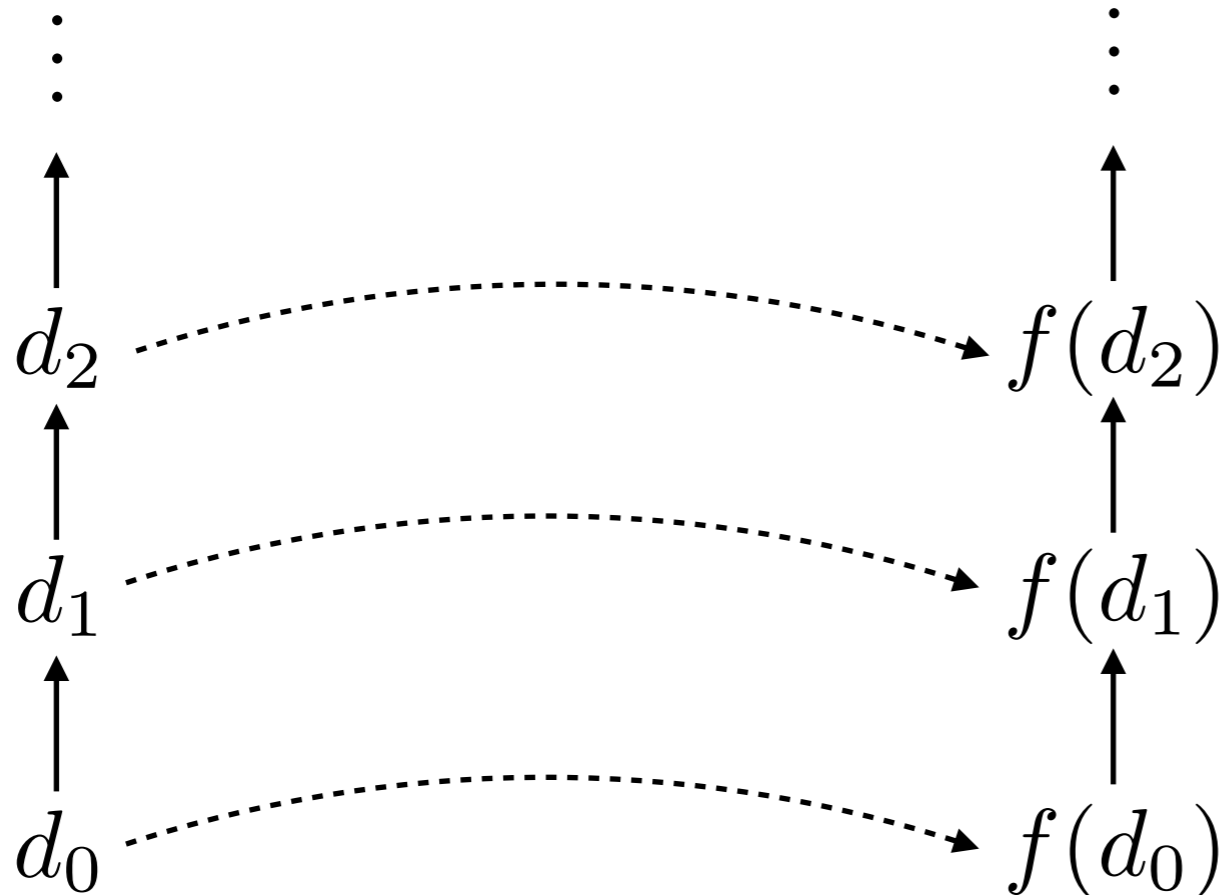
$\uparrow = ?$

$\bigsqcup_{i \in \mathbb{N}} f(d_i)$

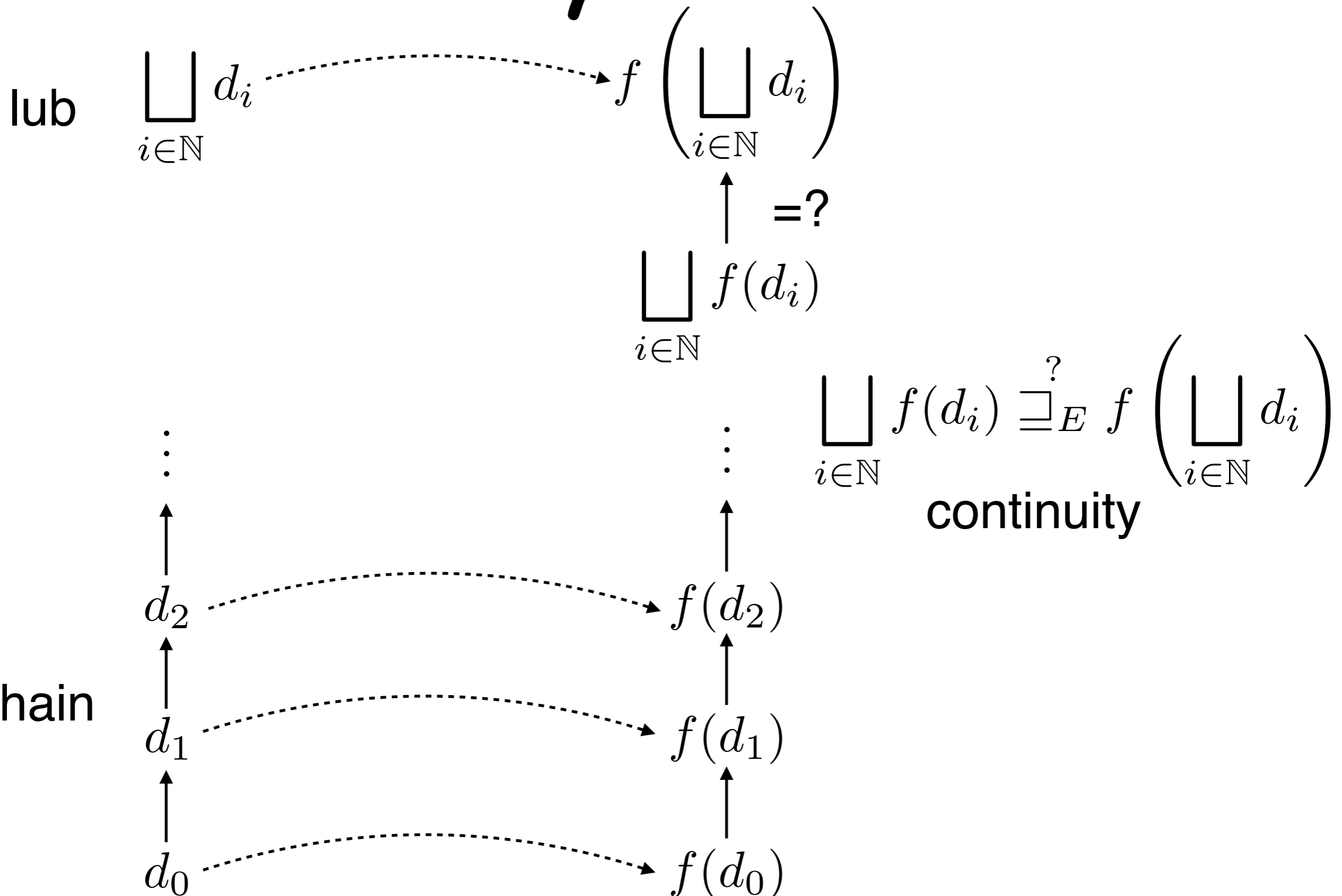
$$\bigsqcup_{i \in \mathbb{N}} f(d_i) \sqsubseteq_E f \left(\bigsqcup_{i \in \mathbb{N}} d_i \right)$$

follows from
monotonicity
(and CPO)

a chain



Continuity illustrated

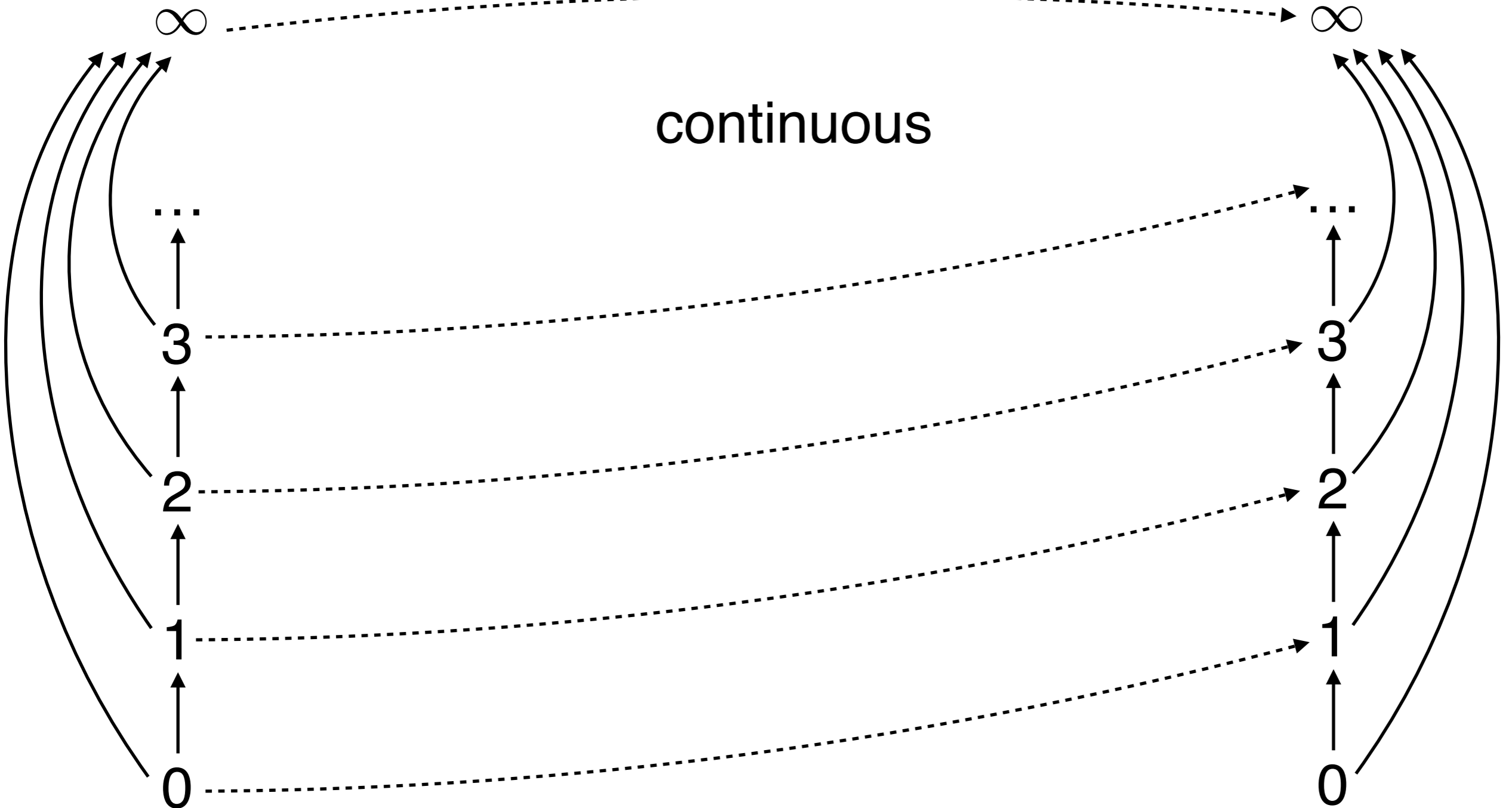


Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$



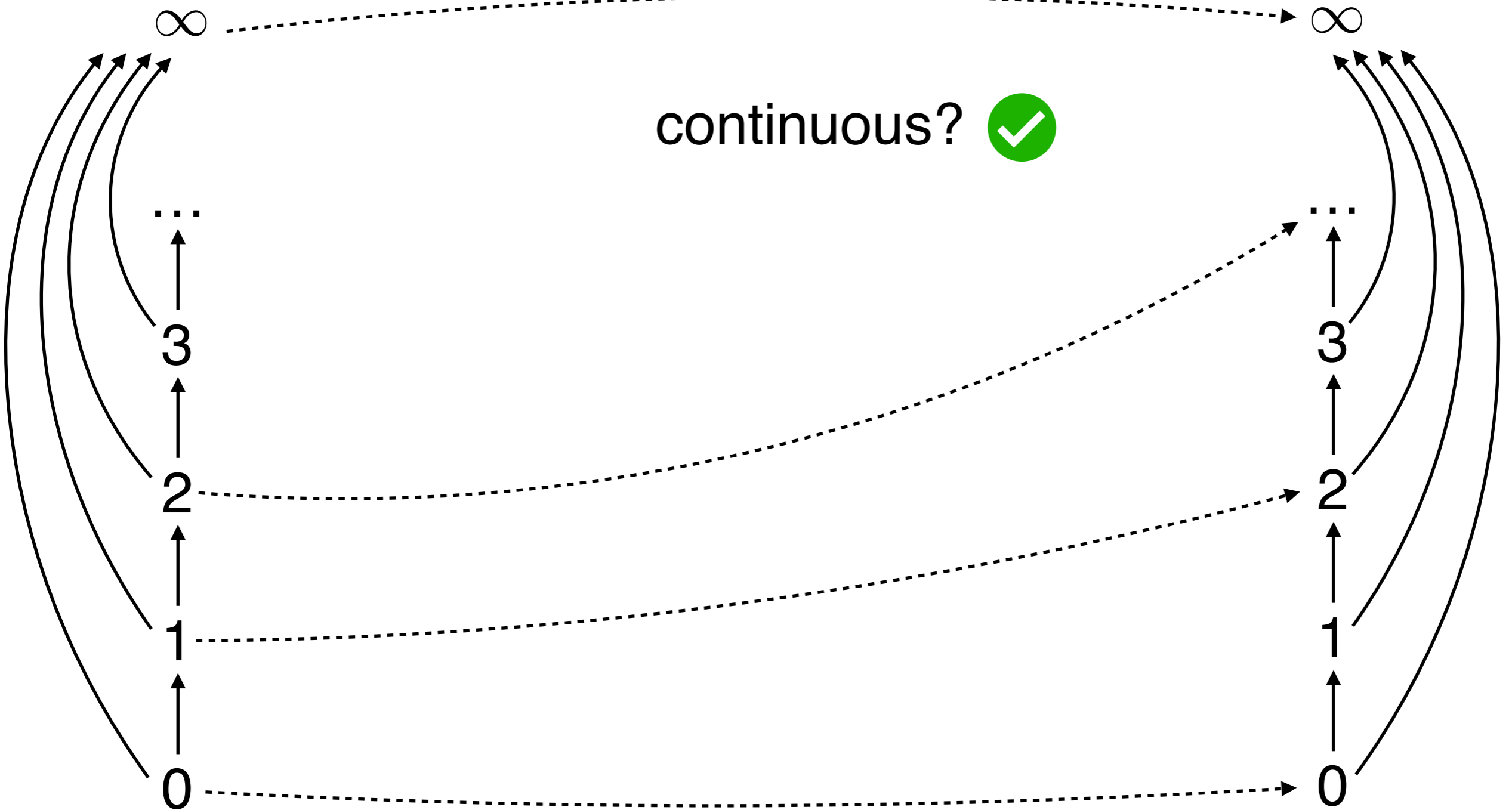


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$

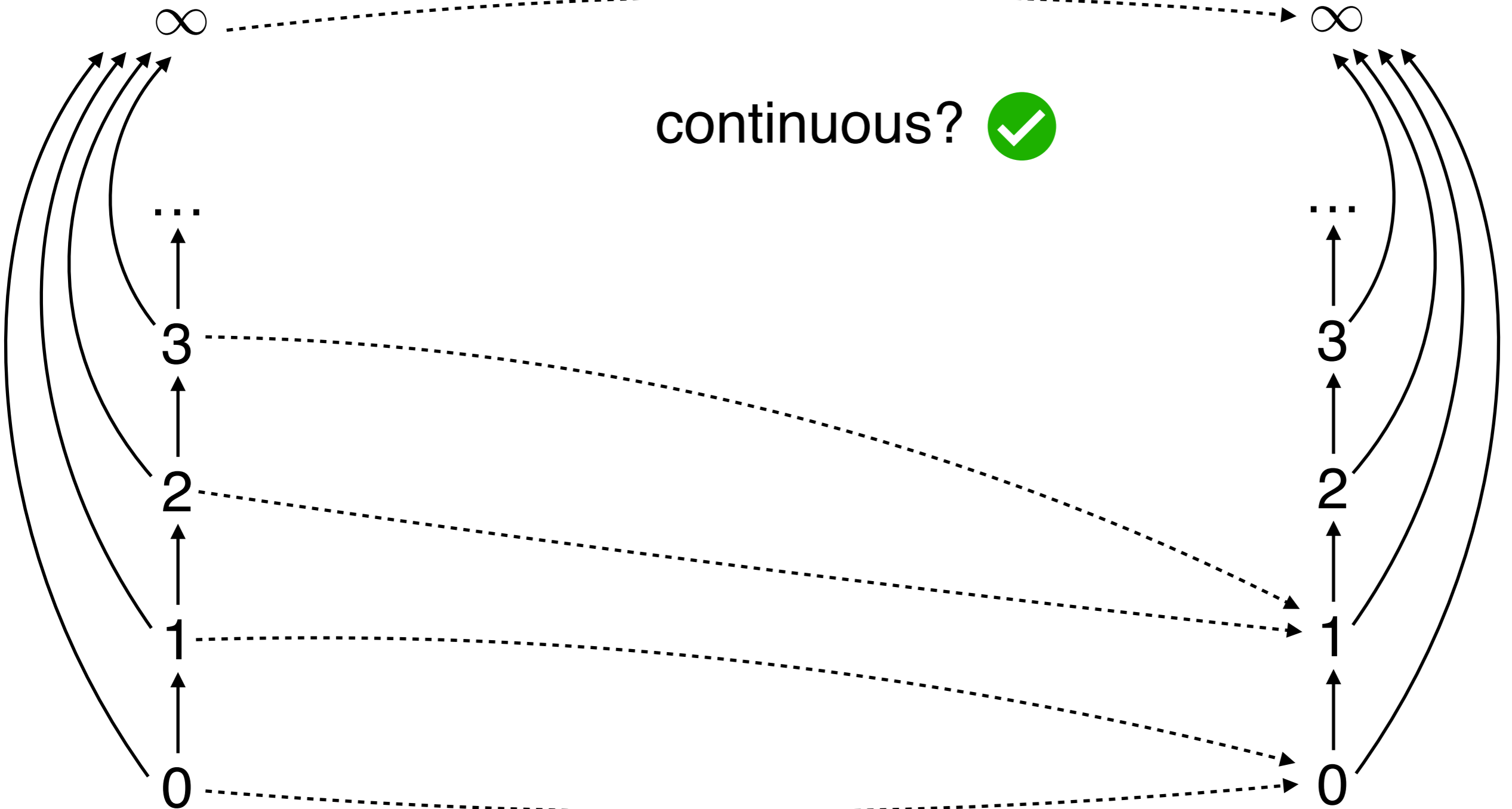


Example

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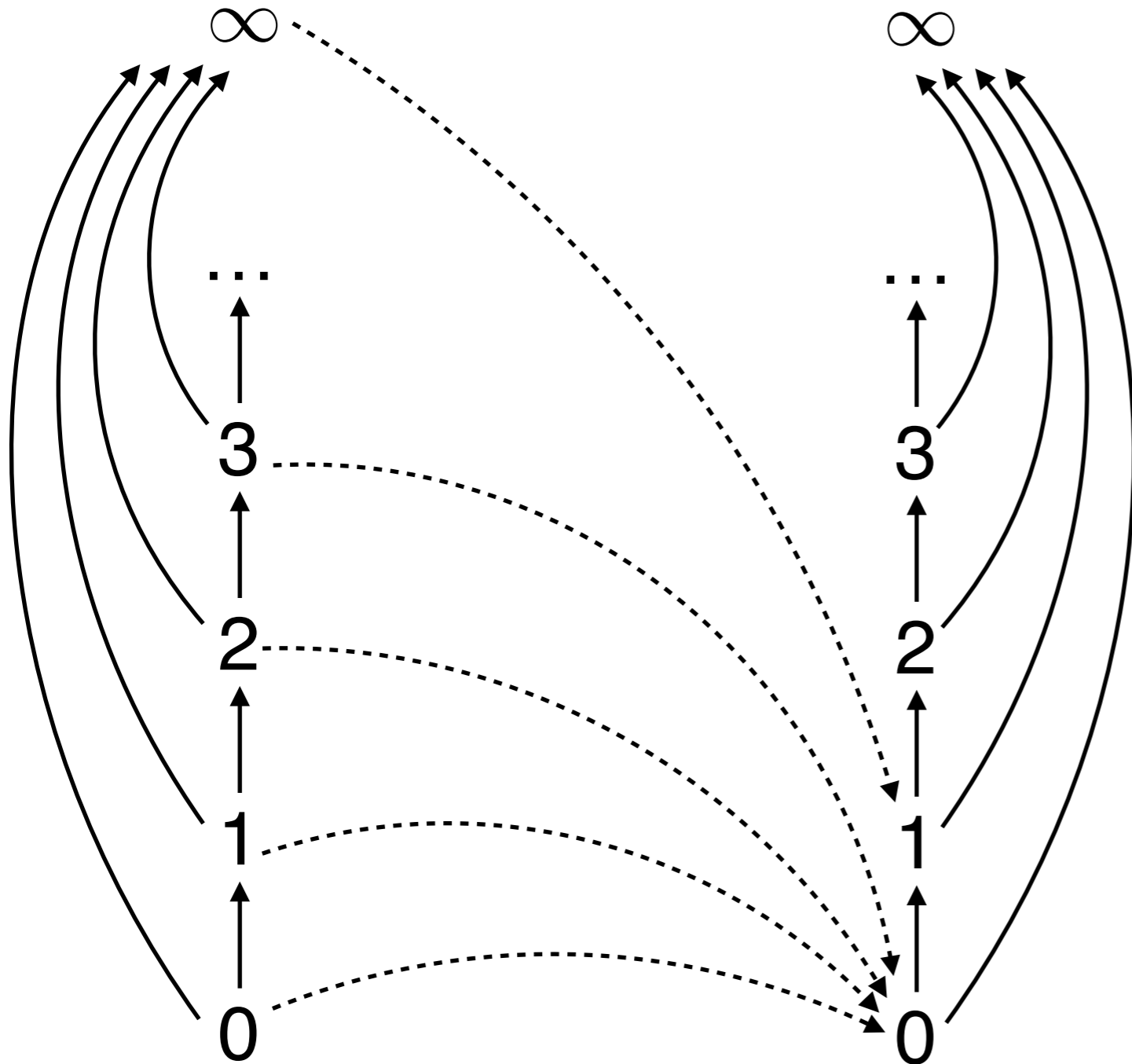
$(\mathbb{N} \cup \{\infty\}, \leq)$



Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

monotone function, not continuous



$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{N} \\ 1 & \text{if } x = \infty \end{cases}$$

$$d_i = 2 \cdot i \quad \bigsqcup_{i \in \mathbb{N}} d_i = \infty$$

$$f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = f(\infty) = 1$$

$$f(d_i) = 0$$

$$\bigsqcup_{i \in \mathbb{N}} f(d_i) = \bigsqcup_{i \in \mathbb{N}} 0 = 0$$



Exercise

$$(\wp(\mathbb{N}), \subseteq) \quad f(S) = \{ m \in \mathbb{N} \mid \exists n \in S, m \leq n \} \quad (\wp(\mathbb{N}), \subseteq)$$

continuous? 



Exercise

$(\wp(\mathbb{N}), \subseteq)$

$$f(S) = \begin{cases} S & \text{if } S \text{ finite} \\ \mathbb{N} & \text{otherwise} \end{cases}$$

$(\wp(\mathbb{N}), \subseteq)$

continuous? 

Lemma

(D, \sqsubseteq_D) CPO
no infinite chains

(E, \sqsubseteq_E) PO

$f : D \rightarrow E$
monotone

\Rightarrow

f
continuous

proof. Take a chain $\{d_i\}_{i \in \mathbb{N}}$

$\{d_i\}_{i \in \mathbb{N}}$ is finite $\Rightarrow \exists k \in \mathbb{N}. \bigsqcup_{i \in \mathbb{N}} d_i = d_k$

\Downarrow

$\{f(d_i)\}_{i \in \mathbb{N}}$ is finite $\Rightarrow \bigsqcup_{i \in \mathbb{N}} f(d_i) = f(d_k) = f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right)$

Composition

TH. Any composition of continuous functions is continuous

$$\begin{array}{l} (D, \sqsubseteq_D) \text{ CPO} \\ (E, \sqsubseteq_E) \text{ CPO} \\ (F, \sqsubseteq_F) \text{ CPO} \end{array} \quad \begin{array}{l} f : D \rightarrow E \text{ continuous} \\ g : E \rightarrow F \text{ continuous} \end{array} \quad \Rightarrow \quad \begin{array}{l} h = g \circ f : D \rightarrow F \\ \text{continuous} \end{array}$$

proof. take a chain $\{d_i\}_{i \in \mathbb{N}}$

we need to prove

$$h \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) = \bigsqcup_{i \in \mathbb{N}} h(d_i)$$

$$\begin{aligned} h \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) &= g \left(f \left(\bigsqcup_{i \in \mathbb{N}} d_i \right) \right) = g \left(\bigsqcup_{i \in \mathbb{N}} f(d_i) \right) = \bigsqcup_{i \in \mathbb{N}} g(f(d_i)) \\ &= \bigsqcup_{i \in \mathbb{N}} h(d_i) \end{aligned}$$

Kleene's fixpoint theorem

Repeated application

$$f : D \rightarrow D$$

$$f^0(d) \triangleq d$$

$$f^{n+1}(d) \triangleq f(f^n(d))$$

$$f^n(d) = \overbrace{f(\cdots (f(d)) \cdots)}^{n \text{ times}}$$

$$f^n : D \rightarrow D$$

Lemma

(D, \sqsubseteq) PO_{\perp} $f : D \rightarrow D$ monotone $\Rightarrow \{f^n(\perp)\}_{n \in \mathbb{N}}$
is a chain

proof. we need to prove $\forall n \in \mathbb{N}. f^n(\perp) \sqsubseteq f^{n+1}(\perp)$
by mathematical induction $P(n) \triangleq f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

$$P(0) \triangleq f^0(\perp) \sqsubseteq f^1(\perp) \qquad f^0(\perp) = \perp \sqsubseteq f^1(\perp)$$

$\forall n \in \mathbb{N}. P(n) \Rightarrow P(n+1)$ take a generic n

assume $P(n) \triangleq f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

we want to prove $P(n+1) \triangleq f^{n+1}(\perp) \sqsubseteq f^{n+2}(\perp)$

$$f^n(\perp) \sqsubseteq f^{n+1}(\perp)$$

\Downarrow

$$f^{n+1}(\perp) = f(f^n(\perp)) \sqsubseteq f(f^{n+1}(\perp)) = f^{n+2}(\perp)$$

Towards Kleene's theo.

when (D, \sqsubseteq) is a CPO_\perp

then $\{f^n(\perp)\}_{n \in \mathbb{N}}$ is a chain

it must have a limit

$\{f^n(d)\}_{n \in \mathbb{N}}$
not necessarily
a chain!

Kleene's fix point theorem states that
if f is continuous, then the limit of the above chain
is the least fixpoint of f

Pre-fixpoints

(D, \sqsubseteq) PO $f : D \rightarrow D$ monotone

fixpoint $p \in D$ $f(p) = p$

pre-fixpoint $p \in D$ $f(p) \sqsubseteq p$

Clearly any fixpoint is also a pre-fixpoint

Kleene's theorem

$(D, \sqsubseteq) \text{ CPO}_\perp$ $f : D \rightarrow D$ continuous

let $\text{fix}(f) \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp)$

1. $\text{fix}(f)$ is a fix point of f

$$f(\text{fix}(f)) = \text{fix}(f)$$

2. $\text{fix}(f)$ is the least pre-fixpoint of f

$$\forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

if d is a pre-fixpoint then $\text{fix}(f)$ is smaller than d

Kleene's theorem: 1

$$1. \quad f(\text{fix}(f)) = \text{fix}(f)$$

proof.

$$\begin{aligned} f(\text{fix}(f)) &= f\left(\bigsqcup_{n \in \mathbb{N}} f^n(\perp)\right) && \text{by def of } \text{fix} \\ &= \bigsqcup_{n \in \mathbb{N}} f(f^n(\perp)) && \text{by continuity} \\ &= \bigsqcup_{n \in \mathbb{N}} f^{n+1}(\perp) && \text{by def of } f^n \\ &= \bigsqcup_{n \in \mathbb{N}} f^n(\perp) && \text{by prefix independence of limits} \\ &= \text{fix}(f) && \text{by def of } \text{fix} \end{aligned}$$

Kleene's theorem: 2

$$2. \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

proof.

we prove that any pre-fixpoint is an upper bound of the chain

$$\{f^n(\perp)\}_{n \in \mathbb{N}}$$

by definition $\text{fix}(f)$ is the lub of the same chain

and thus smaller than any other upper bound

Kleene's theorem: 2

$$2. \forall d \in D. f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d$$

take any $d \in D$ such that $f(d) \sqsubseteq d$

we prove $\forall n \in \mathbb{N}. f^n(\perp) \sqsubseteq d$ (d is an upper bound)

$P(n) \triangleq f^n(\perp) \sqsubseteq d$ by mathematical induction

$$P(0) \triangleq f^0(\perp) \sqsubseteq d \quad f^0(\perp) = \perp \sqsubseteq d$$

$$\forall n \in \mathbb{N}. P(n) \Rightarrow P(n+1)$$

take a generic n

assume $P(n) \triangleq f^n(\perp) \sqsubseteq d$

we want to prove $P(n+1) \triangleq f^{n+1}(\perp) \sqsubseteq d$

$$f^{n+1}(\perp) \stackrel{\text{(by def)}}{=} f(f^n(\perp)) \stackrel{\text{(monot.)} \downarrow}{\sqsubseteq} f(d) \stackrel{\text{(pre-fixpoint)}}{\sqsubseteq} d$$

Example

$$n = 2 \cdot n$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

CPO_⊥

$$\begin{aligned} f(n) &= 2 \cdot n \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 2 \cdot 0 = 0$$

fixpoint reached!

Example

$$n = n + 1$$

$$(\mathbb{N} \cup \{\infty\}, \leq)$$

$$\perp = 0$$

CPO $_{\perp}$

$$\begin{aligned} f(n) &= n + 1 \\ f(\infty) &= \infty \end{aligned}$$

monotone? ok

continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 0 + 1 = 1$$

$$f^2(0) = f(f^1(0)) = f(1) = 1 + 1 = 2$$

$$f^3(0) = f(f^2(0)) = f(2) = 2 + 1 = 3$$

$$\bigsqcup_{n \in \mathbb{N}} f^n(0) = \bigsqcup_{n \in \mathbb{N}} n = \infty \quad \text{fixpoint}$$

Example

$$X = X \cap \{1\}$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset$$

CPO $_{\perp}$

$$f(X) = X \cap \{1\}$$

monotone? ok
continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cap \{1\} = \emptyset$$

fixpoint reached!

Example

$$X = \mathbb{N} \setminus X$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset \quad \text{CPO}_{\perp}$$

$$f(X) = \mathbb{N} \setminus X$$

monotone? NO

the larger X the smaller $f(X)$

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \mathbb{N} \setminus \emptyset = \mathbb{N}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\mathbb{N}) = \mathbb{N} \setminus \mathbb{N} = \emptyset$$

not a chain!

Example

$$X = X \cup \{1\}$$

$$(\wp(\mathbb{N}), \subseteq)$$

$$\perp = \emptyset$$

CPO $_{\perp}$

$$f(X) = X \cup \{1\}$$

monotone? ok
continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cup \{1\} = \{1\}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\{1\}) = \{1\} \cup \{1\} = \{1\}$$

fixpoint reached!

Badge exercise



Let D be a CPO

let $\{d_i\}_{i \in \mathbb{N}}$ be a chain in D

let $\{k_j\}_{j \in \mathbb{N}}$ be an infinite chain in (\mathbb{N}, \leq)

1. Prove that $\{d_{k_j}\}_{j \in \mathbb{N}}$ is a chain in D

2. Prove or disprove that $\bigsqcup_{j \in \mathbb{N}} d_{k_j} = \bigsqcup_{i \in \mathbb{N}} d_i$