Principles for Software Composition

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13c - Continuity theorems
Lifted Domains
Lifted Domains
Lifted Domains

\[ D = (D, \sqsubseteq_D) \quad \text{CPO} \quad \Rightarrow \quad D_\perp = (D_\perp, \sqsubseteq_{D_\perp}) \]

\[ D_\perp \triangleq \{ \perp \} \uplus D \]

\[ = \{(0, \perp)\} \cup (\{1\} \times D) = \{(0, \perp)\} \cup \{(1, d) \mid d \in D\} \]

\[ \perp_{D_\perp} \triangleq (0, \perp) \]

\[ \lfloor \cdot \rfloor : D \rightarrow D_\perp \]

\[ \lfloor d \rfloor \triangleq (1, d) \]

lifting function

how to order lifted elements?

\[ \forall x \in D_\perp. \quad \perp_{D_\perp} \sqsubseteq_{D_\perp} x \]

\[ \forall d_1, d_2 \in D. \quad \lfloor d_1 \rfloor \sqsubseteq_{D_\perp} \lfloor d_2 \rfloor \Leftrightarrow d_1 \sqsubseteq_D d_2 \]
Example

$\mathbb{Z}, (=)$

\[
\begin{array}{ccccccc}
\cdots & -1 & 0 & 1 & \cdots \\
\end{array}
\]

\[
\begin{array}{cc}
\downarrow & \downarrow \\
\mathbb{Z}_\perp & \mathbb{Z}_\perp \\
\end{array}
\]
Lifted Domains

**TH.** \[ \mathcal{D}_\bot = ( D_\bot, \sqsubseteq_{D_\bot} ) \text{ CPO}_\bot \]

try on your own to prove:
PO,
bottom element,
complete

observe that:
\[
\bigsqcup_{i \in \mathbb{N}} [d_i] = \bigsqcup_{i \in \mathbb{N}} d_i
\]

it is an upper bound
it is the least upper bound
Lifting operator

\[(D, \sqsubseteq_D) \text{ CPO}\]
\[(E, \sqsubseteq_E) \text{ CPO}_\bot\]

\[(\cdot)^* : [D \to E] \to [D_\bot \to E]\]

\[\forall f \in [D \to E]. \quad f^*(x) \triangleq \begin{cases} 
\bot_E & \text{if } x = \bot_D
f(d) & \text{if } x = [d]
\end{cases}\]

for the definition to be well-given
we need to prove:

\[f \in [D \to E] \quad \Rightarrow \quad f^* \in [D_\bot \to E]\]

\[f \text{ continuous} \quad \text{implies} \quad f^* \text{ continuous}\]
the lifting operator is well-defined

\textbf{proof.} assume \( f \) continuous, take a chain \( \{x_n\}_{n \in \mathbb{N}} \) in \( D_\perp \)

we need to prove \( f^* \left( \bigsqcup_{n \in \mathbb{N}} x_n \right) = \bigsqcup_{n \in \mathbb{N}} f^*(x_n) \)

if \( \forall n \in \mathbb{N}. \ x_n = \perp_{D_\perp} \) then it is obvious

otherwise, let \( k = \min \{ i \mid x_i \neq \perp_{D_\perp} \} \)
then \( \forall m \geq k. \exists d_m \in D. \ x_m = \lfloor d_m \rfloor \)

and by prefix independence of lub \( \bigsqcup_{n \in \mathbb{N}} x_n = \bigsqcup_{n \in \mathbb{N}} x_{n+k} \)

we can just prove \( f^* \left( \bigsqcup_{n \in \mathbb{N}} x_{n+k} \right) = \bigsqcup_{n \in \mathbb{N}} f^*(x_{n+k}) \)

(see next slide)
(continue) \[ f^* \left( \bigsqcup_{n \in \mathbb{N}} x_{n+k} \right) = \bigsqcup_{n \in \mathbb{N}} f^*(x_{n+k}) \]

\[ f^* \left( \bigsqcup_{n \in \mathbb{N}} x_{n+k} \right) = f^* \left( \bigsqcup_{n \in \mathbb{N}} \lfloor d_{n+k} \rfloor \right) \]

= \[ f^* \left( \bigsqcup_{n \in \mathbb{N}} d_{n+k} \right) \]

= \[ f \left( \bigsqcup_{n \in \mathbb{N}} d_{n+k} \right) \]

by continuity of \( f \)

= \[ \bigsqcup_{n \in \mathbb{N}} f(d_{n+k}) \]

by def of lifting

= \[ \bigsqcup_{n \in \mathbb{N}} f^*(\lfloor d_{n+k} \rfloor) \]

by def of lifting

= \[ \bigsqcup_{n \in \mathbb{N}} f^*(x_{n+k}) \]

by def of \( k \)

by lub in a lifted domain

by def of lifting

by def of \( k \)
**TH.** \((\cdot)^*\) is monotone

(try to prove on your own)

**TH.** \((\cdot)^*\) is continuous

**proof.** take a chain of continuous functions \(\{f_i : D \to E\}_{i \in \mathbb{N}}\)

we need to prove \(\left( \bigsqcup_{i \in \mathbb{N}} f_i \right)^* = \bigsqcup_{i \in \mathbb{N}} f_i^*\)

take a generic \(x \in D_{\perp}\)

we need to prove \(\left( \bigsqcup_{i \in \mathbb{N}} f_i \right) (x) = \left( \bigsqcup_{i \in \mathbb{N}} f_i^* \right) (x)\)

if \(x = \bot_{D_{\perp}}\) it is obvious

if \(x = [d]\) we have… (see next slide)
(continue) \[
\left( \bigcup_{i \in \mathbb{N}} f_i \right)^* ([d]) = \left( \bigcup_{i \in \mathbb{N}} f_i^* \right) ([d])
\]

by def of lifting

\[
\left( \bigcup_{i \in \mathbb{N}} f_i \right)^* ([d]) = \left( \bigcup_{i \in \mathbb{N}} f_i \right) (d)
\]

by def of lifting

\[
= \bigcup_{i \in \mathbb{N}} f_i (d)
\]

by lub in a functional domain

\[
= \bigcup_{i \in \mathbb{N}} f_i^* ([d])
\]

by def of lifting

\[
= \left( \bigcup_{i \in \mathbb{N}} f_i^* \right) ([d])
\]

by lub in a functional domain
Let notation (de-lifting)

\((E, \sqsubseteq_E) \triangleq \text{CPO}_\perp \quad \lambda x. \ e \in [D \to E] \quad t \in D_\perp\)

\[
\text{let } x \leftarrow t. \ e \quad \triangleq \quad \frac{(\lambda x. \ e)^* \begin{array}{c} t \\ [D \to E] \end{array}}{D_\perp} \quad = \quad \begin{cases} \bot_E & \text{if } t = \bot_{D_\perp} \\ e[d/x] & \text{if } t = [d] \end{cases}
\]

intuitively:

if \( t \) is a lifted value \([d]\) then we de-lift the value and assign it to \( x \) in \( e \)

otherwise returns \( \bot_E \)
Continuity theorems
TH. \( (D, \sqsubseteq_D) \) CPO \( (E_i, \sqsubseteq_{E_i}) \) \( f : D \to E_1 \times E_2 \) \( g_i \triangleq \pi_i \circ f \)

\( f \) is continuous \( \iff \) \( g_1, g_2 \) are continuous

**proof.** \( \Rightarrow \) \( f \) is continuous \( \Rightarrow g_i \) is continuous
\( \pi_i \) is continuous

\( \Leftarrow \) note that \( \forall d \in D. \, f(d) = (g_1(d), g_2(d)) \)

assume \( g_1, g_2 \) are continuous
we want to prove \( f \) is continuous
take a chain \( \{d_i\}_{i \in \mathbb{N}} \) in \( D \)
we must prove \( f \left( \bigsqcup_{i \in \mathbb{N}} d_i \right) = \bigsqcup_{i \in \mathbb{N}} f(d_i) \)

(see next slide)
(continue) \[ f \left( \bigcup_{i \in \mathbb{N}} d_i \right) = \bigcup_{i \in \mathbb{N}} f(d_i) \]

\[
f \left( \bigcup_{i \in \mathbb{N}} d_i \right) = \left( g_1 \left( \bigcup_{i \in \mathbb{N}} d_i \right), g_2 \left( \bigcup_{i \in \mathbb{N}} d_i \right) \right) \quad \text{by def } g_1, g_2
\]

\[
= \left( \bigcup_{i \in \mathbb{N}} g_1(d_i), \bigcup_{i \in \mathbb{N}} g_2(d_i) \right) \quad g_1, g_2 \text{ are continuous}
\]

\[
= \bigcup_{i \in \mathbb{N}} (g_1(d_i), g_2(d_i)) \quad \text{by def of lub of pairs}
\]

\[
= \bigcup_{i \in \mathbb{N}} f(d_i) \quad \text{by def } g_1, g_2
\]
\[(D, \sqsubseteq_D) \quad (E, \sqsubseteq_E) \quad (F, \sqsubseteq_F)\]

**TH.** \(
\text{CPO} \quad f : D \times E \to F
\)

\(f\) is continuous \iff \(\forall d \in D. \ f_d\) are continuous \(\forall e \in E. \ f_e\) are continuous

**proof. \(\Rightarrow\)** assume \(f\) is continuous

- take a generic \(d \in D\)
- we want to prove \(f_d\) is continuous
- take a chain \(\{e_i\}_{i \in \mathbb{N}}\) in \(E\)

we prove \(f_d \left( \bigsqcup_{i \in \mathbb{N}} e_i \right) = \bigsqcup_{i \in \mathbb{N}} f_d(e_i)\) (see next slide)
(continue) \[ f_d \left( \bigcup_{i \in \mathbb{N}} e_i \right) = \bigcup_{i \in \mathbb{N}} f_d(e_i) \]

\[ f_d \left( \bigcup_{i \in \mathbb{N}} e_i \right) = f \left( d, \bigcup_{i \in \mathbb{N}} e_i \right) \quad \text{by def of } f_d \]

\[ = f \left( \bigcup_{i \in \mathbb{N}} d, \bigcup_{i \in \mathbb{N}} e_i \right) \quad \text{by lub of constant chain} \]

\[ = f \left( \bigcup_{i \in \mathbb{N}} (d, e_i) \right) \quad \text{by lub of pairs} \]

\[ = \bigcup_{i \in \mathbb{N}} f(d, e_i) \quad \text{by continuity of } f \]

\[ = \bigcup_{i \in \mathbb{N}} f_d(e_i) \quad \text{by def of } f_d \]
TH. \((D, \sqsubseteq_D)\) \((E, \sqsubseteq_E)\) \((F, \sqsubseteq_F)\) CPO \(f : D \times E \to F\)

\[\begin{align*}
  f_d & : E \to F \\
  f_d & \triangleq \lambda e. \ f(d, e) \\
  f_e & : D \to F \\
  f_e & \triangleq \lambda d. \ f(d, e)
\end{align*}\]

\(f\) is continuous \iff \(\forall d \in D. \ f_d\) are continuous \(\forall e \in E. \ f_e\) are continuous

\(\Leftarrow\) assume \(f_d, f_e\) are continuous for all \(d, e\)
we want to prove \(f\) is continuous
take a chain \(\{(d_k, e_k)\}_{k \in \mathbb{N}}\) in \(D \times E\)
we prove \(f \left( \bigsqcup_{k \in \mathbb{N}} (d_k, e_k) \right) = \bigsqcup_{k \in \mathbb{N}} f(d_k, e_k)\)

(see next slide)
\[
(\text{continue}) \quad f \left( \bigsqcup_{k \in \mathbb{N}} (d_k, e_k) \right) = \bigsqcup_{k \in \mathbb{N}} f(d_k, e_k)
\]

\[
f(\bigsqcup_k (d_k, e_k)) = f(\bigsqcup_i d_i, \bigsqcup_j e_j) \quad \text{by def of lub of pairs}
\]

\[
= f_d(\bigsqcup_j e_j) \quad \text{by def of } f_d \text{ with } d \triangleq \bigsqcup_i d_i
\]

\[
= \bigsqcup_j f_d(e_j) \quad \text{by continuity of } f_d
\]

\[
= \bigsqcup_j f(d, e_j) \quad \text{by def of } f_d
\]

\[
= \bigsqcup_j f_{e_j}(d) \quad \text{by def of } f_{e_j}
\]

\[
= \bigsqcup_j f_{e_j}(\bigsqcup_i d_i) \quad \text{by def of } d \triangleq \bigsqcup_i d_i
\]

\[
= \bigsqcup_j \bigsqcup_i f_{e_j}(d_i) \quad \text{by continuity of } f_{e_j}
\]

\[
= \bigsqcup_j \bigsqcup_i f(d_i, e_j) \quad \text{by def of } f_{e_j}
\]

\[
= \bigsqcup_k f(d_k, e_k) \quad \text{by switch lemma (applicable?)}
\]
(continue) \[ f \left( \bigsqcup_{k \in \mathbb{N}} (d_k, e_k) \right) = \bigsqcup_{k \in \mathbb{N}} f(d_k, e_k) \]

if \( i \leq n \land j \leq m \) then \( f(d_i, e_j) \sqsubseteq f(d_n, e_m) \) ? \( \checkmark \)

\[ \Downarrow \]

\[ d_i \sqsubseteq_D d_n \land e_j \sqsubseteq_E e_m \]

\[ f(d_i, e_j) = f_d(e_j) \sqsubseteq f_d(e_m) = f(d_i, e_m) = f_{e_m}(d_i) \sqsubseteq f_{e_m}(d_n) = f(d_n, e_m) \]

\[ f_{d_i} \quad \text{monotone} \quad f_{e_m} \quad \text{monotone} \]

\[ = \bigsqcup_j \bigsqcup_i f(d_i, e_j) \]

\[ = \bigsqcup_k f(d_k, e_k) \quad \text{by switch lemma (applicable?)} \]
Some important functions
Apply

\((D, \sqsubseteq_D)\) CPO
\((E, \sqsubseteq_E)\)

apply \( : [D \to E] \times D \to E \)

apply\((f, d) \triangleq f(d)\)

**TH.** *apply* is monotone

(ttry to prove on your own)

**TH.** *apply* is continuous

*proof.* from a previous theorem, we prove continuity on each parameter separately *apply*\(_{f}\) *apply*\(_{d}\)

1. for any \(f \in [D \to E]\) \(apply_f \triangleq \lambda d. f(d)\) is continuous

2. for any \(d \in D\) \(apply_d \triangleq \lambda f. f(d)\) is continuous

(see next slides)
1. for any \( f \in [D \rightarrow E] \) \( \text{apply}_f \triangleq \lambda d. \ f(d) \) is continuous

take \( f \in [D \rightarrow E] \) and a chain \( \{d_i\}_{i \in \mathbb{N}} \) in \( D \)

we want to prove \( \text{apply}_f \left( \bigsqcup_i d_i \right) = \bigsqcup_i \text{apply}_f(d_i) \)

\[
\text{apply}_f(\bigsqcup_i d_i) = \text{apply}(f, \bigsqcup_i d_i) \quad \text{by def of } \text{apply}_f
\]

\[
= f(\bigsqcup_i d_i) \quad \text{by def of } \text{apply}
\]

\[
= \bigsqcup_i f(d_i) \quad \text{by continuity of } f
\]

\[
= \bigsqcup_i \text{apply}(f, d_i) \quad \text{by def of } \text{apply}
\]

\[
= \bigsqcup_i \text{apply}_f(d_i) \quad \text{by def of } \text{apply}_f
\]
2. for any \( d \in D \) \( \text{apply}_d \triangleq \lambda f. f(d) \) is continuous

Take \( d \in D \) and a chain \( \{ f_i \}_{i \in \mathbb{N}} \) in \([D \to E]\)

We want to prove \( \text{apply}_d \left( \bigsqcup_i f_i \right) = \bigsqcup_i \text{apply}_d(f_i) \)

\[
\text{apply}_d(\bigsqcup_i f_i) = \text{apply}(\bigsqcup_i f_i, d) \quad \text{by def of } \text{apply}_d
\]

\[
= (\bigsqcup_i f_i)(d) \quad \text{by def of } \text{apply}
\]

\[
= \bigsqcup_i f_i(d) \quad \text{by def of lub of functions}
\]

\[
= \bigsqcup_i \text{apply}(f_i, d) \quad \text{by def of } \text{apply}
\]

\[
= \bigsqcup_i \text{apply}_d(f_i) \quad \text{by def of } \text{apply}_d
\]
Apply: recap

\[(D, \sqsubseteq_D) \quad \text{CPO} \quad (E, \sqsubseteq_E)\]

\[\text{apply} : [D \to E] \times D \to E\]

\[\text{apply}(f, d) \triangleq f(d)\]

\[\text{apply} \in [[D \to E] \times D \to E]\]
**Fix**

$(D, \sqsubseteq_D) \text{ CPO}_\bot$

$\text{fix} : [D \to D] \to D$

$\text{fix} \triangleq \lambda f. \bigsqcup_{n \in \mathbb{N}} f^n(\bot_D)$

**TH.** $\text{fix}$ is monotone

(try to prove on your own)

**TH.** $\text{fix}$ is continuous

**proof.** $\text{fix} \triangleq \lambda f. \bigsqcup_{n \in \mathbb{N}} f^n(\bot_D) = \bigsqcup_{n \in \mathbb{N}} \lambda f. f^n(\bot_D)$

by def of lub in functional domains

we prove that $\forall n. \lambda f. f^n(\bot_D)$ is continuous

(by mathematical induction on $n$)

then $\text{fix}$ is continuous because lub of continuous functions

(see next slides)
\( \forall n. \, \lambda f. \, f^n(\bot_D) \)

**base case:** \( \lambda f. \, f^0(\bot_D) = \lambda f. \, \bot_D \)

is a constant function (continuous)

**inductive case:** assume \( g \triangleq \lambda f. \, f^n(\bot_D) \) is continuous

we want to prove \( h \triangleq \lambda f. \, f^{n+1}(\bot_D) \) is continuous

take a chain \( \{f_i\}_{i \in \mathbb{N}} \) in \( [D \to D] \)

we want to prove \( h \left( \bigsqcup_{i \in \mathbb{N}} f_i \right) = \bigsqcup_{i \in \mathbb{N}} h(f_i) \)

(see next slide)
\[(\forall n. \lambda f. \, f^n(\bot_D)) \quad g \triangleq \lambda f. \, f^n(\bot_D) \quad h \triangleq \lambda f. \, f^{n+1}(\bot_D) \quad h\left(\bigcup_{i \in \mathbb{N}} f_i\right) = \bigcup_{i \in \mathbb{N}} h(f_i)\]

\[
h(\bigcup_i f_i) = (\bigcup_i f_i)^{n+1}(\bot_D) \quad \text{by def of } h
\]

\[
= (\bigcup_j f_j)((\bigcup_i f_i)^n(\bot_D)) \quad \text{by def of } (\cdot)^{n+1}
\]

\[
= (\bigcup_j f_j)(g(\bigcup_i f_i)) \quad \text{by def of } g
\]

\[
= (\bigcup_j f_j)(\bigcup_i g(f_i)) \quad \text{by ind. hyp ( } g \text{ continuous)}
\]

\[
= (\bigcup_j f_j)(\bigcup_i f_i^n(\bot_D)) \quad \text{by def of } g
\]

\[
= \bigcup_j \bigcup_i f_j(f_i^n(\bot_D)) \quad \text{by def of lub in functional CPO}
\]

\[
= \bigcup_j \bigcup_i f_j(f_i^n(\bot_D)) \quad \text{by continuity of } f_j
\]

\[
= \bigcup_k f_k(f_k^n(\bot_D)) \quad \text{by switch lemma}
\]

\[
= \bigcup_k f_k^{n+1}(\bot_D) \quad \text{by def of } (\cdot)^{n+1}
\]

\[
= \bigcup_k h(f_k) \quad \text{by def of } h
\]
Fix: recap

\((D, \sqsubseteq_D) \text{ CPO}_\bot\)

\[
\begin{align*}
\text{fix} : [D \to D] & \to D \\
\text{fix} & \triangleq \lambda f. \bigsqcup_{n \in \mathbb{N}} f^n(\bot_D)
\end{align*}
\]

\[
\text{fix} \in \left[ [D \to D] \to D \right]
\]
Curry

\[ (D, \sqsubseteq_D) \]
\[ (E, \sqsubseteq_E) \quad \text{CPO} \]
\[ (F, \sqsubseteq_F) \]

\[ \text{curry} : (D \times E \to F) \to D \to E \to F \]
\[ \text{curry } f \ d \ e \triangleq f(d, e) \]

**TH.** \( f \) continuous \( \Rightarrow \) \( \text{curry}(f) \) continuous

(try to prove on your own)
Uncurry

\[(D, \sqsubseteq_D)\]
\[(E, \sqsubseteq_E)\] CPO
\[(F, \sqsubseteq_F)\]

\[\text{uncurry} : (D \to E \to F) \to (D \times E) \to F\]
\[\text{uncurry } f \ (d, e) \triangleq f \ d \ e\]

**TH.** \(f\) continuous \(\Rightarrow\) \(\text{uncurry}(f)\) continuous

(try to prove on your own)

**TH.** \(\text{uncurry}\) is the inverse of \(\text{curry}\)

(try to prove on your own)
Disjoint Union

\[ \mathcal{D} = (D, \sqsubseteq_D) \]
\[ \mathcal{E} = (E, \sqsubseteq_E) \quad \text{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} + \mathcal{E} = (D \uplus E, \sqsubseteq_{D \uplus E}) \]

\[ D \uplus E \triangleq \{(0, d) \mid d \in D\} \cup \{(1, e) \mid e \in E\} \]

how to order elements?

is there a bottom element?

is it a complete order?

how to define (continuous) injections?

\[ \iota_D : D \to D \uplus E \]
\[ \iota_E : E \to D \uplus E \]