13a - Cartesian domains
A metaphor
Hidden star

Can you spot a regular 5-point star shape inside the picture?

some find it immediately: for them it is impossible that others cannot see it

some spend many efforts in finding it: for them the success is rewarding

some just wait for someone to help

in all cases: once shown, the star will never be hidden again

the metaphor: star = mathematical way to problem solving
The creator
Samuel Loyd (1841-1911)
chess player, puzzle maker, and recreational mathematician

maybe (?) the inventor of the famous 15 puzzle

inventor of “the leaning tower of Pisa” puzzle
HOFL
Towards a denotational semantics
three syntactic categories (types) \( A_{exp}, B_{exp}, Com \)

one interpretation function each

\[ A[\cdot] : A_{exp} \rightarrow \Sigma \rightarrow \mathbb{Z} \]
\[ B[\cdot] : B_{exp} \rightarrow \Sigma \rightarrow \mathbb{B} \]
\[ C[\cdot] : Com \rightarrow \Sigma \rightarrow \Sigma_{\perp} \]

\[ \Sigma \triangleq Ide \rightarrow \mathbb{Z} \]

environment

semantic values

meaning of identifiers

one semantic domain each
one syntactic category for pre-terms  \( T \)

infinitely many types  \( \tau ::= \text{int} \mid \tau_0 \times \tau_1 \mid \tau_0 \rightarrow \tau_1 \)

infinitely many categories for typeable terms  \( T_\tau \)

one semantic domain each  \( D_\tau \)

one parametric interpretation function  \( \llbracket \cdot \rrbracket \)

variables also have different types  \( x : \tau \)

the environment must be type-sensitive  \( \rho \)
Requirements

\[ t : \tau \quad [t] \rho \in D_\tau \]

a domain for each type!

environment \[ \rho : \text{Var} \to \bigcup_{\tau \in \mathcal{T}} D_\tau \]

type consistent assignment of values to variables

\[ x : \tau \Rightarrow \rho(x) \in D_\tau \]

\[ t \text{ may diverge (e.g. } \texttt{rec x. x} \) \Rightarrow D_\tau \]

must include a bottom element \( \bot_{D_\tau} \)
Requirements

\( t : \tau \quad \llbracket t \rrbracket \rho \in D_\tau \)

/ environment
\( \rho : \text{Var} \rightarrow \bigcup_{\tau \in \mathcal{T}} D_\tau \)

\( x : \tau \Rightarrow \rho(x) \in D_\tau \)

type consistent assignment of values to variables

\[ \llbracket \text{rec } x. \ t \rrbracket \rho = \llbracket t \rrbracket \rho \left[ \llbracket \text{rec } x. \ t \rrbracket \rho / x \right] \]

\[ \llbracket \text{rec } x. \ t \rrbracket \rho = \Gamma_{x,t} \left( \llbracket \text{rec } x. \ t \rrbracket \rho \right) \]

\[ \llbracket \text{rec } x. \ t \rrbracket = \text{fix } \Gamma_{x,t} \]

\( \Gamma_{x,t} \triangleq \lambda d. \ \llbracket t \rrbracket \rho^{[d/x]} \)

to solve recursive equations:

\( D_\tau \) must be a CPO \( \bot \)

\( \Gamma_{x,t} \) must be continuous
Requirements

\[ t : \tau \quad \exists [t] \rho \in D_\tau \]

- a domain for each type!

\[ \rho : \text{Var} \rightarrow \bigcup_{\tau \in \mathcal{T}} D_\tau \]

- environment

- type consistent assignment of values to variables

\[ x : \tau \Rightarrow \rho(x) \in D_\tau \]

- we must be able to combine \( \text{CPO}_\bot \)

- using cartesian product and function spaces

\[ \tau ::= \text{int} \mid \tau_0 \ast \tau_1 \mid \tau_0 \rightarrow \tau_1 \]
Requirements

\[ t : \tau \quad [t] \rho \in D_{\tau} \]

/ environment \[ \rho : \text{Var} \rightarrow \bigcup_{\tau \in \mathcal{T}} D_{\tau} \]

type consistent \[ x : \tau \Rightarrow \rho(x) \in D_{\tau} \]
assignment of values to variables

\[ \tau ::= \text{int} \mid \tau_0 \times \tau_1 \mid \tau_0 \rightarrow \tau_1 \]

choose \[ D_{\text{int}} \]

given \[ D_{\tau_0}, D_{\tau_1} \] build \[ D_{\tau_0 \times \tau_1} \quad D_{\tau_0 \rightarrow \tau_1} \]
Flat domain of Integers
Flat domain of Integers

\[ \mathbb{Z} \quad \ldots \quad -1 \quad 0 \quad 1 \quad \ldots \]
Flat domain of Integers

\[ \mathbb{Z}_\perp \quad \ldots \quad -1 \quad 0 \quad 1 \quad \ldots \]

PO: flat order
bottom: any flat order has bottom
completeness: any flat order is complete
   (only finite chains are possible)
Strict extensions

\(\text{op} \in \{+, -, \times\}\)
\(\text{op} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}\)
\(\text{op}_\bot : \mathbb{Z}_\bot \times \mathbb{Z}_\bot \to \mathbb{Z}_\bot\)

\[\begin{align*}
v_1 \quad \text{op}_\bot \quad v_2 & \triangleq \begin{cases} 
v_1 \quad \text{op} \quad v_2 & \text{if } v_1, v_2 \in \mathbb{Z} \\
\bot_{\mathbb{Z}_\bot} & \text{otherwise } (v_1 = \bot_{\mathbb{Z}_\bot} \text{ or } v_2 = \bot_{\mathbb{Z}_\bot})\end{cases}
\end{align*}\]

called strict extension

to prove: \(\text{op}_\bot\) is monotone and continuous

is \(\mathbb{Z}_\bot \times \mathbb{Z}_\bot\) a \(\text{CPO}_\bot\)?
Cartesian product of domains
Cartesian product

\[ \mathcal{D} = (D, \sqsubseteq_D) \]
\[ \mathcal{E} = (E, \sqsubseteq_E) \quad \text{CPO}_\perp \implies \mathcal{D} \times \mathcal{E} = (D \times E, \sqsubseteq_{D \times E}) \]

how to order pairs?

\[(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1) \quad \text{iff} \quad d_0 \sqsubseteq_D d_1 \land e_0 \sqsubseteq_E e_1\]

example \( \mathbb{Z}_\perp \times \mathbb{Z}_\perp \)

\[
\begin{align*}
(0, 1) \quad \text{?} & \quad \sqsubseteq_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (1, 2) \\
(\bot_{\mathbb{Z}_\perp}, 1) \quad \text{?} & \quad \sqsubseteq_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (1, 1) \\
(2, \bot_{\mathbb{Z}_\perp}) \quad \text{?} & \quad \sqsubseteq_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (2, 0) \\
(0, \bot_{\mathbb{Z}_\perp}) \quad \text{?} & \quad \sqsubseteq_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (\bot_{\mathbb{Z}_\perp}, 0)
\end{align*}
\]
Cartesian CPO

\[ \mathcal{D} \times \mathcal{E} = (D \times E, \sqsubseteq_{D \times E}) \]

is it a partial order?

reflexivity, antisymmetry, transitivity of \( \sqsubseteq_{D \times E} \)
follow immediately from those of \( \sqsubseteq_D, \sqsubseteq_E \)

is there a bottom element?

let \( \bot_{D \times E} = (\bot_D, \bot_E) \)

take any pair \( (d, e) \in D \times E \)

since \( \bot_D \sqsubseteq_D d \)
\( \bot_E \sqsubseteq_E e \)

then \( \bot_{D \times E} = (\bot_D, \bot_E) \sqsubseteq_{D \times E} (d, e) \)
Cartesian CPO (ctd)

\[ D \times E = ( D \times E , \sqsubseteq_{D \times E} ) \]

is it complete?

take a chain \( \{ (d_i, e_i) \}_{i \in \mathbb{N}} \) we need to find its lub

we prove its lub is \( \left( \bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right) \)

1. it is an upper bound of the chain
2. it is smaller than or equal to any other upper bound
Cartesian CPO (ctd)

\[ D \times E = ( D \times E, \sqsubseteq_{D \times E} ) \]

take a chain \( \{ (d_i, e_i) \}_{i \in \mathbb{N}} \)

1. \( \left( \bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right) \) is an upper bound of the chain

take a generic element of the chain \((d_j, e_j)\)

\[ d_j \sqsubseteq_D \bigsqcup_{i \in \mathbb{N}} d_i \]

we have

\[ e_j \sqsubseteq_E \bigsqcup_{i \in \mathbb{N}} e_i \]

thus \((d_j, e_j) \sqsubseteq_{D \times E} \left( \bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)\)
Cartesian CPO (ctd)

\[ D \times E = (D \times E, \sqsubseteq_{D \times E}) \]

take a chain \( \{(d_i, e_i)\}_{i \in \mathbb{N}} \)

2. \( \left( \bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right) \) is the least among upper bounds

take a generic upper bound \( (d, e) : \forall i \in \mathbb{N}. (d_i, e_i) \sqsubseteq_{D \times E} (d, e) \)

by def \( \forall i \in \mathbb{N}. d_i \sqsubseteq_D d \land \forall i \in \mathbb{N}. e_i \sqsubseteq_E e \)

i.e., \( d \) is an upper bound of \( \{d_i\}_{i \in \mathbb{N}} \) \( \Rightarrow \) \( \bigsqcup_{i \in \mathbb{N}} d_i \sqsubseteq_D d \)

\( e \) is an upper bound of \( \{e_i\}_{i \in \mathbb{N}} \) \( \Rightarrow \) \( \bigsqcup_{i \in \mathbb{N}} e_i \sqsubseteq_E e \)

hence \( \left( \bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right) \sqsubseteq_{D \times E} (d, e) \)
Cartesian CPO: recap

\[ D \times E = (D \times E, \sqsubseteq_{D \times E}) \]

\[(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1) \text{ iff } d_0 \sqsubseteq_D d_1 \land e_0 \sqsubseteq_E e_1\]

\[\bot_{D \times E} \triangleq (\bot_D, \bot_E)\]

\[\bigsqcup_{i \in \mathbb{N}} (d_i, e_i) \triangleq \left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i\right)\]

is \(\mathbb{Z}_\perp \times \mathbb{Z}_\perp\) a CPO\(\perp\)? \(\checkmark\)
Projections

\[ \pi_1 : D \times E \to D \quad \pi_2 : D \times E \to E \]

\[ \pi_1(d, e) = d \quad \pi_2(d, e) = e \]

**TH.** projections are monotone

**proof.** take \((d_0, e_0) \subseteq_D \times_E (d_1, e_1)\)

we want to prove \(\pi_1(d_0, e_0) \subseteq_D \pi_1(d_1, e_1)\)

\(\pi_2(d_0, e_0) \subseteq_E \pi_2(d_1, e_1)\)

\[ \pi_1(d_0, e_0) = d_0 \subseteq_D d_1 = \pi_1(d_1, e_1) \]

\[ (d_0, e_0) \subseteq_D \times_E (d_1, e_1) \]

the case of \(\pi_2\) is analogous
Projections (ctd)

\[ \pi_1 : D \times E \to D \]
\[ \pi_1(d, e) = d \]
\[ \pi_2 : D \times E \to E \]
\[ \pi_2(d, e) = e \]

**TH.** projections are continuous

**proof.** take \( \{(d_i, e_i)\}_{i \in \mathbb{N}} \)

we want to prove

\[ \pi_1 \left( \bigsqcup_{i \in \mathbb{N}} (d_i, e_i) \right) = \bigsqcup_{i \in \mathbb{N}} \pi_1(d_i, e_i) \]

\[ \pi_1 \left( \bigsqcup_{i \in \mathbb{N}} (d_i, e_i) \right) = \pi_1 \left( \bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right) = \bigsqcup_{i \in \mathbb{N}} d_i = \bigsqcup_{i \in \mathbb{N}} \pi_1(d_i, e_i) \]

by def of lub
by def of \( \pi_1 \)
by def of \( \pi_1 \)

the case of \( \pi_2 \) is analogous
Exercise: smashed prod

\(\mathcal{D} = (D, \sqsubseteq_D)\)  
\(\mathcal{E} = (E, \sqsubseteq_E)\)  
\(\text{CPO}_\perp \Rightarrow \mathcal{D} \otimes \mathcal{E} = (D \otimes E, \sqsubseteq_{D \otimes E})\)

\(D \otimes E \triangleq \{(d, e) \mid (d, e) \in D \times E, d = \perp_D \iff e = \perp_E\}\)

how to order pairs?  
bottom element?  
complete order?