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PSC 2022/23 (375AA, 9CFU)

Principles for Software Composition

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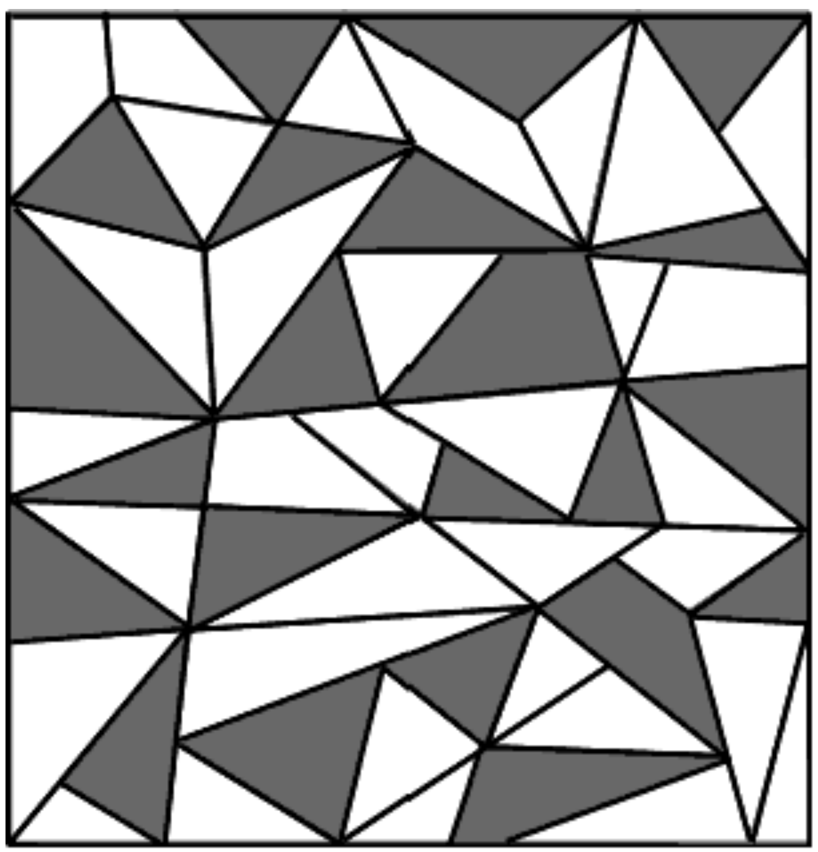
13a - Cartesian domains

A metaphor

Hidden star

Can you spot a regular 5-point star shape inside the picture?



some find it immediately: for them it is impossible that others cannot see it

some spend many efforts in finding it: for them the success is rewarding

some just wait for someone to help

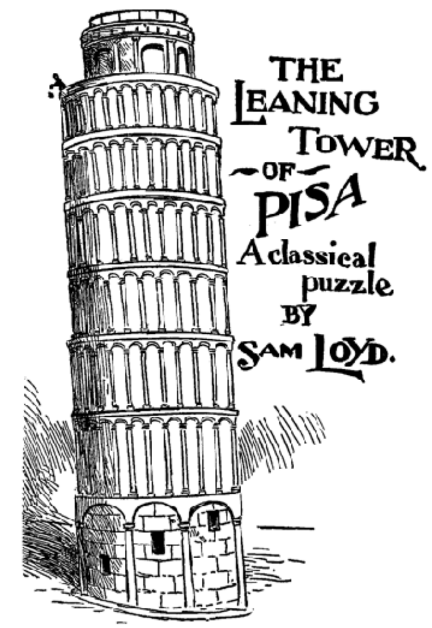
in all cases: once shown, the star will never be hidden again

the metaphor: star = mathematical way to problem solving

The creator

Samuel Loyd (1841-1911)

chess player, puzzle maker, and recreational mathematician



How far does the ball travel?

IF AN elastic ball is dropped from the Leaning Tower of Pisa at a height of 179 feet from the ground, and on each rebound the ball rises exactly one tenth of its previous height, what distance will it travel before it comes to rest?

maybe (?) the inventor of the famous 15 puzzle

inventor of "*the leaning tower of Pisa*" puzzle

HOFL
Towards a denotational semantics

Imp

three syntactic categories (types) $Aexp, Bexp, Com$

one interpretation function each

$$\mathcal{A}[\![\cdot]\!] : Aexp \to \Sigma \to \mathbb{Z}$$

$$\mathcal{B}[\![\cdot]\!] : Bexp \to \Sigma \to \mathbb{B}$$

$$\mathcal{C}[\![\cdot]\!] : Com \to \Sigma \to \Sigma_\perp$$

$$\Sigma \triangleq Ide \to \mathbb{Z}$$

environment

semantic values

meaning of identifiers

syntax

one semantic domain each

HOFL

one syntactic category for pre-terms T

infinitely many types $\tau \;::=\; int \;\mid\; \tau_0 * \tau_1 \;\mid\; \tau_0 \rightarrow \tau_1$

infinitely many categories for typeable terms T_τ

one semantic domain each D_τ

one parametric interpretation function $[\![\cdot]\!]$

variables also have different types $x : \tau$

the environment must be type-sensitive ρ

Requirements

a domain for each type!

$t : \tau \qquad [\![t]\!]\rho \in D_\tau$

environment $\rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $x : \tau \Rightarrow \rho(x) \in D_\tau$

t may diverge (e.g. $\mathbf{rec}\ x.\ x$) $\Rightarrow D_\tau$ must include a bottom element $\bot_{D_\tau}$

Requirements

a domain for each type!

$t : \tau \qquad [\![t]\!]\rho \in D_\tau$

environment $\rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $x : \tau \Rightarrow \rho(x) \in D_\tau$

$$[\![\mathbf{rec}\ x.\ t]\!]\rho = [\![t]\!]\rho[{}^{[\![\mathbf{rec}\ x.\ t]\!]\rho}/_{x}]$$

$$\Gamma_{x,t} \triangleq \lambda d.\ [\![t]\!]\rho[{}^{d}/_{x}]$$

$$[\![\mathbf{rec}\ x.\ t]\!]\rho = \Gamma_{x,t}\ ([\![\mathbf{rec}\ x.\ t]\!]\rho)$$

$$[\![\mathbf{rec}\ x.\ t]\!] = \mathit{fix}\ \Gamma_{x,t}$$

to solve recursive equations:

D_τ must be a $\mathrm{CPO}_\perp$

$\Gamma_{x,t}$ must be continuous

Requirements

a domain for each type!

$t : \tau$ $\quad\quad$ $[\![t]\!]\rho \in D_\tau$

environment $\rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $\quad x : \tau \Rightarrow \rho(x) \in D_\tau$

$$\tau \ ::= \ int \ \mid \ \tau_0 * \tau_1 \ \mid \ \tau_0 \to \tau_1$$

we must be able to combine $\mathrm{CPO}_\perp$

using cartesian product and function spaces

Requirements

a domain for each type!

$t : \tau$ $\qquad [\![t]\!]\rho \in D_\tau$

environment $\quad \rho : Var \to \bigcup_{\tau \in \mathcal{T}} D_\tau$

type consistent assignment of values to variables $\quad x : \tau \Rightarrow \rho(x) \in D_\tau$

$$\tau \;::=\; int \;\mid\; \tau_0 * \tau_1 \;\mid\; \tau_0 \to \tau_1$$

choose D_{int}

given D_{τ_0}, D_{τ_1} build $D_{\tau_0 * \tau_1}$ $\quad D_{\tau_0 \to \tau_1}$

Flat domain of Integers

Flat domain of Integers

$\mathbb{Z}$ … -1 0 1 …

Flat domain of Integers

$$\mathbb{Z}_\perp \qquad \begin{tikzcd} \cdots & -1 & 0 & 1 & \cdots \\ & & \perp_{\mathbb{Z}_\perp} \arrow[ul] \arrow[u] \arrow[ur] & & \end{tikzcd}$$

PO: flat order

bottom: any flat order has bottom

completeness: any flat order is complete
(only finite chains are possible)

Strict extensions

$\mathsf{op} \in \{+, - \times\}$

$\mathsf{op} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$\underline{\mathsf{op}}_{\perp} : \mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp} \to \mathbb{Z}_{\perp}$

$1 \underline{+}_{\perp} \perp = \perp$

$\perp \underline{\times}_{\perp} 5 = \perp$

$\perp \underline{-}_{\perp} \perp = \perp$

$$v_1 \ \underline{\mathsf{op}}_{\perp} \ v_2 \triangleq \begin{cases} v_1 \ \underline{\mathsf{op}} \ v_2 & \text{if } v_1, v_2 \in \mathbb{Z} \\ \perp_{\mathbb{Z}_{\perp}} & \text{otherwise } (v_1 = \perp_{\mathbb{Z}_{\perp}} \text{ or } v_2 = \perp_{\mathbb{Z}_{\perp}}) \end{cases}$$

called *strict* extension

to prove: $\underline{\mathsf{op}}_{\perp}$ is monotone and continuous

is $\mathbb{Z}_{\perp} \times \mathbb{Z}_{\perp}$ a $\mathrm{CPO}_{\perp}$?

Cartesian product of domains

Cartesian product

$$\mathcal{D} = (D, \sqsubseteq_D) \qquad \mathcal{E} = (E, \sqsubseteq_E) \qquad \text{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

how to order pairs?

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1) \quad \text{iff} \quad d_0 \sqsubseteq_D d_1 \wedge e_0 \sqsubseteq_E e_1$$

example $\mathbb{Z}_\perp \times \mathbb{Z}_\perp$

$$(0, 1) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (1, 2)$$ ✗

$$(\perp_{\mathbb{Z}_\perp}, 1) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (1, 1)$$ ✓

$$(2, \perp_{\mathbb{Z}_\perp}) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (2, 0)$$ ✓

$$(0, \perp_{\mathbb{Z}_\perp}) \overset{?}{\sqsubseteq}_{\mathbb{Z}_\perp \times \mathbb{Z}_\perp} (\perp_{\mathbb{Z}_\perp}, 0)$$ ✗

Cartesian CPO

$$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

is it a partial order?

reflexivity, antisymmetry, transitivity of $\sqsubseteq_{D \times E}$
follow immediately from those of $\sqsubseteq_D \quad \sqsubseteq_E$

is there a bottom element?

let $\perp_{D \times E} = (\perp_D, \perp_E)$

take any pair $(d, e) \in D \times E$

since $\begin{array}{l} \perp_D \sqsubseteq_D d \\ \perp_E \sqsubseteq_E e \end{array}$ then $\perp_{D \times E} = (\perp_D, \perp_E) \sqsubseteq_{D \times E} (d, e)$

Cartesian CPO (ctd)

$$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

is it complete?

take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$ we need to find its lub

we prove its lub is $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$

1. it is an upper bound of the chain
2. it is smaller than or equal to any other upper bound

Cartesian CPO (ctd)

$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$ take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$

1. $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$ is an upper bound of the chain

take a generic element of the chain (d_j, e_j)

we have
$$d_j \sqsubseteq_D \bigsqcup_{i \in \mathbb{N}} d_i$$
$$e_j \sqsubseteq_E \bigsqcup_{i \in \mathbb{N}} e_i$$
thus $(d_j, e_j) \sqsubseteq_{D \times E} \left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$

Cartesian CPO (ctd)

$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$ take a chain $\{(d_i, e_i)\}_{i \in \mathbb{N}}$

2. $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$ is the least among upper bounds

take a generic upper bound (d, e): $\forall i \in \mathbb{N}.\ (d_i, e_i) \sqsubseteq_{D \times E} (d, e)$

by def $\forall i \in \mathbb{N}.\ d_i \sqsubseteq_D d \ \wedge\ \forall i \in \mathbb{N}.\ e_i \sqsubseteq_E e$

i.e., d is an upper bound of $\{d_i\}_{i \in \mathbb{N}}$

e is an upper bound of $\{e_i\}_{i \in \mathbb{N}}$

$\Rightarrow$ $\bigsqcup_{i \in \mathbb{N}} d_i \sqsubseteq_D d$, $\bigsqcup_{i \in \mathbb{N}} e_i \sqsubseteq_E e$

hence $\left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right) \sqsubseteq_{D \times E} (d, e)$

Cartesian CPO: recap

$$\mathcal{D} \times \mathcal{E} = (\ D \times E\ ,\ \sqsubseteq_{D \times E}\ )$$

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1) \quad \text{iff} \quad d_0 \sqsubseteq_D d_1 \wedge e_0 \sqsubseteq_E e_1$$

$$\bot_{D \times E} \triangleq (\bot_D, \bot_E)$$

$$\bigsqcup_{i \in \mathbb{N}} (d_i, e_i) \triangleq \left(\bigsqcup_{i \in \mathbb{N}} d_i, \bigsqcup_{i \in \mathbb{N}} e_i \right)$$

is $\mathbb{Z}_\bot \times \mathbb{Z}_\bot$ a $\mathrm{CPO}_\bot$? ✓

Projections

$$\pi_1 : D \times E \to D \qquad \pi_2 : D \times E \to E$$

$$\pi_1(d, e) = d \qquad \pi_2(d, e) = e$$

TH. projections are monotone

proof. take $(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1)$

we want to prove $\pi_1(d_0, e_0) \sqsubseteq_D \pi_1(d_1, e_1)$

$\pi_2(d_0, e_0) \sqsubseteq_E \pi_2(d_1, e_1)$

$$\pi_1(d_0, e_0) = d_0 \sqsubseteq_D d_1 = \pi_1(d_1, e_1)$$

$$\Uparrow$$

$$(d_0, e_0) \sqsubseteq_{D \times E} (d_1, e_1)$$

the case of π_2 is analogous

Projections (ctd)

$\pi_1 : D \times E \to D$ $\qquad$ $\pi_2 : D \times E \to E$

$\pi_1(d,e) = d$ $\qquad$ $\pi_2(d,e) = e$

TH. projections are continuous

proof. take $\{(d_i, e_i)\}_{i\in\mathbb{N}}$

we want to prove $\pi_1\left(\bigsqcup_{i\in\mathbb{N}}(d_i, e_i)\right) = \bigsqcup_{i\in\mathbb{N}} \pi_1(d_i, e_i)$

$$\pi_1\left(\bigsqcup_{i\in\mathbb{N}}(d_i, e_i)\right) = \pi_1\left(\bigsqcup_{i\in\mathbb{N}} d_i, \bigsqcup_{i\in\mathbb{N}} e_i\right) = \bigsqcup_{i\in\mathbb{N}} d_i = \bigsqcup_{i\in\mathbb{N}} \pi_1(d_i, e_i)$$

by def of lub $\qquad$ by def of π_1 $\qquad$ by def of π_1

the case of π_2 is analogous

Exercise: smashed prod

$$\mathcal{D} = (D, \sqsubseteq_D) \qquad \mathcal{E} = (E, \sqsubseteq_E) \qquad \mathrm{CPO}_\perp \quad \Rightarrow \quad \mathcal{D} \otimes \mathcal{E} = (\ D \otimes E\ ,\ \sqsubseteq_{D \otimes E}\ )$$

$$D \otimes E \triangleq \{(d, e) \mid (d, e) \in D \times E, d = \perp_D \Leftrightarrow e = \perp_E\}$$

how to order pairs?

bottom element?

complete order?