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PSC 2021/22 (375AA, 9CFU)

Principles for Software Composition

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18b - CCS strong bisimulaton

CCS syntax

p, q	$::=$	$\mathbf{nil}$	inactive process
	$\mid$	x	process variable (for recursion)
	$\mid$	$\mu.p$	action prefix
	$\mid$	$p\backslash\alpha$	restricted channel
	$\mid$	$p[\phi]$	channel relabelling
	$\mid$	$p + q$	nondeterministic choice (sum)
	$\mid$	$p \mid q$	parallel composition
	$\mid$	$\mathbf{rec}\ x.\ p$	recursion

(operators are listed in order of precedence)

CCS op. semantics

Act) $$\frac{}{\mu.p \xrightarrow{\mu} p}$$

Res) $$\frac{p \xrightarrow{\mu} q \quad \mu \notin \{\alpha, \overline{\alpha}\}}{p \backslash \alpha \xrightarrow{\mu} q \backslash \alpha}$$

Rel) $$\frac{p \xrightarrow{\mu} q}{p[\phi] \xrightarrow{\phi(\mu)} q[\phi]}$$

SumL) $$\frac{p_1 \xrightarrow{\mu} q}{p_1 + p_2 \xrightarrow{\mu} q}$$

SumR) $$\frac{p_2 \xrightarrow{\mu} q}{p_1 + p_2 \xrightarrow{\mu} q}$$

ParL) $$\frac{p_1 \xrightarrow{\mu} q_1}{p_1 | p_2 \xrightarrow{\mu} q_1 | p_2}$$

Com) $$\frac{p_1 \xrightarrow{\lambda} q_1 \quad p_2 \xrightarrow{\overline{\lambda}} q_2}{p_1 | p_2 \xrightarrow{\tau} q_1 | q_2}$$

ParR) $$\frac{p_2 \xrightarrow{\mu} q_2}{p_1 | p_2 \xrightarrow{\mu} p_1 | q_2}$$

Rec) $$\frac{p[{}^{\mathbf{rec}\ x.\ p}/_{x}] \xrightarrow{\mu} q}{\mathbf{rec}\ x.\ p \xrightarrow{\mu} q}$$

Bisimulation game

Bisimulation game

two processes *p,q* and two opposing players

Alice, the attacker, aims to prove *p* and *q* are not equivalent

Bob, the defender, aims to prove *p* and *q* are equivalent

the game is turn based, at each turn:

Alice chooses one process and one of its outgoing transitions

Bob must reply with a transition of the other process, matching the label of the transition chosen by Alice

at the next turn, if any, the players will consider the equivalence of the target processes of the chosen transitions

Bisimulation game

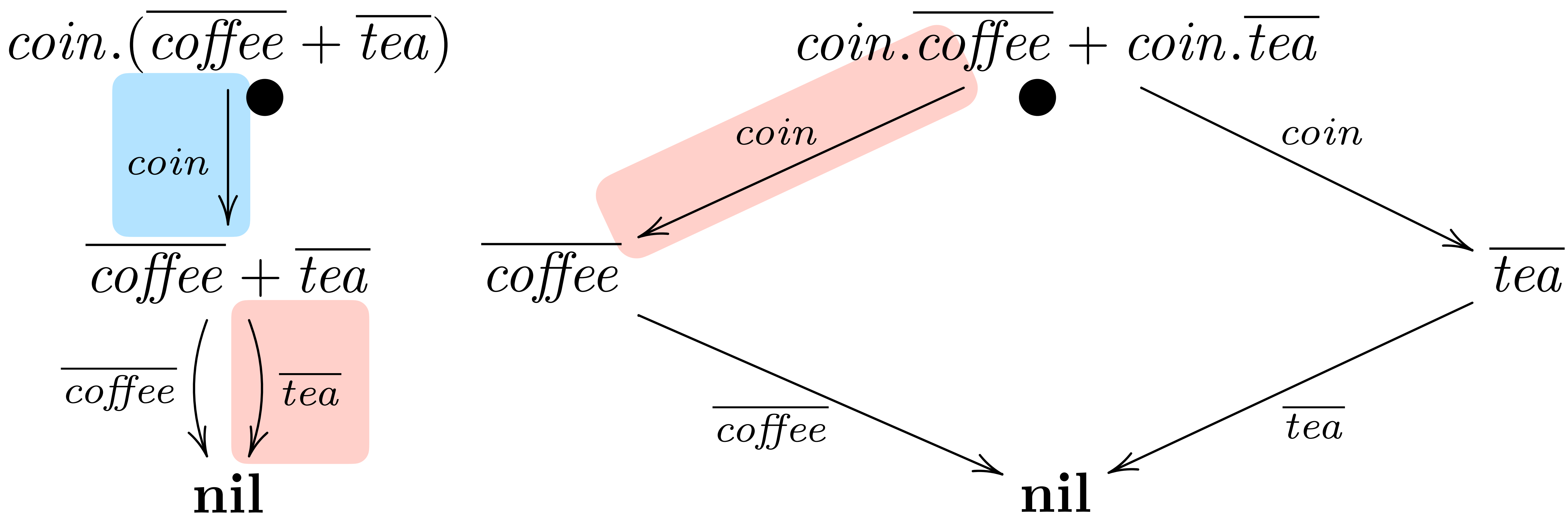
Alice wins if, at some stage,
she can make a move that Bob cannot match

Bob wins in all other cases
- if Alice cannot find a move
- if the game does not terminate

Alice has a winning strategy
if she can make a move that Bob cannot match;
or if she can make a move that no matter what Bob replies,
at the next turn she wins;
or so the like after any (finite) number of moves…

Alice has a winning strategy if she can disprove the equivalence of *p* and *q* in a finite number of moves

Bisimulation game



Alice plays $coin.\overline{coffee} + coin.\overline{tea} \xrightarrow{coin} \overline{coffee}$

Bob can only reply $coin.(\overline{coffee} + \overline{tea}) \xrightarrow{coin} \overline{coffee} + \overline{tea}$

Alice plays $\overline{coffee} + \overline{tea} \xrightarrow{\overline{tea}} \mathbf{nil}$

Bob cannot reply $\overline{coffee} \not\xrightarrow{\overline{tea}}$ Alice wins!

CCS
Strong bisimulation

Strong bisimulation

the notion of bisimulation is not restricted to CCS processes

it applies to any LTS

in the following we recall Milner's original definition of *strong bisimulation relation*

to keep in mind

there are many strong bisimulation relations

we are interested in the largest such relation, called *strong bisimilarity*

to prove that two processes are strong bisimilar
it is enough to show they are related by a strong bisimulation

Strong bisimulation

$\mathcal{P}$ set of processes

$\mathbf{R} \subseteq \mathcal{P} \times \mathcal{P}$ a binary relation

we write $p \mathbf{R} q$ when $(p, q) \in \mathbf{R}$

$\mathbf{R}$ is a strong bisimulation if

$$\forall p, q.\ (p, q) \in \mathbf{R} \boxed{\Rightarrow} \begin{cases} \forall \mu, p'.\ p \xrightarrow{\mu} p' & \Rightarrow & \exists q'.\ q \xrightarrow{\mu} q' \wedge p' \mathbf{R} q' \\ \wedge & & \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' & \Rightarrow & \exists p'.\ p \xrightarrow{\mu} p' \wedge p' \mathbf{R} q' \end{cases}$$

intuitively: if two processes are related, then for any move of Alice, Bob can find a move that leads to related processes i.e., Bob has a winning strategy

Example

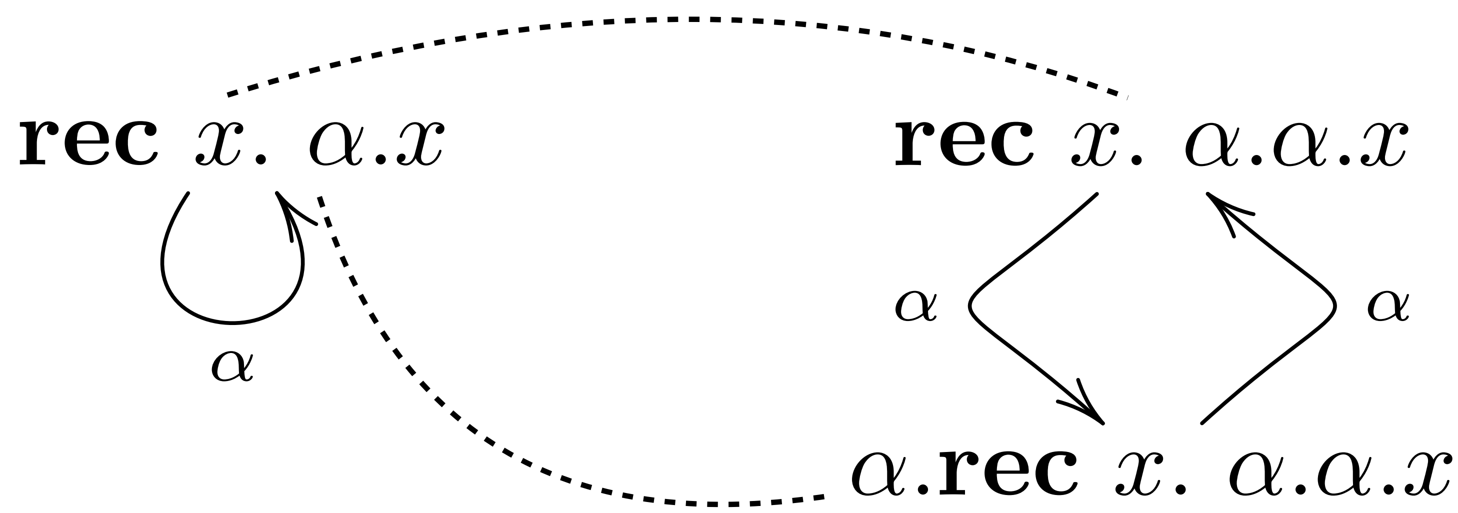
$\emptyset$ is a strong bisimulation

$Id \triangleq \{(p,p) \mid p \in \mathcal{P}\}$ is a strong bisimulation

any graph isomorphism defines a strong bisimulation

$$\mathbf{R}_f \triangleq \{(p, f(p))\}$$

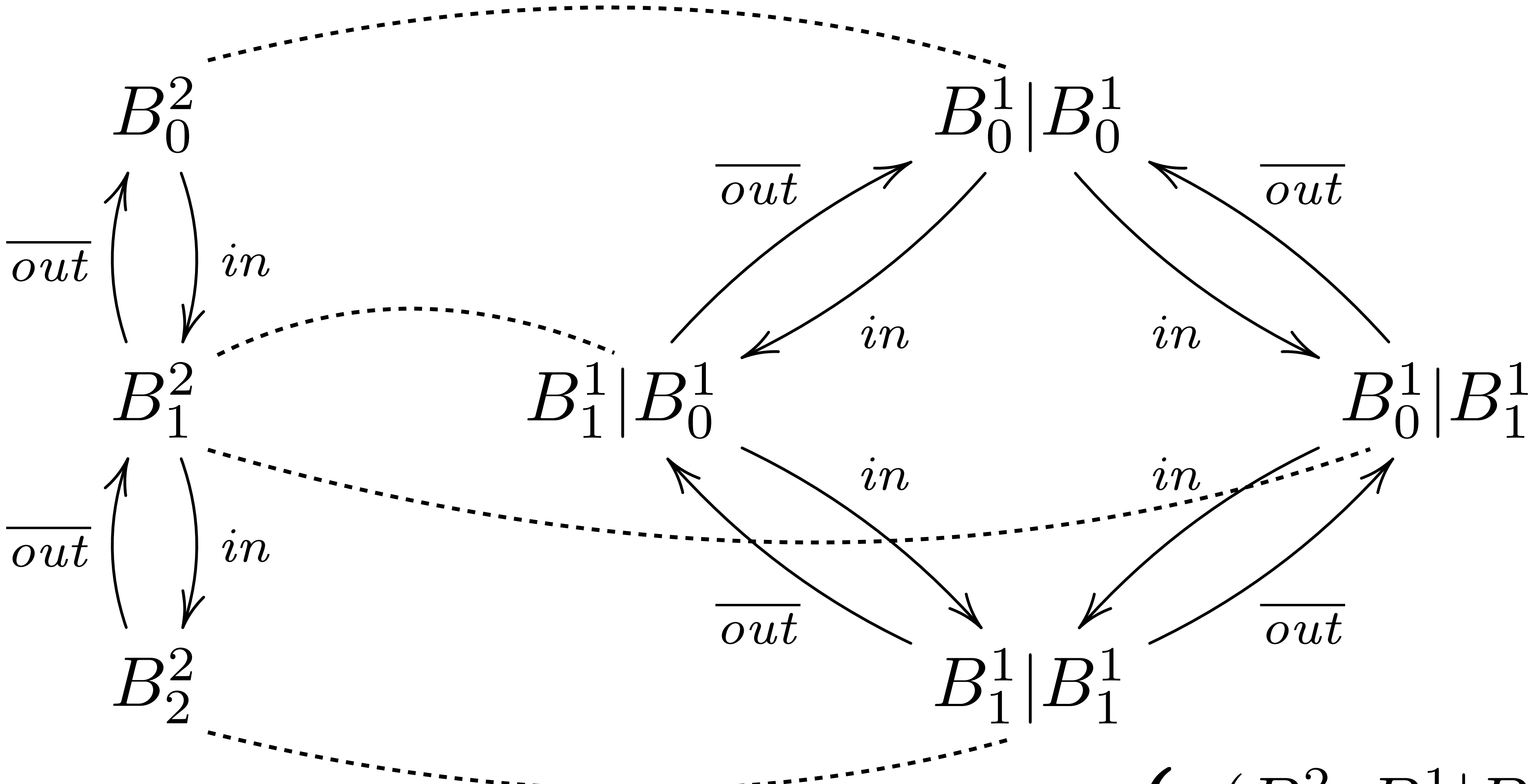
Example



$$\mathbf{R} \triangleq \left\{ \begin{array}{l} (\mathbf{rec}\ x.\ \alpha.x, \mathbf{rec}\ x.\ \alpha.\alpha.x), \\ (\mathbf{rec}\ x.\ \alpha.x, \alpha.\mathbf{rec}\ x.\ \alpha.\alpha.x) \end{array} \right\}$$

unlike graph isomorphisms,
the same process can be related to many processes

Example



$$\mathbf{R} \triangleq \left\{ \begin{array}{l} (B^2_0, B^1_0|B^1_0), \\ (B^2_1, B^1_1|B^1_0), \\ (B^2_1, B^1_0|B^1_1), \\ (B^2_2, B^1_1|B^1_1) \end{array} \right\}$$

Union

Lemma If $\mathbf{R}_1$ and $\mathbf{R}_2$ are strong bisimulations,
then $\mathbf{R}_1 \cup \mathbf{R}_2$ is a strong bisimulation

proof. take $(p, q) \in \mathbf{R}_1 \cup \mathbf{R}_2$

take $p \xrightarrow{\mu} p'$ we want to find $q \xrightarrow{\mu} q'$ with $(p', q') \in \mathbf{R}_1 \cup \mathbf{R}_2$

since $(p, q) \in \mathbf{R}_1 \cup \mathbf{R}_2$ we have $p \; \mathbf{R}_i \; q$ for some $i \in \{1, 2\}$

since $\mathbf{R}_i$ is a strong bisimulation and $p \xrightarrow{\mu} p'$

we have $q \xrightarrow{\mu} q'$ with $p' \; \mathbf{R}_i \; q'$ and hence $(p', q') \in \mathbf{R}_1 \cup \mathbf{R}_2$

take $q \xrightarrow{\mu} q'$ we want to find $p \xrightarrow{\mu} p'$ with $(p', q') \in \mathbf{R}_1 \cup \mathbf{R}_2$

analogous to the previous case

Inverse

Lemma If $\mathbf{R}$ is a strong bisimulation,
then $\mathbf{R}^{-1} \triangleq \{(q,p) \mid p \mathbf{R} q\}$ is a strong bisimulation

proof. take $(q,p) \in \mathbf{R}^{-1}$

take $q \xrightarrow{\mu} q'$ we want to find $p \xrightarrow{\mu} p'$ with $(q',p') \in \mathbf{R}^{-1}$

since $(q,p) \in \mathbf{R}^{-1}$ we have $p \mathbf{R} q$

since $\mathbf{R}$ is a strong bisimulation and $q \xrightarrow{\mu} q'$

we have $p \xrightarrow{\mu} p'$ with $p' \mathbf{R} q'$ and hence $(q',p') \in \mathbf{R}^{-1}$

take $p \xrightarrow{\mu} p'$ we want to find $q \xrightarrow{\mu} q'$ with $(q',p') \in \mathbf{R}^{-1}$

analogous to the previous case

Composition

Lemma If $\mathbf{R}_1$ and $\mathbf{R}_2$ are strong bisimulations,
then $\mathbf{R}_2 \circ \mathbf{R}_1 \triangleq \{(p, q) \mid \exists r.\ p \ \mathbf{R}_1 \ r \wedge r \ \mathbf{R}_2 \ q\}$
is a strong bisimulation

proof. take $(p, q) \in \mathbf{R}_2 \circ \mathbf{R}_1$

take $p \xrightarrow{\mu} p'$ we want to find $q \xrightarrow{\mu} q'$ with $(p', q') \in \mathbf{R}_2 \circ \mathbf{R}_1$

since $(p, q) \in \mathbf{R}_2 \circ \mathbf{R}_1$ we have $p \ \mathbf{R}_1 \ r \wedge r \ \mathbf{R}_2 \ q$ for some r

since $\mathbf{R}_1$ is a strong bisimulation and $p \xrightarrow{\mu} p'$

we have $r \xrightarrow{\mu} r'$ with $p' \ \mathbf{R}_1 \ r'$

since $\mathbf{R}_2$ is a strong bisimulation and $r \xrightarrow{\mu} r'$

we have $q \xrightarrow{\mu} q'$ with $r' \ \mathbf{R}_2 \ q'$ and hence $(p', q') \in \mathbf{R}_2 \circ \mathbf{R}_1$

take $q \xrightarrow{\mu} q'$ we want to find $p \xrightarrow{\mu} p'$ with $(p', q') \in \mathbf{R}_2 \circ \mathbf{R}_1$

analogous to the previous case

Notation

$$\mathbf{R}_2 \circ \mathbf{R}_1 \triangleq \{(p, q) \mid \exists r.\ p\ \mathbf{R}_1\ r \wedge r\ \mathbf{R}_2\ q\}$$

sometimes written

$$\mathbf{R}_1\mathbf{R}_2$$

CCS
Strong bisimilarity

Strong bisimilarity

$\simeq$ often denoted $\sim$ in the literature

we use $\simeq$ to remark it is a congruence relation

$p \simeq q \quad \text{iff} \quad \exists \mathbf{R}$ a strong bisimulation with $(p, q) \in \mathbf{R}$

i.e. Bob has a winning strategy

i.e. $\simeq \;\triangleq\; \bigcup_{\mathbf{R}\ \text{s.b.}} \mathbf{R}$

a strong bisimulation is not necessarily an equivalence

is strong bisimilarity an equivalence relation?

Equivalence relation

$\equiv$

Reflexive	$\forall p \in \mathcal{P}$	$p \equiv p$
Symmetric	$\forall p, q \in \mathcal{P}$	$p \equiv q \Rightarrow q \equiv p$
Transitive	$\forall p, q, r \in \mathcal{P}$	$p \equiv q \wedge q \equiv r \Rightarrow p \equiv r$

Induced equivalence

Any relation $\mathbf{R}$ induces an equivalence relation $\equiv_{\mathbf{R}}$

$\equiv_{\mathbf{R}}$ is the smallest equivalence that contains $\mathbf{R}$

$$\frac{p \;\mathbf{R}\; q}{p \equiv_{\mathbf{R}} q} \qquad \frac{}{p \equiv_{\mathbf{R}} p} \qquad \frac{p \equiv_{\mathbf{R}} q}{q \equiv_{\mathbf{R}} p} \qquad \frac{p \equiv_{\mathbf{R}} q \quad q \equiv_{\mathbf{R}} r}{p \equiv_{\mathbf{R}} r}$$

Lemma if $\mathbf{R}$ is a strong bisimulation,

then $\equiv_{\mathbf{R}}$ is a strong bisimulation

Induced partition

Any equivalence relation induces a partition of processes into equivalence classes

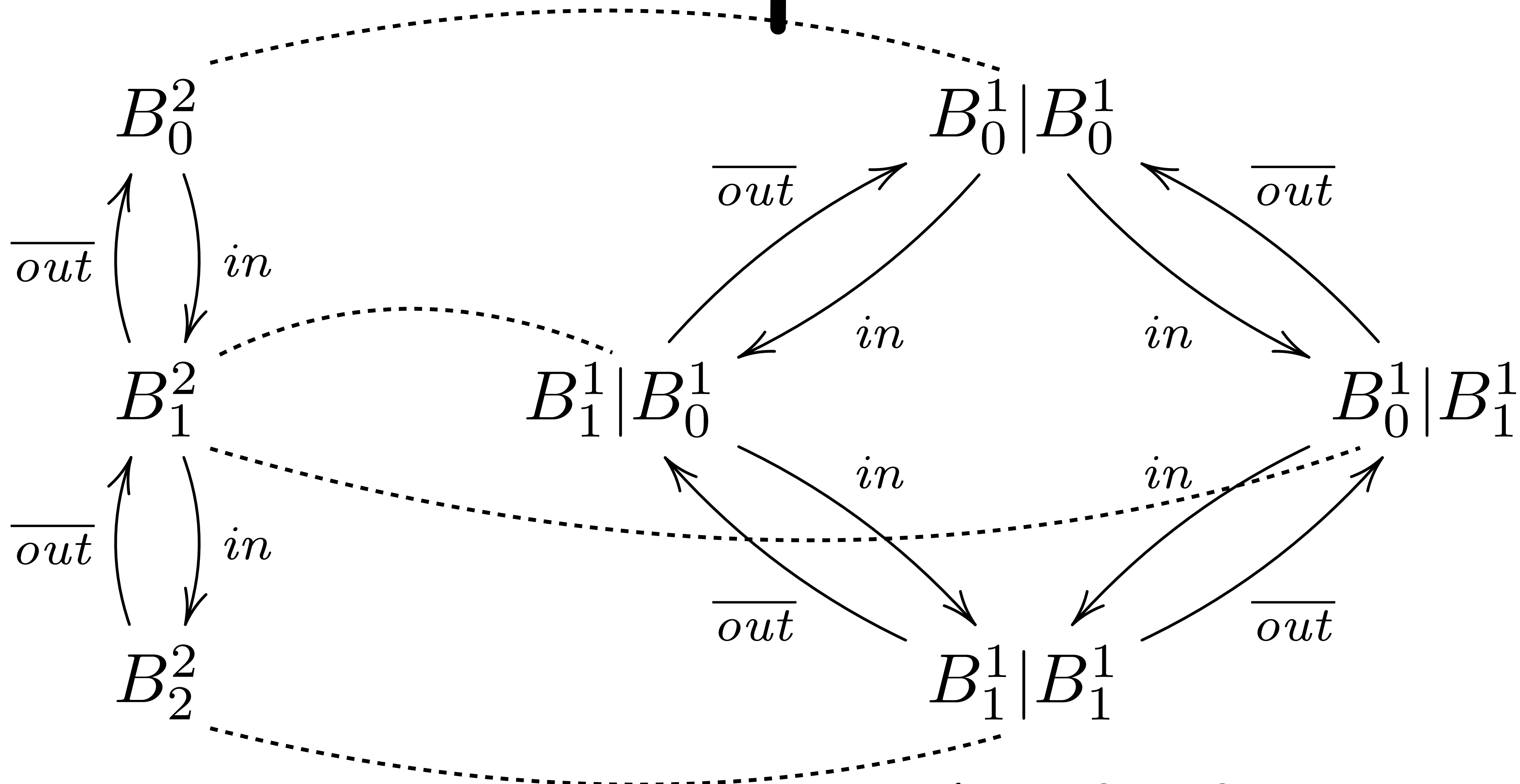
$$[p]_{\equiv} = \{q \mid p \equiv q\}$$

if $\equiv_{\mathbf{R}}$ is a strong bisimulation

$$q \in [p]_{\equiv_{\mathbf{R}}} \wedge p \xrightarrow{\mu} p' \;\; \Rightarrow \;\; \exists q' \in [p']_{\equiv_{\mathbf{R}}}.\ q \xrightarrow{\mu} q'$$

instead of listing all pairs of $\equiv_{\mathbf{R}}$
we list only its equivalence classes

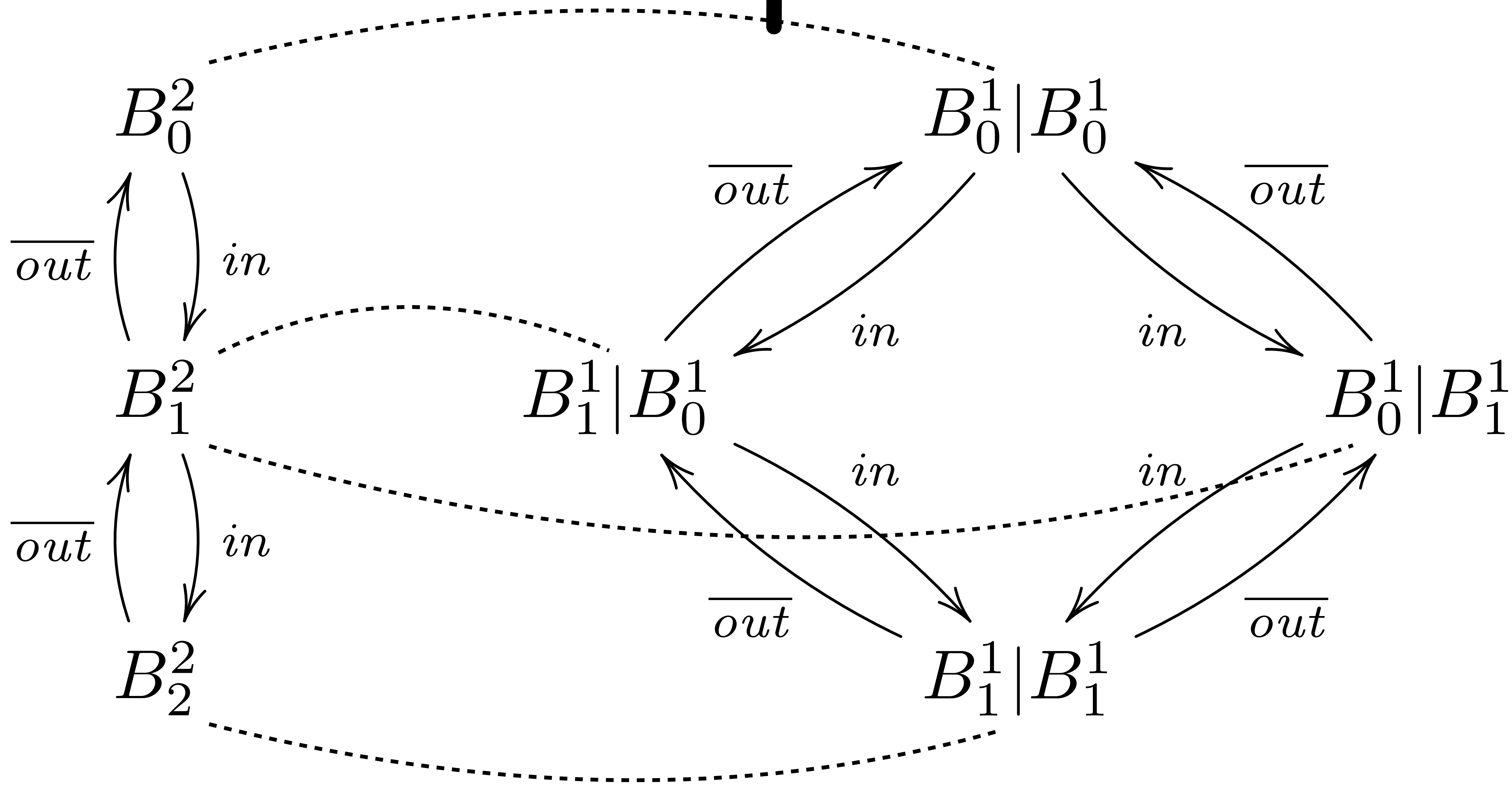
Example



$$\mathbf{R} \triangleq \left\{ \begin{array}{l} (B_0^2, B_0^1|B_0^1), \\ (B_1^2, B_1^1|B_0^1), \\ (B_1^2, B_0^1|B_1^1), \\ (B_2^2, B_1^1|B_1^1) \end{array} \right\}$$

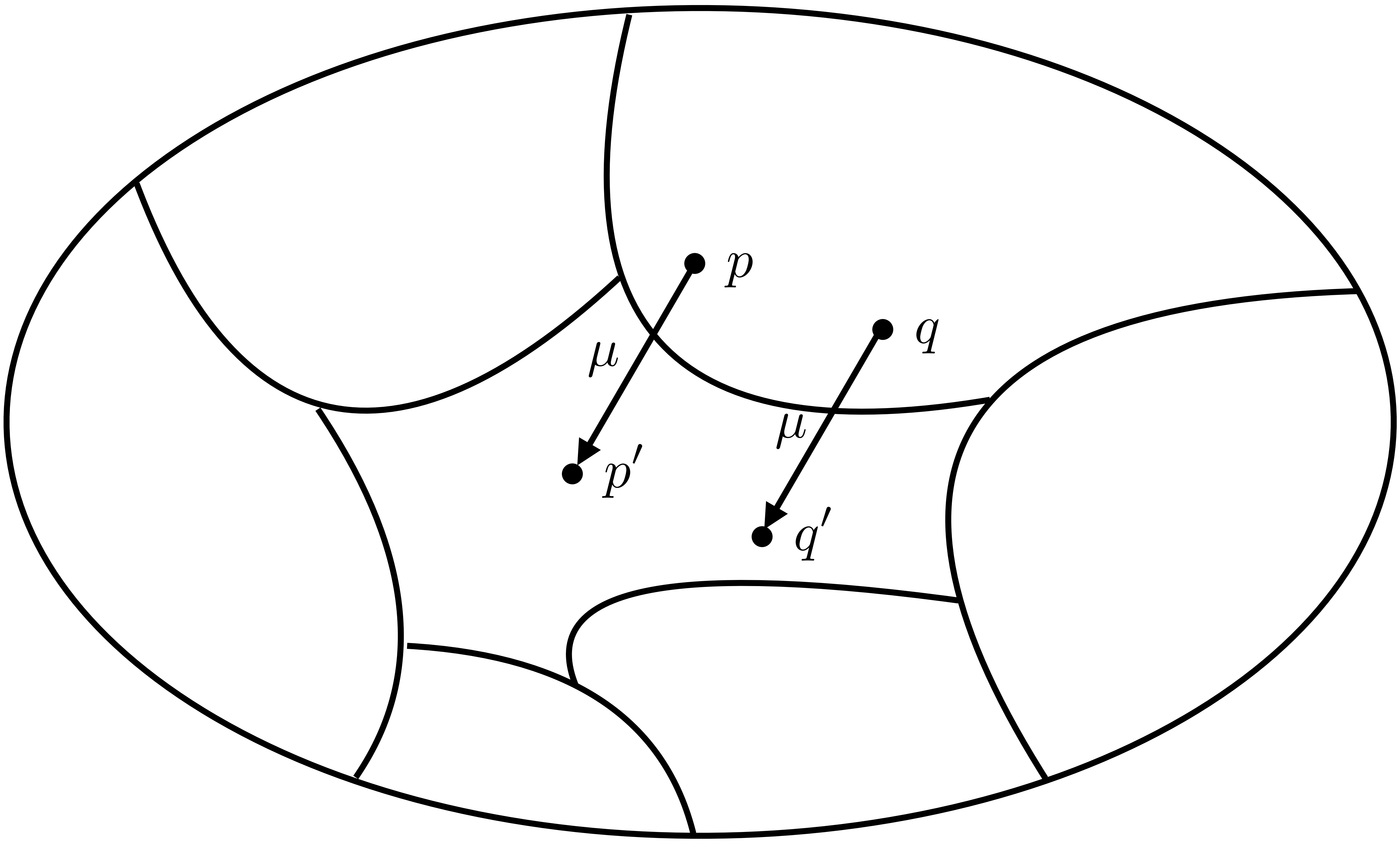
$$\equiv_{\mathbf{R}} \triangleq \left\{ \begin{array}{l} (B_0^2, B_0^2), \\ (B_0^2, B_0^1|B_0^1), \\ (B_0^1|B_0^1, B_0^2), \\ (B_0^1|B_0^1, B_0^1|B_0^1), \\ (B_1^2, B_1^2), \\ \ldots \end{array} \right\}$$

Example

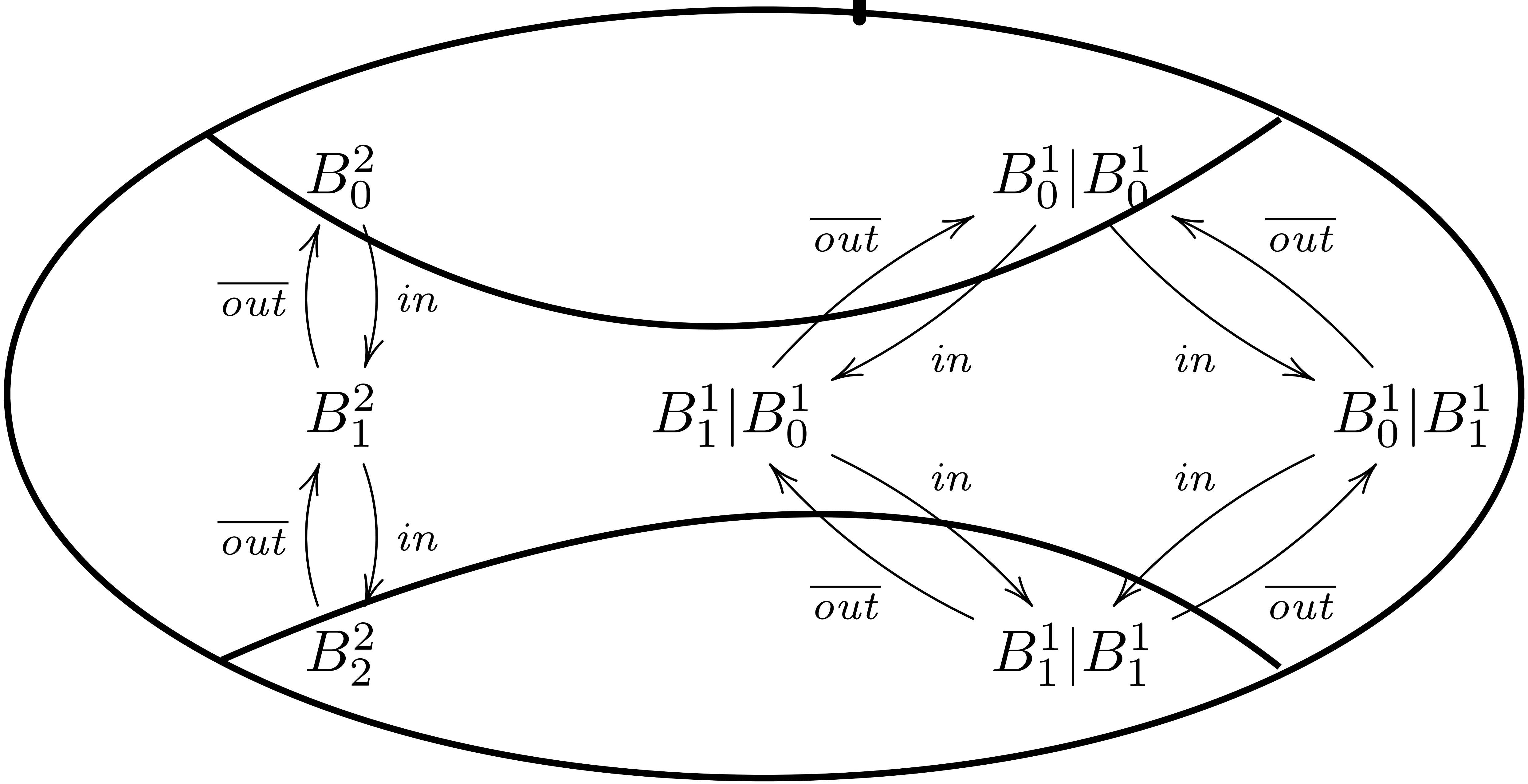


$$\mathbf{R} \triangleq \left\{ \begin{array}{c} (B^2_0, B^1_0|B^1_0), \\ (B^2_1, B^1_1|B^1_0), \\ (B^2_1, B^1_0|B^1_1), \\ (B^2_2, B^1_1|B^1_1) \end{array} \right\} \qquad \equiv_{\mathbf{R}} \triangleq \left\{ \begin{array}{l} \{B^2_0, B^1_0|B^1_0\}, \\ \{B^2_1, B^1_0|B^1_1, B^1_1|B^1_0\}, \\ \{B^2_2, B^1_1|B^1_1\} \end{array} \right\}$$

Bisimulation check



Example



$$\equiv_{\mathbf{R}} \triangleq \left\{ \begin{array}{l} \{B_0^2, B_0^1|B_0^1\}, \\ \{B_1^2, B_0^1|B_1^1, B_1^1|B_0^1\}, \\ \{B_2^2, B_1^1|B_1^1\} \end{array} \right\}$$

TH. Strong bisimilarity is an equivalence relation

proof.

reflexive $Id \subseteq \simeq$

symmetric assume $p \simeq q$ we want to prove $q \simeq p$

$p \simeq q$ means there is a s.b. $\mathbf{R}$ with $(p, q) \in \mathbf{R}$

then $(q, p) \in \mathbf{R}^{-1}$ and $\mathbf{R}^{-1}$ is a s.b.

thus $(q, p) \in \mathbf{R}^{-1} \subseteq \simeq$ i.e. $q \simeq p$

transitive assume $p \simeq q$ $q \simeq r$ we want to prove $p \simeq r$

$p \simeq q$ means there is a s.b. $\mathbf{R}_1$ with $(p, q) \in \mathbf{R}_1$

$q \simeq r$ means there is a s.b. $\mathbf{R}_2$ with $(q, r) \in \mathbf{R}_2$

then $(p, r) \in \mathbf{R}_2 \circ \mathbf{R}_1$ and $\mathbf{R}_2 \circ \mathbf{R}_1$ is a s.b.

thus $(p, r) \in \mathbf{R}_2 \circ \mathbf{R}_1 \subseteq \simeq$ i.e. $p \simeq r$

TH. Strong bisimilarity is a strong bisimulation

proof.

take $p \simeq q$

take $p \xrightarrow{\mu} p'$ we want to find $q \xrightarrow{\mu} q'$ with $p' \simeq q'$

$p \simeq q$ means there is a s.b. $\mathbf{R}$ with $(p, q) \in \mathbf{R}$

since $\mathbf{R}$ is a strong bisimulation and $p \xrightarrow{\mu} p'$

we have $q \xrightarrow{\mu} q'$ with $(p', q') \in \mathbf{R}$

since $\mathbf{R} \subseteq \simeq$ we have $p' \simeq q'$

take $q \xrightarrow{\mu} q'$ we want to find $p \xrightarrow{\mu} p'$ with $p' \simeq q'$

follows from previous case (strong bisimilarity is symmetric)

Cor. Strong bisimilarity is the **largest** strong bisimulation

proof.

strong bisimilarity is a strong bisimulation (previous TH.)

by definition

$$\simeq \;\triangleq\; \bigcup_{\mathbf{R}\ \mathrm{s.b.}} \mathbf{R}$$

any other strong bisimulation is included in $\simeq$

TH. Recursive definition of strong bisimilarity

$$\forall p, q.\ p \simeq q \Leftrightarrow \begin{cases} \forall \mu, p'.\ p \xrightarrow{\mu} p' \quad \Rightarrow \quad \exists q'.\ q \xrightarrow{\mu} q' \wedge p' \simeq q' \\ \wedge \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' \quad \Rightarrow \quad \exists p'.\ p \xrightarrow{\mu} p' \wedge p' \simeq q' \end{cases}$$

proof.

$\Rightarrow)$ follows immediately because $\simeq$ is a strong bisimulation

$\Leftarrow)$ take *p,q* s.t. $\begin{cases} \forall \mu, p'.\ p \xrightarrow{\mu} p' \quad \Rightarrow \quad \exists q'.\ q \xrightarrow{\mu} q' \wedge p' \simeq q' \\ \wedge \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' \quad \Rightarrow \quad \exists p'.\ p \xrightarrow{\mu} p' \wedge p' \simeq q' \end{cases}$

we want to prove $p \simeq q$

this is done by proving that $\mathbf{R} \triangleq \{(p, q)\} \cup \simeq$ is a s.b.

(see next slide)

TH. Recursive definition of strong bisimilarity (continue)

$$\mathbf{R} \triangleq \{(p,q)\} \cup \simeq \quad \text{is a s.b.}$$

take $(r,s) \in \mathbf{R}$

take $r \xrightarrow{\mu} r'$ we want to find $s \xrightarrow{\mu} s'$ with $(r',s') \in \mathbf{R}$

if $r \simeq s$ then we can find $s \xrightarrow{\mu} s'$ with $(r',s') \in \simeq \subseteq \mathbf{R}$

because $\simeq$ is a strong bisimulation

if $(r,s) = (p,q)$ then $p \xrightarrow{\mu} r'$ and $\begin{cases} \forall \mu, p'.\ p \xrightarrow{\mu} p' & \Rightarrow & \exists q'.\ q \xrightarrow{\mu} q' \wedge p' \simeq q' \\ \wedge & & \\ \forall \mu, q'.\ q \xrightarrow{\mu} q' & \Rightarrow & \exists p'.\ p \xrightarrow{\mu} p' \wedge p' \simeq q' \end{cases}$

thus we can find $q \xrightarrow{\mu} s'$ with $(r',s') \in \simeq \subseteq \mathbf{R}$

take $s \xrightarrow{\mu} s'$ we want to find $r \xrightarrow{\mu} r'$ with $(r',s') \in \mathbf{R}$

analogous to the previous case

CCS
Compositionality

Compositionality

recall that an equivalence $\equiv$ is a congruence when

$$\forall \mathbb{C}[\cdot].\ \forall p, q.\ \boxed{p \equiv q \ \Rightarrow\ \mathbb{C}[p] \equiv \mathbb{C}[q]}$$

we can replace equivalent processes in any context without changing the abstract semantics

TH. Strong bisimilarity is a congruence

1. $\forall p, q.\ p \simeq q \;\Rightarrow\; \forall \mu.\ \mu.p \simeq \mu.q$

2. $\forall p, q.\ p \simeq q \;\Rightarrow\; \forall \alpha.\ p \backslash \alpha \simeq q \backslash \alpha$

3. $\forall p, q.\ p \simeq q \;\Rightarrow\; \forall \phi.\ p[\phi] \simeq q[\phi]$

4. $\forall p_0, q_0, p_1, q_1.\ p_0 \simeq q_0 \wedge p_1 \simeq q_1 \;\Rightarrow\; p_0 + p_1 \simeq q_0 + q_1$

5. $\forall p_0, q_0, p_1, q_1.\ p_0 \simeq q_0 \wedge p_1 \simeq q_1 \;\Rightarrow\; p_0 | p_1 \simeq q_0 | q_1$

let us omit quantification to make the statement more readable

TH. Strong bisimilarity is a congruence

1. $p \simeq q \;\Rightarrow\; \mu.p \simeq \mu.q$

2. $p \simeq q \;\Rightarrow\; p\backslash\alpha \simeq q\backslash\alpha$

3. $p \simeq q \;\Rightarrow\; p[\phi] \simeq q[\phi]$

4. $p_0 \simeq q_0 \wedge p_1 \simeq q_1 \;\Rightarrow\; p_0 + p_1 \simeq q_0 + q_1$

5. $p_0 \simeq q_0 \wedge p_1 \simeq q_1 \;\Rightarrow\; p_0|p_1 \simeq q_0|q_1$

proof technique:
“guess” a relation large enough to contain all pairs of interest;
show that it is a bisimulation relation;
then it is contained in the strong bisimilarity relation

TH. Strong bisimilarity is a congruence (3)

take $\mathbf{R} \triangleq \{(p[\phi], q[\phi]) \mid p \simeq q\}$

we show that $\mathbf{R}$ is a strong bisimulation relation

take $(p[\phi], q[\phi]) \in \mathbf{R}$ (i.e. with $p \simeq q$)

take $p[\phi] \xrightarrow{\mu} p'$ we want to find $q[\phi] \xrightarrow{\mu} q'$ with $(p', q') \in \mathbf{R}$

by rule rel) it must be $p \xrightarrow{\mu'} p''$ $\quad \mu = \phi(\mu')$ $\quad p' = p''[\phi]$

since $p \simeq q$ then $q \xrightarrow{\mu'} q''$ with $p'' \simeq q''$

by rule rel) $q[\phi] \xrightarrow{\phi(\mu')} q''[\phi]$

take $q' = q''[\phi]$ so that $(p', q') = (p''[\phi], q''[\phi]) \in \mathbf{R}$

take $q[\phi] \xrightarrow{\mu} q'$ we want to find $p[\phi] \xrightarrow{\mu} p'$ with $(p', q') \in \mathbf{R}$

analogous to the previous case

TH. Strong bisimilarity is a congruence (4)

take $\mathbf{R} \triangleq \{(p_0 + p_1, q_0 + q_1) \mid p_0 \simeq q_0 \wedge p_1 \simeq q_1\}$

we show that $\mathbf{R}$ is a strong bisimulation relation

take $(p_0 + p_1, q_0 + q_1) \in \mathbf{R}$ (i.e. with $p_0 \simeq q_0$ $p_1 \simeq q_1$)

take $p_0 + p_1 \xrightarrow{\mu} p'$ we need $q_0 + q_1 \xrightarrow{\mu} q'$ with $(p', q') \in \mathbf{R}$

if rule suml) was used: $p_0 \xrightarrow{\mu} p'$

since $p_0 \simeq q_0$ then $q_0 \xrightarrow{\mu} q'$ with $p' \simeq q'$

by rule suml) $q_0 + q_1 \xrightarrow{\mu} q'$

but unfortunately $(p', q') \in \simeq$ not necessarily $(p', q') \in \mathbf{R}$

how can we repair the proof?

TH. Strong bisimilarity is a congruence (4)

take $\mathbf{R} \triangleq \{(p_0 + p_1, q_0 + q_1) \mid p_0 \simeq q_0 \wedge p_1 \simeq q_1\} \boxed{\cup \simeq}$

we show that $\mathbf{R}$ is a strong bisimulation relation

take $(p_0 + p_1, q_0 + q_1) \in \mathbf{R}$ (i.e. with $p_0 \simeq q_0$ $p_1 \simeq q_1$)

take $p_0 + p_1 \xrightarrow{\mu} p'$ we need $q_0 + q_1 \xrightarrow{\mu} q'$ with $(p', q') \in \mathbf{R}$

if rule suml) was used: $p_0 \xrightarrow{\mu} p'$

since $p_0 \simeq q_0$ then $q_0 \xrightarrow{\mu} q'$ with $p' \simeq q'$

by rule suml) $q_0 + q_1 \xrightarrow{\mu} q'$

then $(p', q') \in \simeq \subseteq \mathbf{R}$

how can we repair the proof?

(no need to check the pairs in $\simeq$)

fill in the missing details

- sumr)
- $q_0 + q_1$ moves

CCS: some laws

$$p + \mathbf{nil} \simeq p \qquad p|\mathbf{nil} \simeq p$$

$$p + q \simeq q + p \qquad p|q \simeq q|p$$

$$p + (q + r) \simeq (p + q) + r \qquad p|(q|r) \simeq (p|q)|r$$

$$p + p \simeq p$$

how to prove them? find a strong bisimulation for each of them

$$\mathbf{nil}\backslash\alpha \simeq \mathbf{nil}$$

$$(\mu.p)\backslash\alpha \simeq \mathbf{nil} \quad \text{if } \mu \in \{\alpha, \overline{\alpha}\}$$

$$(\mu.p)\backslash\alpha \simeq \mu.(p\backslash\alpha) \quad \text{if } \mu \notin \{\alpha, \overline{\alpha}\}$$

$$(p + q)\backslash\alpha \simeq (p\backslash\alpha) + (q\backslash\alpha)$$

$$p\backslash\alpha\backslash\alpha \simeq p\backslash\alpha$$

$$p\backslash\alpha\backslash\beta \simeq p\backslash\beta\backslash\alpha$$

$$\mathbf{nil}[\phi] \simeq \mathbf{nil}$$

$$(\mu.p)[\phi] \simeq \phi(\mu).(p[\phi])$$

$$(p + q)[\phi] \simeq (p[\phi]) + (q[\phi])$$

$$p[\phi][\eta] \simeq p[\eta \circ \phi]$$