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PSC 2021/22 (375AA, 9CFU)

Principles for Software Composition

Roberto Bruni
http://www.di.unipi.it/~bruni/

08b - Kleene's fixed point theorem

Partial functions

Partial functions

$D = (A \rightharpoonup B) = \mathbf{Pf}(A, B) = \{f : A \rightharpoonup B\}$ partial functions

$f \sqsubseteq g$ if $f(a)$ is defined, $g(a)$ is defined and $g(a) = f(a)$

but $g(a)$ can be defined when $f(a)$ is not

if we see partial functions as relations

$$\{(x, f(x)) \mid f(x) \neq \bot\} \subseteq A \times B$$

$f \sqsubseteq g$ means essentially $f \subseteq g$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \bot & \text{otherwise} \end{cases}$$

$$f = \{ \quad \begin{array}{l} \quad n \quad f(n) \\ (0, 0), \\ (2, 1), \\ (4, 2), \\ (6, 3), \\ \ldots \\ (2k, k), \\ \ldots \end{array} \quad \}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ 2 \cdot n & \text{otherwise} \end{cases}$$

$$\begin{aligned} g = \{ \quad & (0,0), (1,2), \\ & (2,1), (3,6), \\ & (4,2), (5,10), \\ & (6,3), (7,14), \\ & \ldots \\ & (2k,k), (1+2k, 2+4k), \\ & \ldots \qquad \qquad \qquad \qquad \qquad \} \end{aligned}$$

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ 2 \cdot n & \text{otherwise} \end{cases}$$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ \perp & \text{otherwise} \end{cases}$$

$$g = \{ \begin{array}{l} (0,0),(1,2), \\ (2,1),(3,6), \\ (4,2),(5,10), \\ (6,3),(7,14), \\ \ldots \\ (2k,k),(1+2k,2+4k), \\ \ldots \end{array} \}$$

$$f = \{ \begin{array}{l} (0,0), \\ (2,1), \\ (4,2), \\ (6,3), \\ \ldots \\ (2k,k), \\ \ldots \end{array} \}$$

$f \sqsubseteq g$? ✓

$g \sqsubseteq f$? ✗

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq$ { (0,0) } $\sqsubseteq$ { (0,0), (1,1) } $\sqsubseteq$ …

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{\ (0,0)\ \} \sqsubseteq \left\{\begin{array}{l}(0,0),\\ (1,1)\end{array}\right\} \sqsubseteq \left\{\begin{array}{l}(0,0),\\ (1,1),\\ (2,2)\end{array}\right\} \sqsubseteq \cdots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,0) \} \sqsubseteq \left\{ \begin{array}{l} (0,0), \\ (1,1) \end{array} \right\} \sqsubseteq \left\{ \begin{array}{l} (0,0), \\ (1,1), \\ (2,2) \end{array} \right\} \sqsubseteq \left\{ \begin{array}{l} (0,0), \\ (1,1), \\ (2,2), \\ (3,3) \end{array} \right\} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,0) \} \sqsubseteq \begin{Bmatrix} (0,0), \\ (1,1) \end{Bmatrix} \sqsubseteq \begin{Bmatrix} (0,0), \\ (1,1), \\ (2,2) \end{Bmatrix} \sqsubseteq \begin{Bmatrix} (0,0), \\ (1,1), \\ (2,2), \\ (3,3) \end{Bmatrix} \sqsubseteq \begin{Bmatrix} (0,0), \\ (1,1), \\ (2,2), \\ (3,3), \\ (4,4) \end{Bmatrix} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$\emptyset \sqsubseteq$ { (0,1) } $\sqsubseteq$ { (0,1), (1,1) } $\sqsubseteq$ { (0,1), (1,1), (2,2) } $\sqsubseteq$ ...

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \left\{ \begin{array}{l} (0,1), \\ (1,1) \end{array} \right\} \sqsubseteq \left\{ \begin{array}{l} (0,1), \\ (1,1), \\ (2,2) \end{array} \right\} \sqsubseteq \left\{ \begin{array}{l} (0,1), \\ (1,1), \\ (2,2), \\ (3,6) \end{array} \right\} \sqsubseteq \ldots$$

which function(s) are we approximating?

Example

$\mathbf{Pf}(\mathbb{N}, \mathbb{N})$

$$\emptyset \sqsubseteq \{ (0,1) \} \sqsubseteq \{ (0,1), (1,1) \} \sqsubseteq \{ (0,1), (1,1), (2,2) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6) \} \sqsubseteq \{ (0,1), (1,1), (2,2), (3,6), (4,24) \} \sqsubseteq \ldots$$

which function(s) are we approximating?

Functional property

$\mathbf{Pf}(A,B) = \{f : A \rightharpoonup B\}$ partial functions

$$\mathbf{Pf}(A,B) = \{f \subseteq A \times B \mid \boxed{\forall a \in A. \forall b_1, b_2 \in B. (a,b_1) \in f \wedge (a,b_2) \in f \Rightarrow b_1 = b_2}\}$$

functional property

$f(a)\downarrow \triangleq \exists b \in B.\ (a,b) \in f$ function f is defined on a

$$f \sqsubseteq g \Leftrightarrow (\forall a \in A. \quad f(a)\downarrow \Rightarrow g(a)\downarrow \wedge f(a) = g(a))$$
$$\Leftrightarrow f \subseteq g$$

$(\mathbf{Pf}(A,B), \sqsubseteq)$ is a PO with bottom
what is bottom?
is it complete?

the empty relation
(the function always undefined)

Is Pf complete?

$(\mathbf{Pf}(A,B), \sqsubseteq)$

complete?

Given a chain $\{f_i\}_{i\in\mathbb{N}}$ let us consider $\bigcup_{i\in\mathbb{N}} f_i \subseteq A \times B$

we want to prove that $\bigcup_{i\in\mathbb{N}} f_i \in \mathbf{Pf}(A,B)$

i.e. that $f = \bigcup_{i\in\mathbb{N}} f_i$ satisfies the functional property

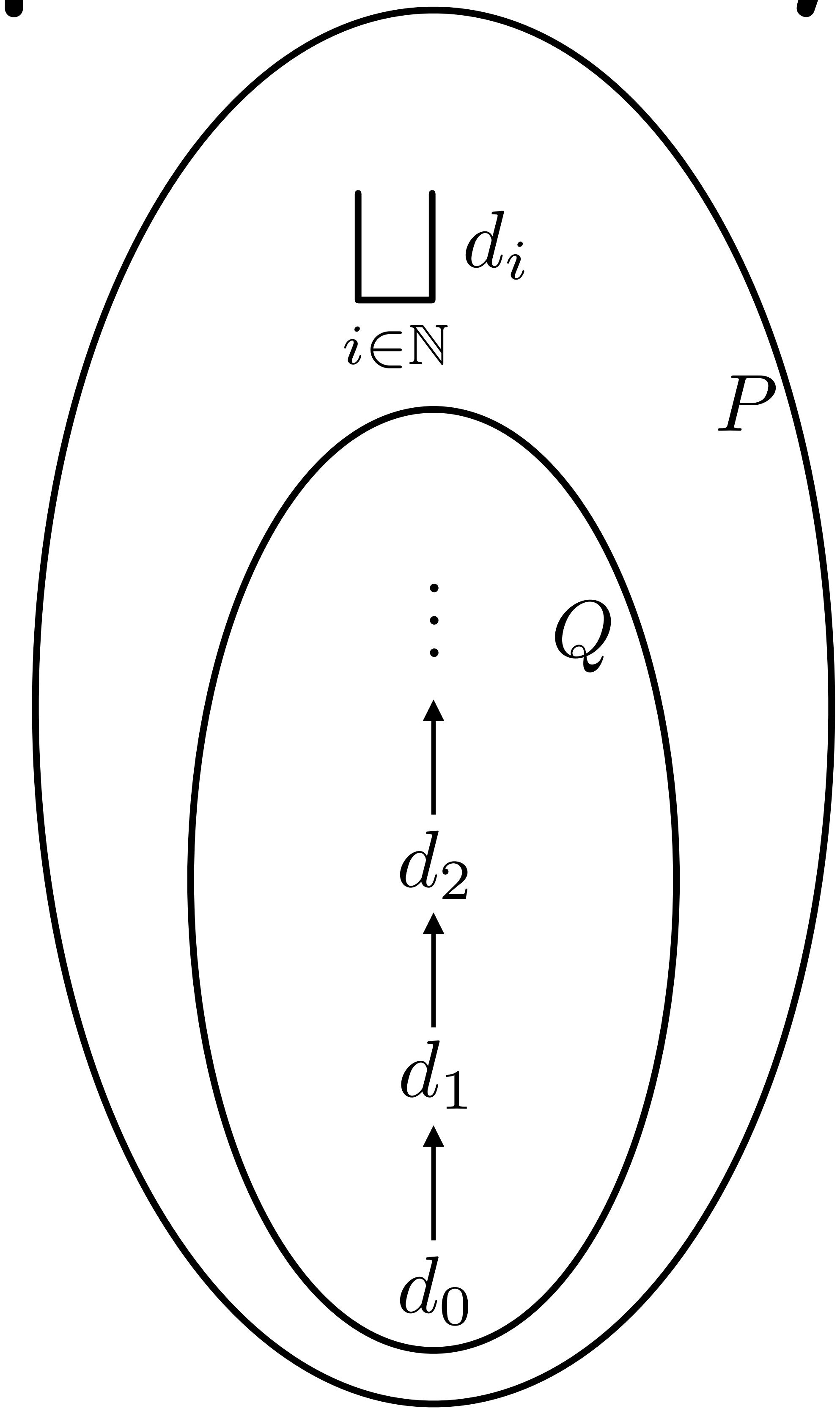
we know that each f_i is functional

$$\forall i \in \mathbb{N}. \forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f_i \wedge (a, b_2) \in f_i \Rightarrow b_1 = b_2$$

we need to prove f is functional

$$\forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2$$

pictorially



is the limit in Q ?

Pf is complete

we need to prove f is functional

$$\forall a \in A. \forall b_1, b_2 \in B. (a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2$$

Take $a \in A, b_1, b_2 \in B$ such that $(a, b_1) \in f \wedge (a, b_2) \in f$

we need to prove $b_1 = b_2$

$$(a, b_1) \in f = \bigcup_{i \in \mathbb{N}} f_i \Leftrightarrow \exists k \in \mathbb{N}. (a, b_1) \in f_k$$

$$m \triangleq \max\{k, h\}$$

$$(a, b_2) \in f = \bigcup_{i \in \mathbb{N}} f_i \Leftrightarrow \exists h \in \mathbb{N}. (a, b_2) \in f_h$$

Clearly $f_k \subseteq f_m$ $\quad$ $f_h \subseteq f_m$ $\quad$ f_m is functional

$$(a, b_1) \in f_m \qquad (a, b_2) \in f_m \qquad \Rightarrow \qquad b_1 = b_2$$

Example

$\mathbf{Pf}(\mathbb{N},\mathbb{N})$

$$
\begin{array}{rl}
\overset{f_0}{\emptyset} & \subseteq \{(0,1)\}^{f_1} \\
& \subseteq \{(0,1),(1,1)\}^{f_2} \\
& \subseteq \{(0,1),(1,1),(2,2)\}^{f_3} \\
& \subseteq \{(0,1),(1,1),(2,2),(3,6)\}^{f_4} \\
& \subseteq \{(0,1),(1,1),(2,2),(3,6),(4,24)\}^{f_5} \\
& \subseteq \ldots
\end{array}
$$

$\bigcup_{i\in\mathbb{N}} f_i$ is (maybe) the factorial function

note: the limit of partial functions can be a total function

Total functions

$\mathbf{Tf}(A,B) = (A \to B)$ total functions

$\mathbf{Pf}(A,B) \equiv \mathbf{Tf}(A,B_\perp)$ $B_\perp \triangleq B \uplus \{\perp\}$

$\sqsubseteq_{B_\perp} \triangleq$ flat order

$$f \sqsubseteq g \Leftrightarrow \forall a \in A.\ f(a) \sqsubseteq_{B_\perp} g(a)$$

PO? immediate to check

bottom? $f_\perp(a) = \perp$ for any $a \in A$

complete? we will prove it later
(as an instance of a more general result)

$$(\bigsqcup_{i\in\mathbb{N}} f_i)(a) \triangleq \bigsqcup_{i\in\mathbb{N}} f_i(a)$$ (flat order, limit exists)

Monotone functions

Monotone function

$(D, \sqsubseteq_D)$ PO $(E, \sqsubseteq_E)$ PO $f : D \to E$

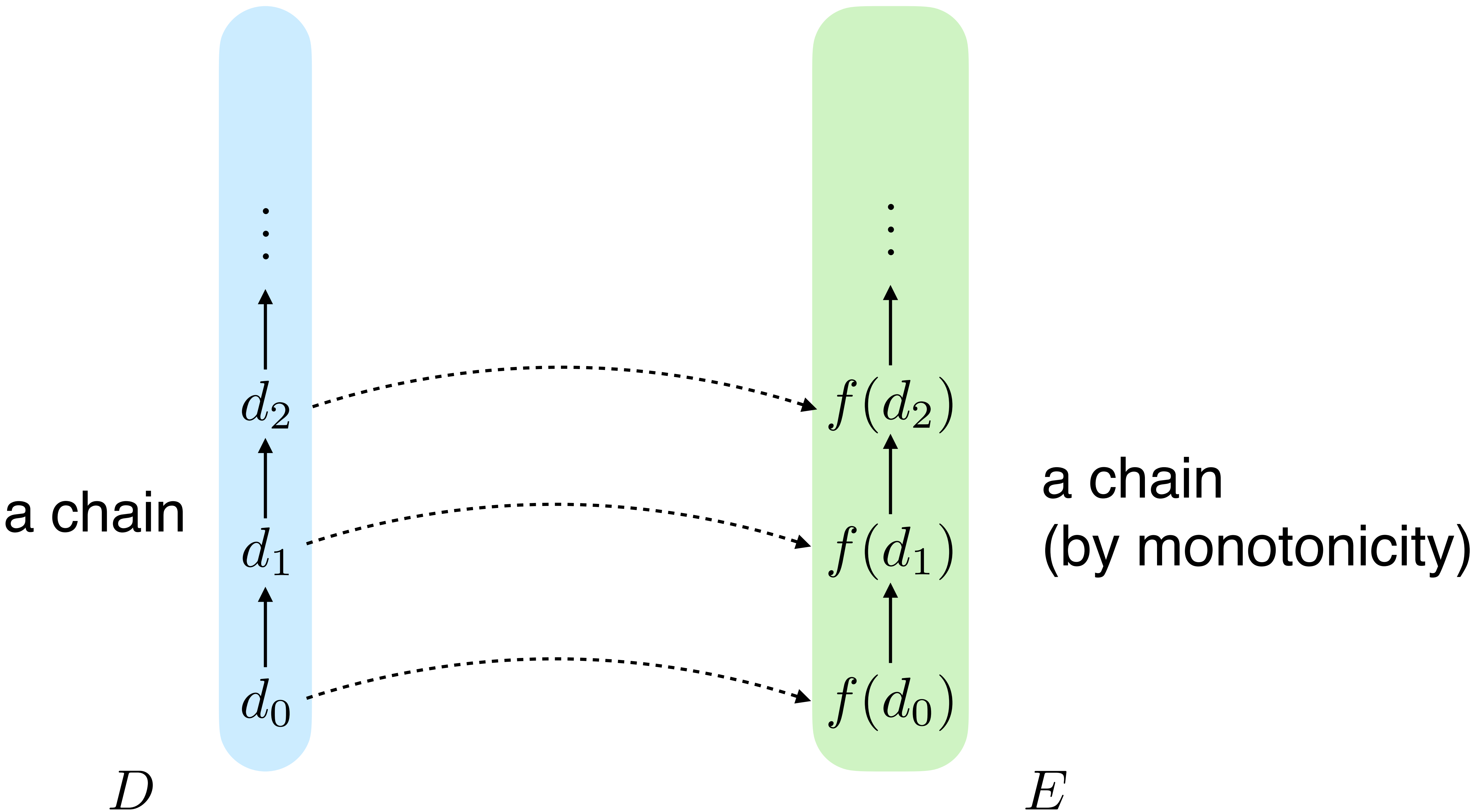
f is monotone if $\forall d_1, d_2 \in D.\ d_1 \sqsubseteq_D d_2 \Rightarrow f(d_1) \sqsubseteq_E f(d_2)$

Monotone = Order preserving

$$\left.\begin{array}{l} \{d_i\}_{i\in\mathbb{N}} \text{ a chain in } D \\ f \text{ monotone} \end{array}\right\} \Rightarrow \{f(d_i)\}_{i\in\mathbb{N}} \text{ a chain in } E$$

When $D = E$ we say $f : D \to D$ is a function on D

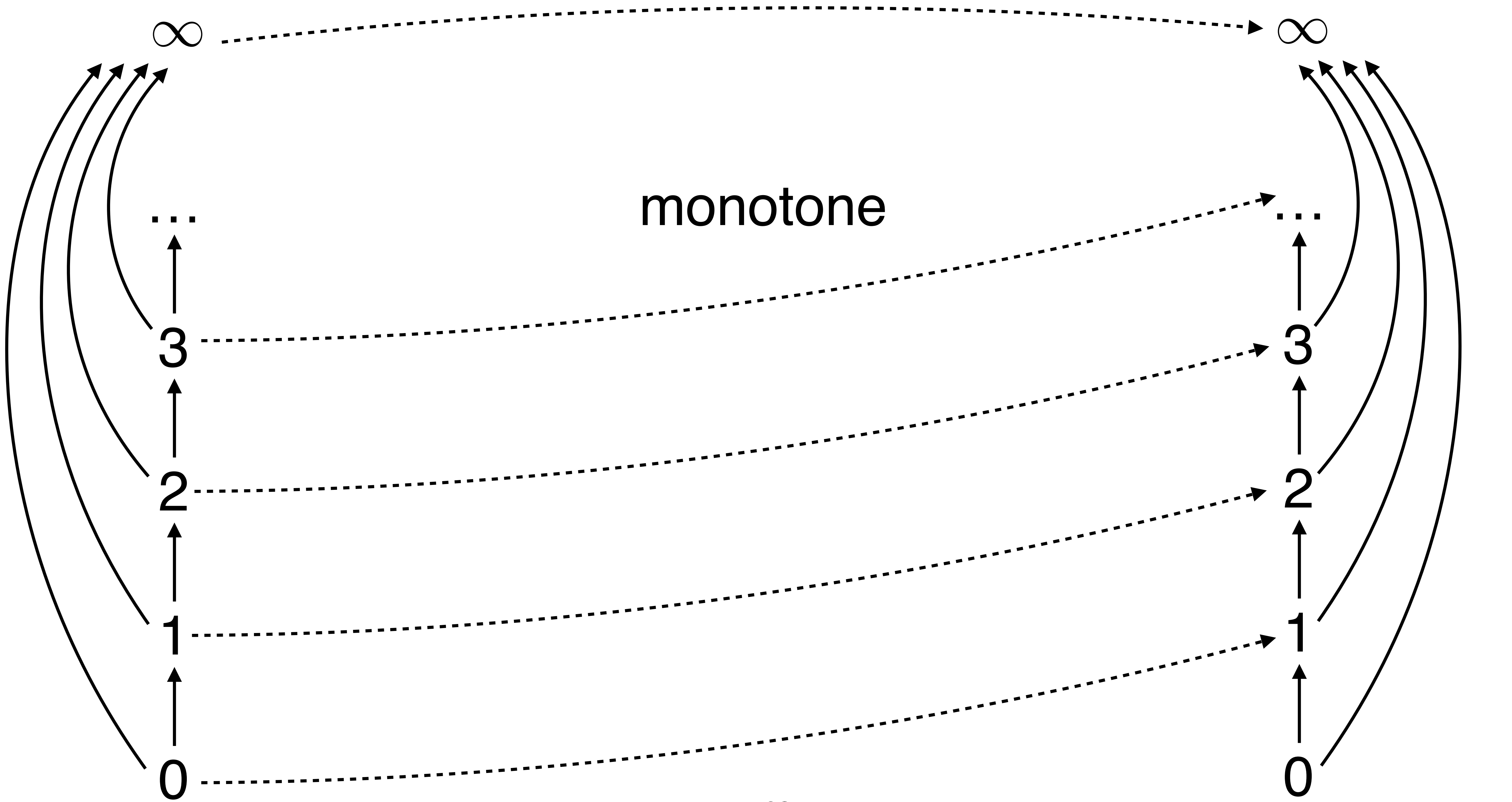
Monotonicity illustrated



Example

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$ $(\mathbb{N} \cup \{\infty\}, \leq)$

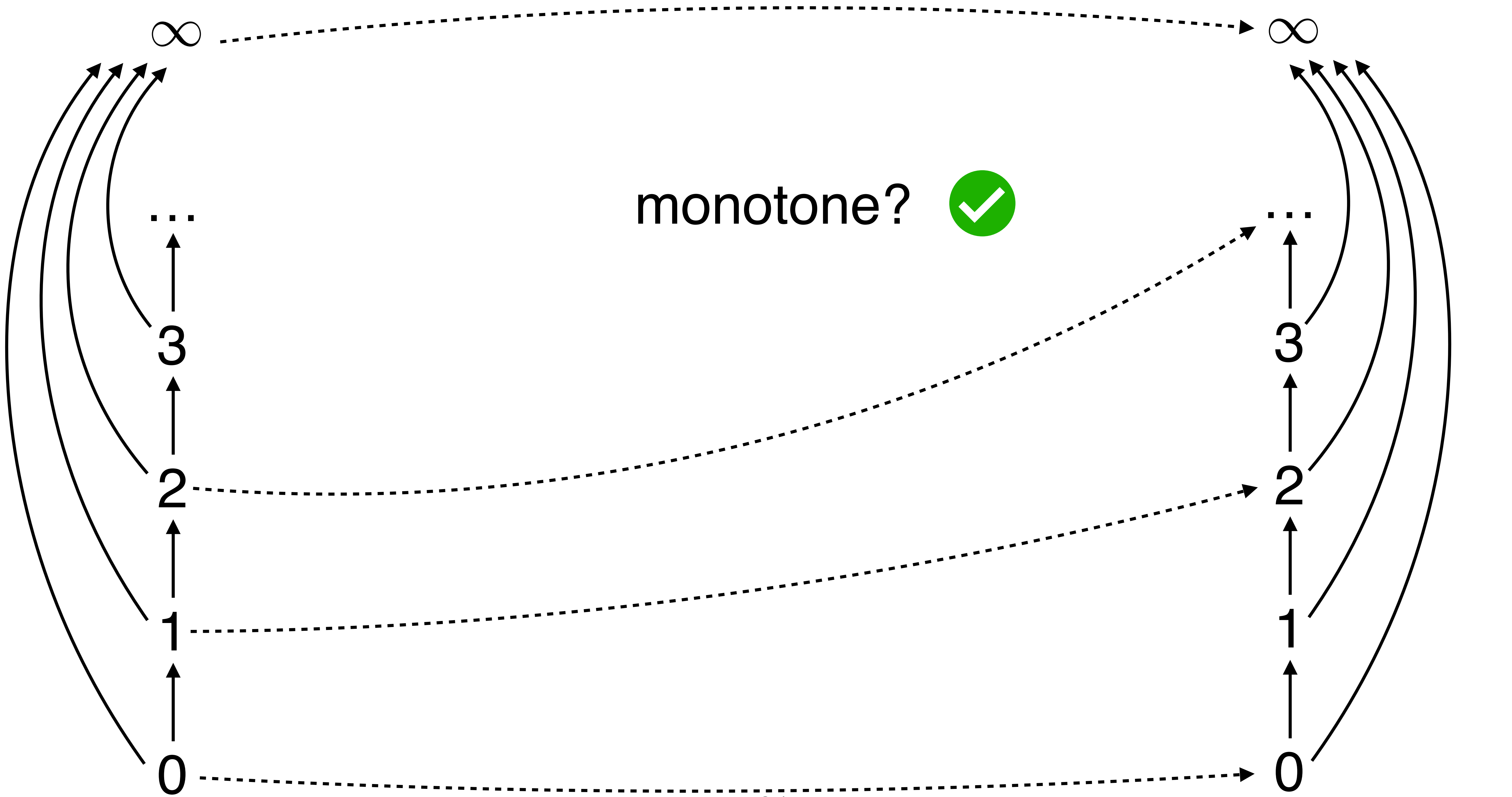


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$



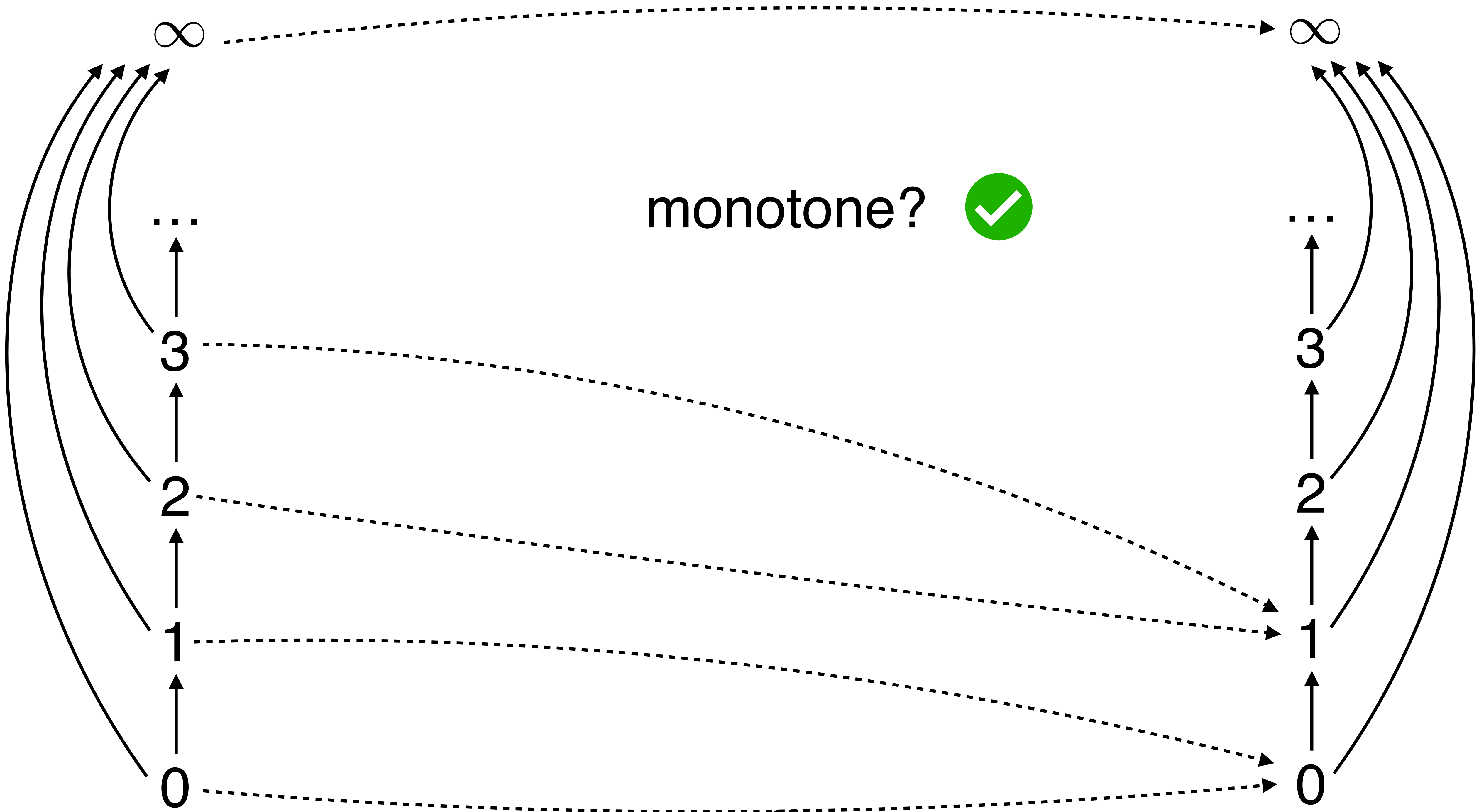
monotone?

Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n/2$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$



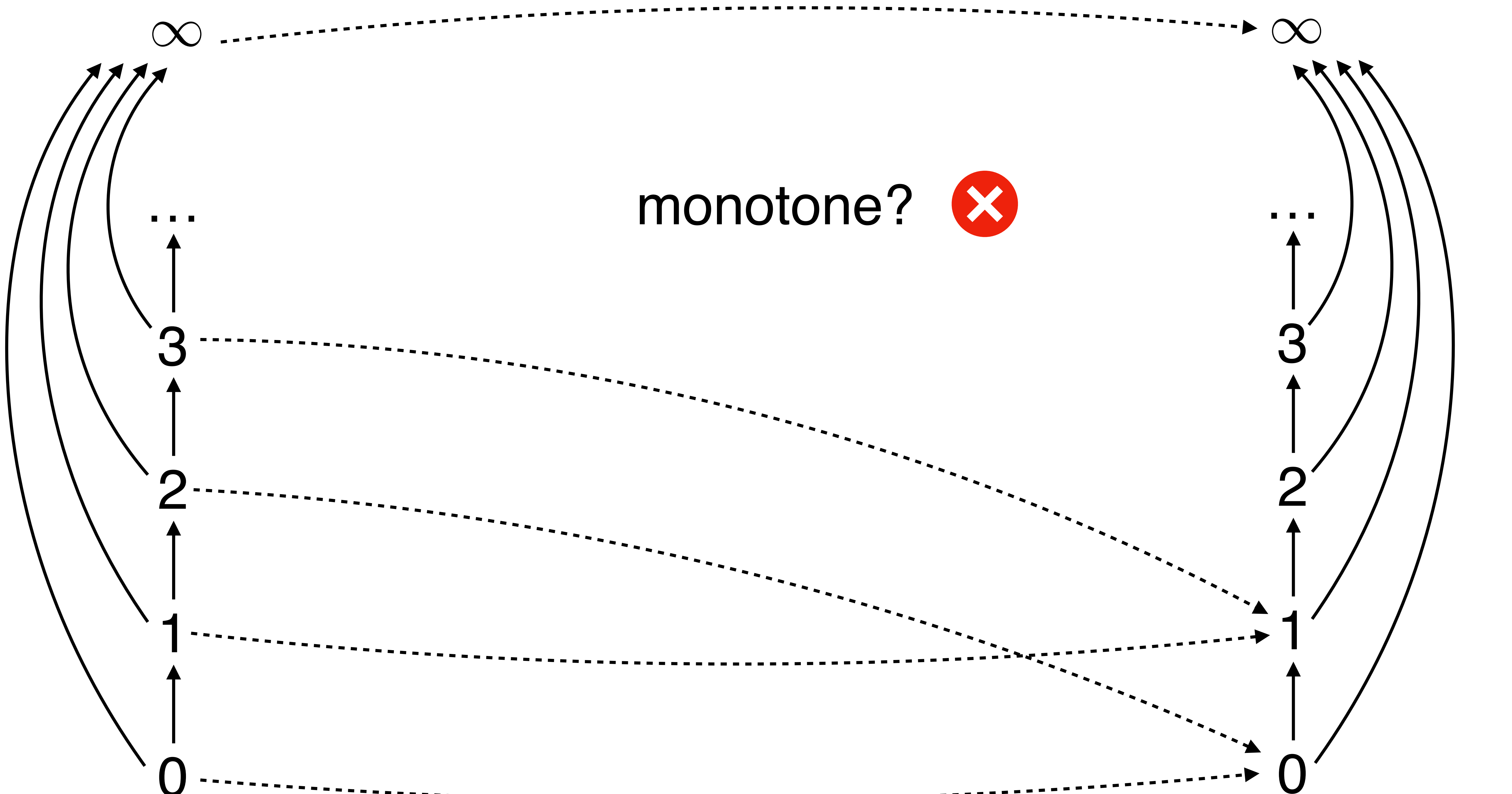
monotone?

Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

$$f(n) = n \ \% \ 2$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$

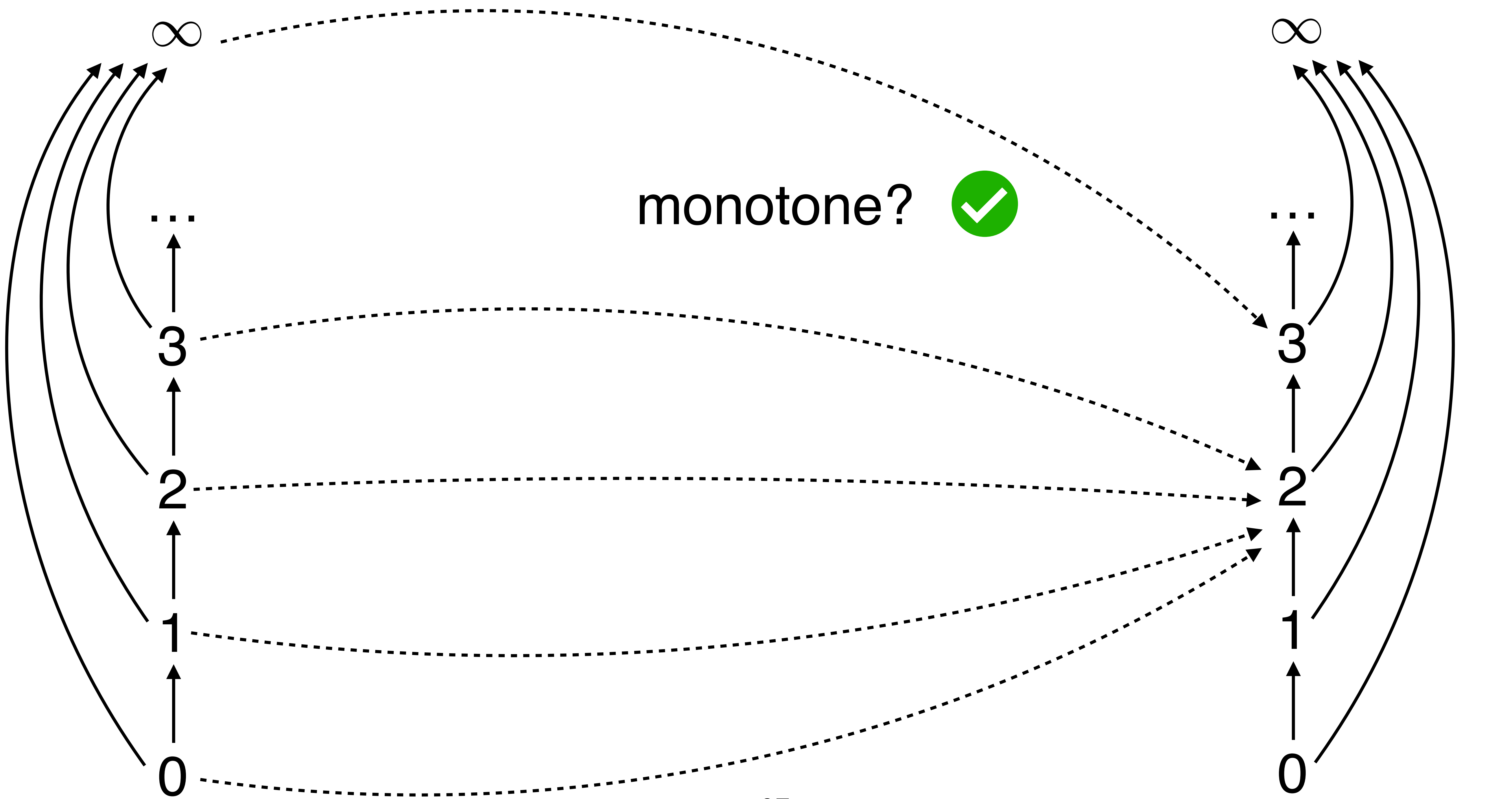


Exercise

$(\mathbb{N} \cup \{\infty\}, \leq)$

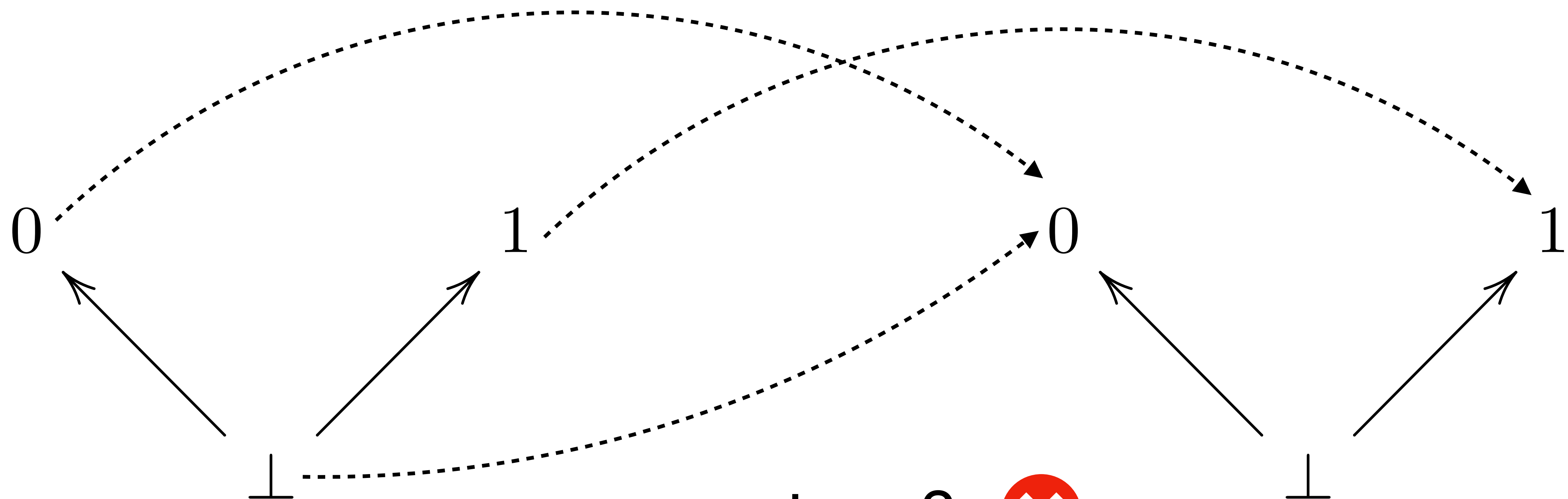
$$f(n) = 2$$
$$f(\infty) = 3$$

$(\mathbb{N} \cup \{\infty\}, \leq)$



monotone?

Exercise



monotone? ❌

D $\qquad f : D \to D \qquad$ D

$$f(\perp) = f(0) = 0$$
$$f(1) = 1$$

$\perp \sqsubseteq 1$ $\qquad\qquad$ $f(\perp) = 0 \not\sqsubseteq 1 = f(1)$

Composition

TH. Any composition of monotone function is monotone

$(D, \sqsubseteq_D)$ PO
$(E, \sqsubseteq_E)$ PO
$(F, \sqsubseteq_F)$ PO

$f : D \to E$ monotone
$g : E \to F$ monotone

$\Rightarrow$ $h = g \circ f : D \to F$ monotone

proof. we need to prove $\forall x, y \in D.\ x \sqsubseteq_D y \Rightarrow h(x) \sqsubseteq_F h(y)$

take $x \sqsubseteq_D y$

we want to prove $h(x) \sqsubseteq_F h(y)$

then $f(x) \sqsubseteq_E f(y)$ because f is monotone

then $g(f(x)) \sqsubseteq_F g(f(y))$ because g is monotone

$$\underbrace{g(f(x))}_{= \, h(x)} \sqsubseteq_F \underbrace{g(f(y))}_{= \, h(y)}$$

Continuous functions

Continuous function

$(D, \sqsubseteq_D)$ CPO $(E, \sqsubseteq_E)$ CPO $f : D \to E$ monotone

f is continuous if $\forall \{d_i\}_{i \in \mathbb{N}}.\ f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)$

chain

limit in D limit in E

Continuous = limit preserving

Continuity Illustrated

$$
\begin{tikzcd}
\text{lub} & \bigsqcup_{i\in\mathbb{N}} d_i \arrow[r, dashed] & f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) & \begin{array}{l}\text{u.b.}\\ \text{(by monotonicity)}\end{array} \\
& & \bigsqcup_{i\in\mathbb{N}} f(d_i) \arrow[u, "=?"'] & \text{lub} \\
& \vdots & \vdots & \\
& d_2 \arrow[u] \arrow[r, dashed] & f(d_2) \arrow[u] & \\
\text{a chain} & d_1 \arrow[u] \arrow[r, dashed] & f(d_1) \arrow[u] & \begin{array}{l}\text{a chain}\\ \text{(by monotonicity)}\end{array} \\
& d_0 \arrow[u] \arrow[r, dashed] & f(d_0) \arrow[u] &
\end{tikzcd}
$$

Continuity illustrated

lub

a chain

$$
\begin{tikzcd}
\bigsqcup_{i\in\mathbb{N}} d_i \arrow[r, dashed] & f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) \\
 & \bigsqcup_{i\in\mathbb{N}} f(d_i) \arrow[u, "=?"'] \\
\vdots & \vdots \\
d_2 \arrow[u] \arrow[r, dashed] & f(d_2) \arrow[u] \\
d_1 \arrow[u] \arrow[r, dashed] & f(d_1) \arrow[u] \\
d_0 \arrow[u] \arrow[r, dashed] & f(d_0) \arrow[u]
\end{tikzcd}
$$

$$\bigsqcup_{i\in\mathbb{N}} f(d_i) \sqsubseteq_E f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$$

follows from monotonicity (and CPO)

Continuity illustrated

$$
\begin{tikzcd}
\text{lub} & \bigsqcup_{i\in\mathbb{N}} d_i \arrow[r, dashed] & f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right) \\
 & & \bigsqcup_{i\in\mathbb{N}} f(d_i) \arrow[u, "=?"'] \\
 & \vdots & \vdots \\
 & d_2 \arrow[u] \arrow[r, dashed] & f(d_2) \arrow[u] \\
\text{a chain} & d_1 \arrow[u] \arrow[r, dashed] & f(d_1) \arrow[u] \\
 & d_0 \arrow[u] \arrow[r, dashed] & f(d_0) \arrow[u]
\end{tikzcd}
$$

$$\bigsqcup_{i\in\mathbb{N}} f(d_i) \overset{?}{\sqsupseteq}_E f\left(\bigsqcup_{i\in\mathbb{N}} d_i\right)$$

continuity

Lemma

$(D, \sqsubseteq_D)$ CPO
no infinite chains

$(E, \sqsubseteq_E)$ PO

$f : D \to E$
monotone

$\Rightarrow$

f
continuous

proof. Take a chain $\{d_i\}_{i \in \mathbb{N}}$

$\{d_i\}_{i \in \mathbb{N}}$ is finite $\Rightarrow$ $\exists k \in \mathbb{N}.\ \bigsqcup_{i \in \mathbb{N}} d_i = d_k$

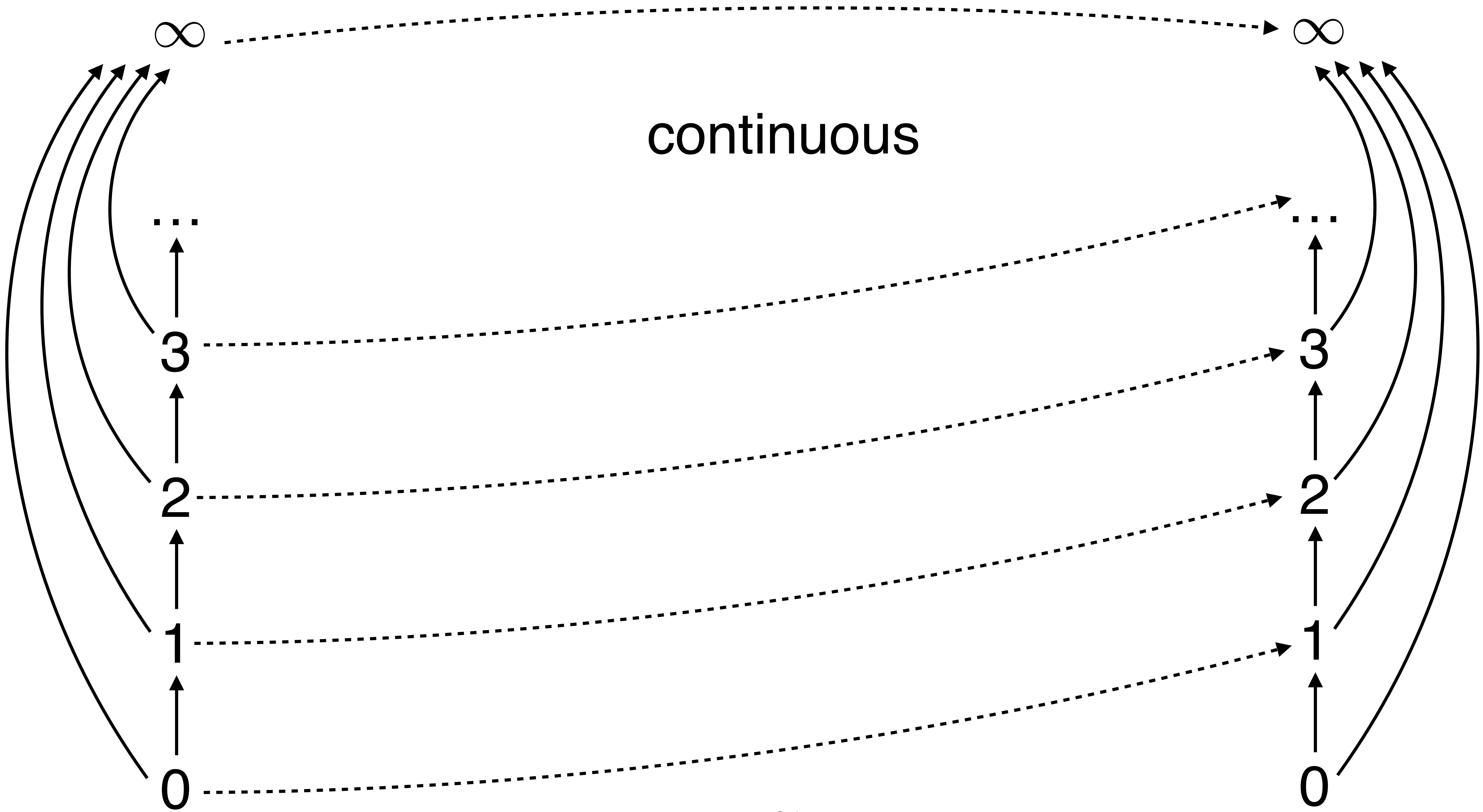
$\Downarrow$

$\{f(d_i)\}_{i \in \mathbb{N}}$ is finite $\Rightarrow$ $\bigsqcup_{i \in \mathbb{N}} f(d_i) = f(d_k) = f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right)$

Example

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

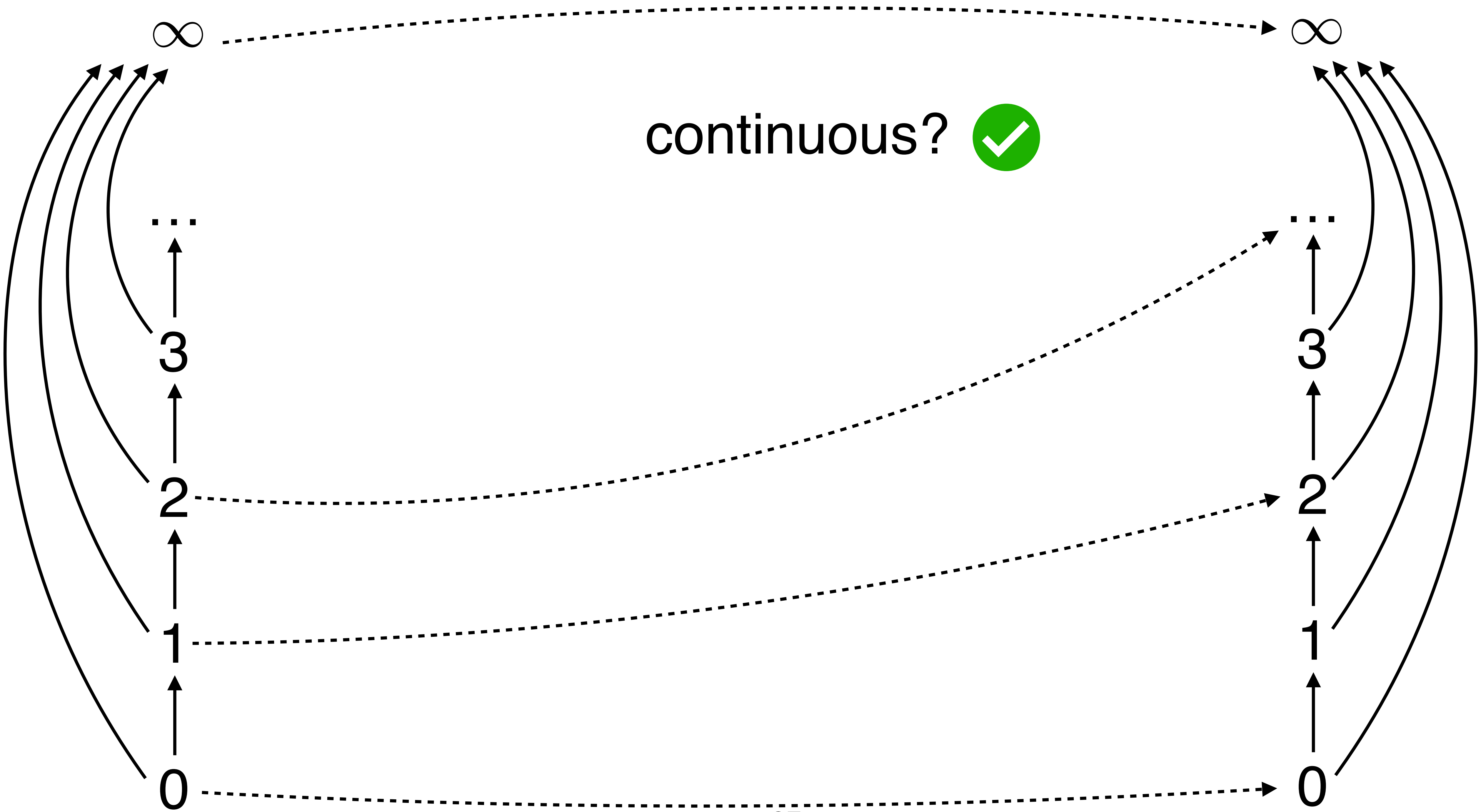
$(\mathbb{N} \cup \{\infty\}, \leq)$ $(\mathbb{N} \cup \{\infty\}, \leq)$



Exercise

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

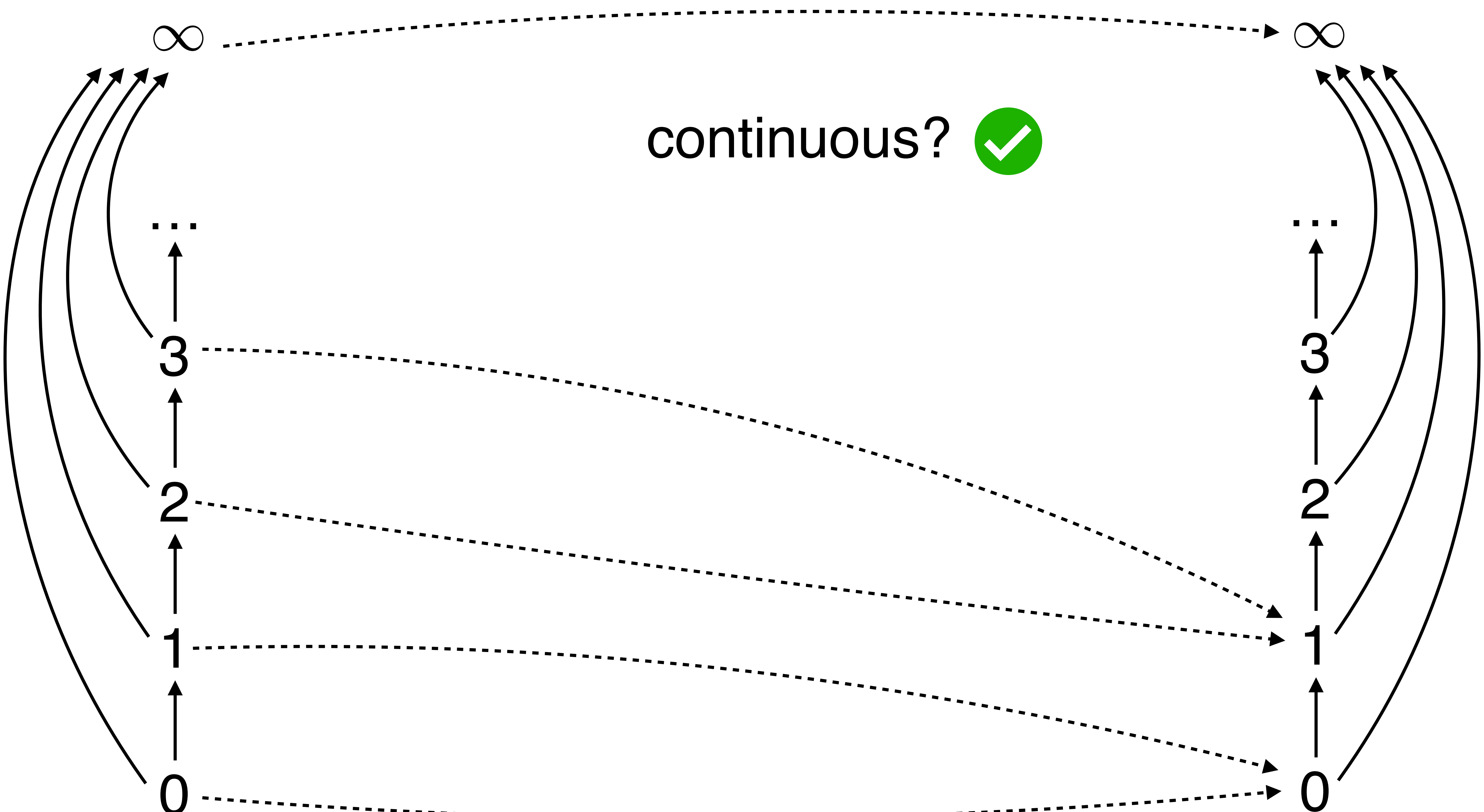
$(\mathbb{N} \cup \{\infty\}, \leq)$ $(\mathbb{N} \cup \{\infty\}, \leq)$



Example

$$f(n) = n/2$$
$$f(\infty) = \infty$$

$(\mathbb{N} \cup \{\infty\}, \leq)$ $(\mathbb{N} \cup \{\infty\}, \leq)$

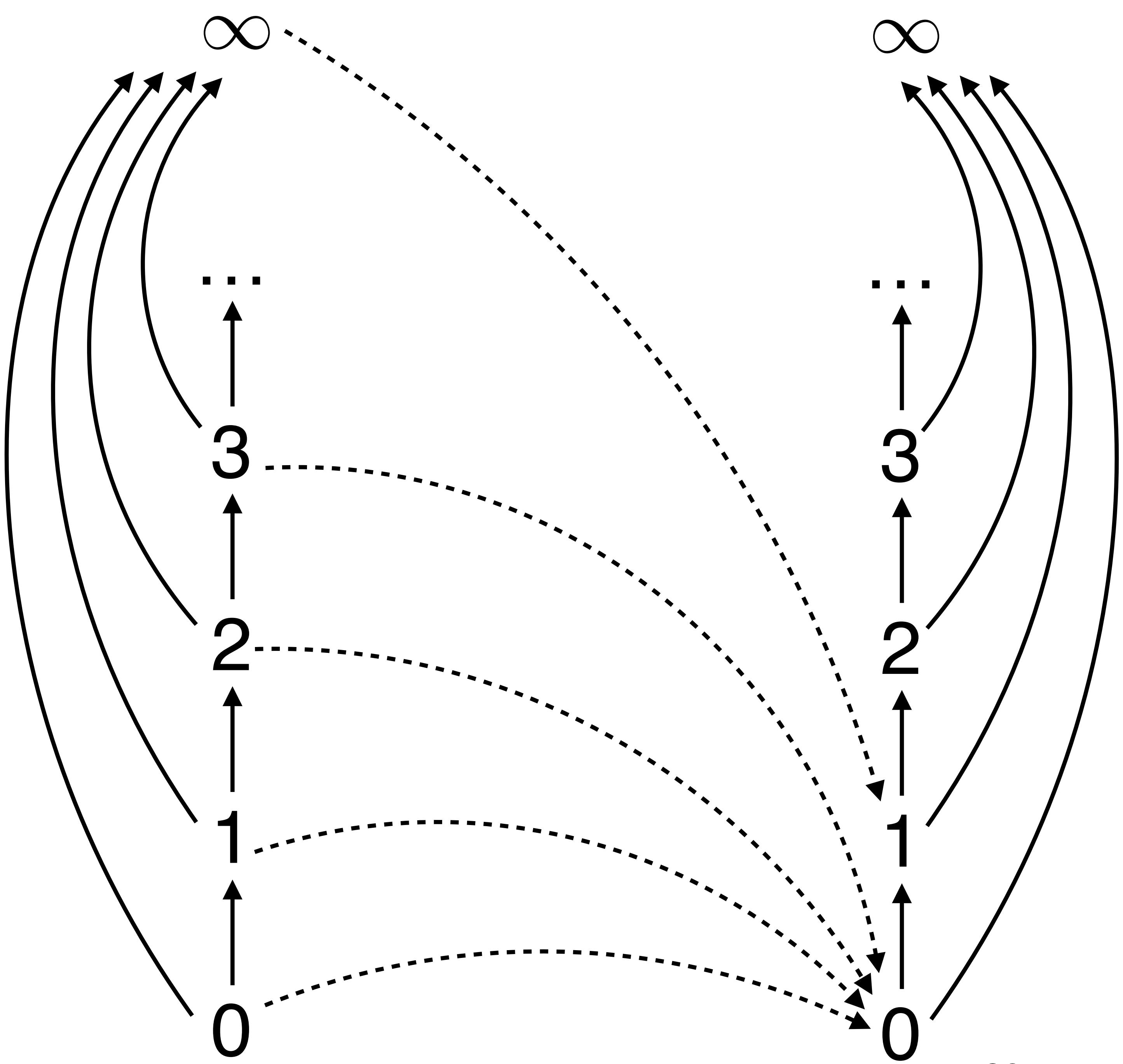


continuous? ✓

Example

$(\mathbb{N} \cup \{\infty\}, \leq)$

monotone function, not continuous



$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{N} \\ 1 & \text{if } x = \infty \end{cases}$$

$$d_i = 2 \cdot i \qquad \bigsqcup_{i \in \mathbb{N}} d_i = \infty$$

$$f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = f(\infty) = 1$$

$$f(d_i) = 0$$

$$\bigsqcup_{i \in \mathbb{N}} f(d_i) = \bigsqcup_{i \in \mathbb{N}} 0 = 0$$

Composition

TH. Any composition of continuous function is continuous

$(D, \sqsubseteq_D)$ CPO
$(E, \sqsubseteq_E)$ CPO
$(F, \sqsubseteq_F)$ CPO

$f : D \to E$ continuous
$g : E \to F$ continuous

$\Rightarrow$ $h = g \circ f : D \to F$ continuous

proof. take a chain $\{d_i\}_{i \in \mathbb{N}}$

we need to prove $h\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = \bigsqcup_{i \in \mathbb{N}} h(d_i)$

$$h\left(\bigsqcup_{i \in \mathbb{N}} d_i\right) = g\left(f\left(\bigsqcup_{i \in \mathbb{N}} d_i\right)\right) = g\left(\bigsqcup_{i \in \mathbb{N}} f(d_i)\right) = \bigsqcup_{i \in \mathbb{N}} g(f(d_i))$$

$$= \bigsqcup_{i \in \mathbb{N}} h(d_i)$$

Kleene's fixpoint theorem

Repeated application

$$f : D \to D$$

$$f^0(d) \triangleq d$$

$$f^{n+1}(d) \triangleq f(f^n(d))$$

$$f^n(d) = \overbrace{f(\cdots(f}^{n \text{ times}}(d))\cdots)$$

$$f^n : D \to D$$

Lemma

$(D, \sqsubseteq)$ $\mathrm{PO}_\perp$ $f : D \to D$ monotone $\Rightarrow$ $\{f^n(\perp)\}_{n\in\mathbb{N}}$ is a chain

proof. we need to prove $\forall n \in \mathbb{N}.\ f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

by mathematical induction $P(n) \triangleq f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

$P(0) \triangleq f^0(\perp) \sqsubseteq f^1(\perp)$ $f^0(\perp) = \perp \sqsubseteq f^1(\perp)$

$\forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)$ take a generic n

assume $P(n) \triangleq f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

we want to prove $P(n+1) \triangleq f^{n+1}(\perp) \sqsubseteq f^{n+2}(\perp)$

$$f^n(\perp) \sqsubseteq f^{n+1}(\perp)$$
$$\Downarrow$$
$$f^{n+1}(\perp) = f(f^n(\perp)) \sqsubseteq f(f^{n+1}(\perp)) = f^{n+2}(\perp)$$

Towards Kleene's theo.

when $(D, \sqsubseteq)$ is a $\mathrm{CPO}_\perp$

then $\{f^n(\perp)\}_{n\in\mathbb{N}}$ is a chain

it must have a limit

$\{f^n(d)\}_{n\in\mathbb{N}}$
not necessarily
a chain!

Kleene's fix point theorem states that
if f is continuous, then the limit of the above chain
is the least fixpoint of f

Pre-fixpoints

$(D, \sqsubseteq)$ PO $\qquad f : D \to D$ monotone

fixpoint $\qquad p \in D \qquad f(p) = p$

pre-fixpoint $\qquad p \in D \qquad f(p) \sqsubseteq p$

Clearly any fixpoint is also a pre-fixpoint

Kleene's theorem

$(D, \sqsubseteq)\ \mathrm{CPO}_\perp \quad f : D \to D$ continuous

let $\mathit{fix}(f) \triangleq \bigsqcup_{n \in \mathbb{N}} f^n(\perp)$

1. $\mathit{fix}(f)$ is a fix point of f

$$f(\mathit{fix}(f)) = \mathit{fix}(f)$$

2. $\mathit{fix}(f)$ is the least pre-fixpoint of f

$$\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d$$

if d is a pre-fixpoint then $\mathit{fix}(f)$ is smaller than d

Kleene's theorem: 1

1. $f(\mathit{fix}(f)) = \mathit{fix}(f)$

proof.

$$f(\mathit{fix}(f)) = f\left(\bigsqcup_{n\in\mathbb{N}} f^n(\bot)\right)$$ by def of fix

$$= \bigsqcup_{n\in\mathbb{N}} f(f^n(\bot))$$ by continuity

$$= \bigsqcup_{n\in\mathbb{N}} f^{n+1}(\bot)$$ by def of f^n

$$= \bigsqcup_{n\in\mathbb{N}} f^n(\bot)$$ by prefix independence of limits

$$= \mathit{fix}(f)$$ by def of fix

Kleene's theorem: 2

2. $\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d$

proof.

we prove that any pre-fixpoint is an upper bound of the chain

$$\{f^n(\perp)\}_{n\in\mathbb{N}}$$

by definition $\mathit{fix}(f)$ is the lub of the same chain

and thus smaller than any other upper bound

Kleene's theorem: 2

2. $\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d$

take any $d \in D$ such that $f(d) \sqsubseteq d$

we prove $\forall n \in \mathbb{N}.\ f^n(\bot) \sqsubseteq d$ (d is an upper bound)

$P(n) \triangleq f^n(\bot) \sqsubseteq d$ by mathematical induction

$P(0) \triangleq f^0(\bot) \sqsubseteq d$ $\quad f^0(\bot) = \bot \sqsubseteq d$

$\forall n \in \mathbb{N}.\ P(n) \Rightarrow P(n+1)$

take a generic n

assume $P(n) \triangleq f^n(\bot) \sqsubseteq d$

we want to prove $P(n+1) \triangleq f^{n+1}(\bot) \sqsubseteq d$

$$f^n(\bot) \sqsubseteq d$$

$$\text{(monot.)}\Downarrow$$

$$f^{n+1}(\bot) \overset{\text{(by def)}}{=} f(f^n(\bot)) \sqsubseteq f(d) \overset{\text{(pre-fixpoint)}}{\sqsubseteq} d$$

Example

$n = 2 \cdot n$ $(\mathbb{N} \cup \{\infty\}, \leq)$ $\bot = 0$ $\mathrm{CPO}_\bot$

$$f(n) = 2 \cdot n$$
$$f(\infty) = \infty$$

monotone? ok
continuous? ok

$$f^0(0) = 0$$

$$f^1(0) = f(0) = 2 \cdot 0 = 0$$

fixpoint reached!

Example

$n = n + 1$ $(\mathbb{N} \cup \{\infty\}, \leq)$ $\perp = 0$ $\mathrm{CPO}_\perp$

$$f(n) = n + 1$$
$$f(\infty) = \infty$$

monotone? ok
continuous? ok

$$f^0(0) = 0$$
$$f^1(0) = f(0) = 0 + 1 = 1$$
$$f^2(0) = f(f^1(0)) = f(1) = 1 + 1 = 2$$
$$f^3(0) = f(f^2(0)) = f(2) = 2 + 1 = 3$$

$$\bigsqcup_{n \in \mathbb{N}} f^n(0) = \bigsqcup_{n \in \mathbb{N}} n = \infty$$ fixpoint

Example

$X = X \cap \{1\}$ $(\wp(\mathbb{N}), \subseteq)$ $\bot = \emptyset$ $\mathrm{CPO}_\bot$

$f(X) = X \cap \{1\}$

monotone? ok

continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cap \{1\} = \emptyset$$

fixpoint reached!

Example

$X = \mathbb{N} \setminus X$ $\quad (\wp(\mathbb{N}), \subseteq)$ $\quad \bot = \emptyset \quad \mathrm{CPO}_\bot$

$$f(X) = \mathbb{N} \setminus X$$

monotone? NO

the larger X the smaller $f(X)$

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \mathbb{N} \setminus \emptyset = \mathbb{N}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\mathbb{N}) = \mathbb{N} \setminus \mathbb{N} = \emptyset$$

not a chain!

Example

$X = X \cup \{1\}$ $(\wp(\mathbb{N}), \subseteq)$ $\perp = \emptyset$ $\mathrm{CPO}_\perp$

$f(X) = X \cup \{1\}$

monotone? ok
continuous? ok

$$f^0(\emptyset) = \emptyset$$

$$f^1(\emptyset) = f(\emptyset) = \emptyset \cup \{1\} = \{1\}$$

$$f^2(\emptyset) = f(f^1(\emptyset)) = f(\{1\}) = \{1\} \cup \{1\} = \{1\}$$

fixpoint reached!

Badge exercise

Let D be a CPO

let $\{d_i\}_{i \in \mathbb{N}}$ be a chain in D

let $\{k_j\}_{j \in \mathbb{N}}$ be an infinite chain in $(\mathbb{N}, \leq)$

1. Prove that $\{d_{k_j}\}_{j \in \mathbb{N}}$ is a chain in D
2. Prove or disprove that $\bigsqcup_{j \in \mathbb{N}} d_{k_j} = \bigsqcup_{i \in \mathbb{N}} d_i$