PSC 2020/21 (375AA, 9CFU)

Principles for Software Composition

Roberto Bruni
http://www.di.unipi.it/~bruni/

http://didawiki.di.unipi.it/doku.php/magistraleinformatica/psc/

25b - CTMC
Exponential distribution
Probability law

Cumulative distribution function (probability law)

\[ X : \Omega \rightarrow \mathbb{R} \]

\[ F_X(x) \triangleq P(X \leq x) = P(\{\omega \in \Omega \mid X(\omega) \leq x\}) \]

\[ P(X \leq a) = F_X(a) \]

\[ P(X > a) = 1 - F_X(a) \]

\[ P(a < X \leq b) = F_X(b) - F_X(a) \]
Probability density

\[ X : \Omega \to \mathbb{R} \]

integrable \( f_X : \mathbb{R} \to [0, +\infty) \)

such that \( F_X(a) = \int_{-\infty}^{a} f_X(x) \, dx \)

i.e. \( P(a < X \leq b) = \int_{a}^{b} f_X(x) \, dx \)

note that \( P(X = a) \) is usually 0 when \( X \) is continuous
(Negative) Exp distribution

rate $\lambda$

$$f_X(x) \triangleq \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad F_X(x) \triangleq \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

![Graph showing exponential probability laws with different rates](image1)

![Graph showing exponential density distributions with different rates](image2)

expected value (mean) $1/\lambda$ \quad variance $1/\lambda^2$
Properties

memoryless \( P(X > a + b | X > a) = P(X > b) \)
(it is the only memoryless distribution)

\((X_1, \lambda_1) (X_2, \lambda_2)\) (independent, exponentially distributed)

\[ X(\omega) = \min\{X_1(\omega), X_2(\omega)\} \quad (X, \lambda_1 + \lambda_2) \]

\[ P(X \leq x) \triangleq 1 - e^{-(\lambda_1 + \lambda_2)x} \]
related to sojourn time
in CTMC
exploited in PEPA

\[ P(X_1 < X_2) \triangleq \frac{\lambda_1}{\lambda_1 + \lambda_2} \]
related to embedded DTMC
exploited in PEPA
(homogeneous) CTMC
(\Omega, \mathcal{A}, P)$ probability space

$\{X_t\}_{t \in \mathbb{R}}$ homogeneous Markov chain

$P(X_{t_n} + \Delta t = x | X_{t_n} = x_n, \cdots, X_{t_0} = x_0) = P(X_{\Delta t} = x | X_0 = x_n)$

let $T_i$ be the time spent in state $i$ before making a transition to any other state

$P(T_i > t + \Delta t | T_i > t) = P(T_i > \Delta t)$

$P(T_i > 15 | T_i > 10) = P(T_i > 5)$

the random variable $T_i$ is memoryless

it must be exponentially distributed

rate $\lambda_i$, mean $1/\lambda_i$
CTMC

\((\Omega, \mathcal{A}, P)\) probability space \(\{X_t\}_{t \in \mathbb{R}}\) homogeneous Markov chain

\[ P(X_{t_n + \Delta t} = x | X_{t_n} = x_n, \ldots, X_{t_0} = x_0) = P(X_{\Delta t} = x | X_0 = x_n) \]

let \(T_{i,j}\) be the time spent in state \(i\) before making a transitions to state \(j\)

\[ P(T_{i,j} > t + \Delta t | T_{i,j} > t) = P(T_{i,j} > \Delta t) \]

the random variable \(T_{i,j}\) is memoryless

it must be exponentially distributed
rate \(\lambda_{i,j}\), mean \(1/\lambda_{i,j}\)

\[ T_i = \min_{j \neq i} T_{i,j} \quad \lambda_i = \sum_{j \neq i} \lambda_{i,j} \]
Embedded DTMC

\((\Omega, \mathcal{A}, P)\) probability space

\(\{X_t\}_{t \in \mathbb{R}}\) homogeneous Markov chain

\[ T_i = \min_{j \neq i} T_{i,j} \]

\[ \lambda_i = \sum_{j \neq i} \lambda_{i,j} \]

\[ p_{i,j} = P \left( \bigwedge_{k \neq i,j} T_{i,j} < T_{i,k} \right) = P \left( T_{i,j} < \min_{k \neq i,j} T_{i,k} \right) = \frac{\lambda_{i,j}}{\lambda_i} \]

\[
P = \begin{bmatrix}
0 & p_{1,2} & \cdots & p_{1,N} \\
p_{2,1} & 0 & \cdots & p_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
p_{N,1} & p_{N,2} & \cdots & 0
\end{bmatrix}
= \begin{bmatrix}
0 & \lambda_{1,2}/\lambda_1 & \cdots & \lambda_{1,N}/\lambda_1 \\
\lambda_{2,1}/\lambda_2 & 0 & \cdots & \lambda_{2,N}/\lambda_2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{N,1}/\lambda_N & \lambda_{N,2}/\lambda_N & \cdots & 0
\end{bmatrix}
\]
Embedded DTMC

\((\Omega, \mathcal{A}, P)\) probability space

\(\{X_t\}_{t \in \mathbb{R}}\) homogeneous Markov chain

\[
P = \begin{bmatrix}
0 & p_{1,2} & \cdots & p_{1,N} \\
p_{2,1} & 0 & \cdots & p_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
p_{N,1} & p_{N,2} & \cdots & 0
\end{bmatrix} = \begin{bmatrix}
0 & \lambda_{1,2}/\lambda_1 & \cdots & \lambda_{1,N}/\lambda_1 \\
\lambda_{2,1}/\lambda_2 & 0 & \cdots & \lambda_{2,N}/\lambda_2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{N,1}/\lambda_N & \lambda_{N,2}/\lambda_N & \cdots & 0
\end{bmatrix}
\]

\[
\pi = \pi \cdot P
\]

\[
\sum_{i=1}^{N} \pi_i = 1
\]

(if ergodic) we get a steady state distribution

but the embedded DTMC ignores the amount of time spent in each state
Balance equations

\((\Omega, \mathcal{A}, P)\) probability space \(\{X_t\}_{t \in \mathbb{R}}\) homogeneous Markov chain

let \(p_i\) be the long time proportion of time spent in state \(i\) with respect to the time spent in other states

the flow in/out of each state \(i\) must balance

\[\lambda_i p_i\quad \text{outgoing flow}\]
rate at which transitions out of state \(i\) occur

\[\lambda_{k,i} p_k\quad \text{rate at which transitions into state } i \text{ occur from state } k\]

\[\sum_{k \neq i} \lambda_{k,i} p_k\quad \text{incoming flow}\]
rate at which transitions into state \(i\) occur from other states
Infinitesimal matrix gen

\( (\Omega, \mathcal{A}, P) \) probability space

\[ \{X_t\}_{t \in \mathbb{R}} \] homogeneous Markov chain

outgoing flow \( \lambda_i p_i = \sum_{k \neq i} \lambda_{k,i} p_k \) incoming flow

\[ \lambda_i = \sum_{j \neq i} \lambda_{i,j} \]

\[ Q = \begin{bmatrix}
    -\lambda_1 & \lambda_{1,2} & \cdots & \lambda_{1,N} \\
    \lambda_{2,1} & -\lambda_2 & \cdots & \lambda_{2,N} \\
    \vdots & \vdots & \ddots & \vdots \\
    \lambda_{N,1} & \lambda_{N,2} & \cdots & -\lambda_N
\end{bmatrix} \]

\[
\begin{aligned}
    p \cdot Q &= 0 \\
    \sum_{i=1}^{N} p_i &= 1
\end{aligned}
\]
Stationary distributions

\[ P = \begin{bmatrix}
0 & p_{1,2} & \cdots & p_{1,N} \\
p_{2,1} & 0 & \cdots & p_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
p_{N,1} & p_{N,2} & \cdots & 0
\end{bmatrix} \]

\[ Q = \begin{bmatrix}
-\lambda_1 & \lambda_{1,2} & \cdots & \lambda_{1,N} \\
\lambda_{2,1} & -\lambda_2 & \cdots & \lambda_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{N,1} & \lambda_{N,2} & \cdots & -\lambda_N
\end{bmatrix} \]

\[ \pi = \pi \cdot P \]

\[ \sum_{i=1}^{N} \pi_i = 1 \]

\[ p \cdot Q = 0 \]

\[ \sum_{i=1}^{N} p_i = 1 \]

\( \pi_i \) proportion of transitions into state \( i \)

\( 1/\lambda_i \) mean time spent into state \( i \)

if \( \forall i, j. \lambda_i = \lambda_j \) then \( \forall i. p_i = \pi_i \)

\[ p_i = \frac{\pi_i/\lambda_i}{\sum_j \pi_j/\lambda_j} \]
Example

A server can serve up to two requests
one/two new requests arrive with rate $\lambda$
one/two requests are served with rate $\mu$
represent the system as a CTMC, by defining its
infinitesimal generator matrix
embedded DTMC
find the steady state distribution when $\lambda = \mu$
Example

A server can serve up to two requests
one/two new requests arrive with rate $\lambda$
one/two requests are served with rate $\mu$
represent the system as a CTMC, by defining its
infinitesimal generator matrix
embedded DTMC
find the steady state distribution when $\lambda = \mu$

\[
Q = \begin{bmatrix}
\end{bmatrix}
\quad P = \begin{bmatrix}
\end{bmatrix}
\]
Example

A server can serve up to two requests
one/two new requests arrive with rate $\lambda$
one/two requests are served with rate $\mu$
represent the system as a CTMC, by defining its
infinitesimal generator matrix
embedded DTMC
find the steady state distribution when $\lambda = \mu$

\[
Q = \begin{bmatrix}
-2\lambda & \lambda & \lambda \\
\mu & -\lambda - \mu & \lambda \\
\mu & \mu & -2\mu
\end{bmatrix}
\]

\[
P = \begin{bmatrix}
0 & 1/2 & 1/2 \\
\mu/\lambda + \mu & 0 & \lambda/\lambda + \mu \\
1/2 & 1/2 & 0
\end{bmatrix}
\]
Example

find the steady state distribution when \( \lambda = \mu \)

\[
Q = \begin{bmatrix}
-2\lambda & \lambda & \lambda & \lambda \\
\lambda & -2\lambda & \mu & \lambda \\
\lambda & \lambda & -2\lambda & 2\mu \\
\lambda & \lambda & -2\lambda & 2\mu
\end{bmatrix}
\]

\[
P = \begin{bmatrix}
0 & 0 & 1/2 & 1/2 \\
\mu/2\lambda + \rho & 0 & 0/2 & \lambda/\lambda + \mu \\
1/2 & 21/2 & 1/0 & 0
\end{bmatrix}
\]
Example

find the steady state distribution when $\lambda = \mu$

\[
Q = \begin{bmatrix}
-2\lambda & \lambda & \lambda \\
\lambda & -2\lambda & \lambda \\
\lambda & \lambda & -2\lambda
\end{bmatrix}
\quad P = \begin{bmatrix}
0 & 1/2 & 1/2 \\
1/2 & 0 & 1/2 \\
1/2 & 1/2 & 0
\end{bmatrix}
\]

\[
\begin{align*}
-2\lambda p_1 + \lambda p_2 + \lambda p_3 &= 0 \\
\lambda p_1 - 2\lambda p_2 + \lambda p_3 &= 0 \\
\lambda p_1 + \lambda p_2 - 2\lambda p_3 &= 0 \\
p_1 + p_2 + p_3 &= 1
\end{align*}
\]

\[
p = \begin{bmatrix}
1/3 \\
1/3 \\
1/3
\end{bmatrix}
\quad \pi = \begin{bmatrix}
1/3 \\
1/3 \\
1/3
\end{bmatrix}
\]
CTMC as LTS

\[ \alpha_C : S \rightarrow S \rightarrow \mathbb{R} \]

embedded DTMC

\[ \alpha_D : S \rightarrow \mathbb{D}(S) \]

\[ \alpha_D i j = \begin{cases} 
\frac{\lambda_{i,j}}{\sum_{k \neq i} \lambda_{i,k}} & \text{if } i \neq j \\
0 & \text{otherwise}
\end{cases} \]

can we derive a notion of equivalence between states?