PSC 2020/21 (375AA, 9CFU)
Principles for Software Composition
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13c - Continuity theorems
Lifted Domains
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Lifted Domains

\[ D = (D, \sqsubseteq_D) \text{ CPO} \Rightarrow D_\bot = (D_\bot, \sqsubseteq_{D_\bot}) \]

\[ D_\bot \triangleq \{ \bot \} \uplus D = \{(0, \bot)\} \uplus (\{1\} \times D) = \{(0, \bot)\} \uplus \{(1, d) \mid d \in D\} \]

\[ \bot_{D_\bot} \triangleq (0, \bot) \]

\[ \cdot : D \to D_\bot \]

\[ [d] \triangleq (1, d) \]

how to order lifted elements?

\[ \forall x \in D_\bot. \; \bot_{D_\bot} \sqsubseteq_{D_\bot} x \]

\[ \forall d_1, d_2 \in D. \; [d_1] \sqsubseteq_{D_\bot} [d_2] \iff d_1 \sqsubseteq_D d_2 \]
Example

\((\mathbb{Z}, =)\)

\[ \ldots \cdots \quad -1 \quad 0 \quad 1 \quad \ldots \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ \mathbb{Z}_{\perp} \quad \mathbb{Z}_{\perp} \]

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Lifted Domains

**TH.** \[ D_{\perp} = (D_{\perp}, \sqsubseteq_{D_{\perp}}) \text{ CPO}_{\perp} \]

try on your own to prove:
- PO,
- bottom element,

complete observe that: \[ \bigsqcup_{i \in \mathbb{N}} [d_i] = \bigsqcup_{i \in \mathbb{N}} d_i \]

it is an upper bound
it is the least upper bound
Lifting operator

\[(D, \sqsubseteq_D) \text{ CPO} \quad (E, \sqsubseteq_E) \text{ CPO}_\perp\]

\[(\cdot)^*: [D \to E] \to [D_\perp \to E]\]

\[\forall f \in [D \to E]. \quad f^*(x) \triangleq \begin{cases} 
\perp_E & \text{if } x = \perp_{D_\perp} \\
 f(d) & \text{if } x = [d]
\end{cases}\]

for the definition to be well-given we need to prove:

\[f \in [D \to E] \implies f^* \in [D_\perp \to E]\]

\[f \text{ continuous} \implies f^* \text{ continuous}\]
**TH.** the lifting operator is well-defined

**proof.** assume $f$ continuous, take a chain $\{x_n\}_{n \in \mathbb{N}}$ in $D_\perp$

we need to prove $f^* \left( \bigsqcup_{n \in \mathbb{N}} x_n \right) = \bigsqcup_{n \in \mathbb{N}} f^*(x_n)$

if $\forall n \in \mathbb{N}. \ x_n = \bot_{D_\perp}$ then it is obvious

otherwise, let $k = \min\{i \mid x_i \neq \bot_{D_\perp}\}$

then $\forall m \geq k. \ \exists d_m \in D. \ x_m = \lfloor d_m \rfloor$

and by prefix independence of lub

$\bigsqcup_{n \in \mathbb{N}} x_n = \bigsqcup_{n \in \mathbb{N}} x_{n+k}$

$\bigsqcup_{n \in \mathbb{N}} f^*(x_n) = \bigsqcup_{n \in \mathbb{N}} f^*(x_{n+k})$

we can just prove $f^* \left( \bigsqcup_{n \in \mathbb{N}} x_{n+k} \right) = \bigsqcup_{n \in \mathbb{N}} f^*(x_{n+k})$

(see next slide)
(continue) \[ f^* \left( \bigsqcup_{n \in \mathbb{N}} x_{n+k} \right) = \bigsqcup_{n \in \mathbb{N}} f^*(x_{n+k}) \]

\[ f^* \left( \bigsqcup_{n \in \mathbb{N}} x_{n+k} \right) = f^* \left( \bigsqcup_{n \in \mathbb{N}} \lfloor d_{n+k} \rfloor \right) \]

\[ = f^* \left( \bigsqcup_{n \in \mathbb{N}} d_{n+k} \right) \]

\[ = f \left( \bigsqcup_{n \in \mathbb{N}} d_{n+k} \right) \]

\[ = \bigsqcup_{n \in \mathbb{N}} f(d_{n+k}) \]

\[ = \bigsqcup_{n \in \mathbb{N}} f^*(\lfloor d_{n+k} \rfloor) \]

\[ = \bigsqcup_{n \in \mathbb{N}} f^*(x_{n+k}) \]

by def of \( k \)

by lub in a lifted domain

by def of lifting

by continuity of \( f \)

by def of lifting

by def of \( k \)
\textbf{TH.} $(\cdot)^*$ is monotone

(try to prove on your own)

\textbf{TH.} $(\cdot)^*$ is continuous

\textit{proof.} take a chain of continuous functions $\{f_i : D \to E\}_{i \in \mathbb{N}}$

we need to prove $\left( \bigcup_{i \in \mathbb{N}} f_i \right)^* = \bigcup_{i \in \mathbb{N}} f_i^*$

take a generic $x \in D_{\bot}$

we need to prove $\left( \bigcup_{i \in \mathbb{N}} f_i \right)^*(x) = \left( \bigcup_{i \in \mathbb{N}} f_i^* \right)(x)$

if $x = \bot_{D_{\bot}}$ it is obvious

if $x = \lfloor d \rfloor$ we have... (see next slide)
(continue)

\[
\left( \bigsqcup_{i \in \mathbb{N}} f_i \right)^* ([d]) = \left( \bigsqcup_{i \in \mathbb{N}} f_i^* \right)([d])
\]

by def of lifting

\[
\left( \bigsqcup_{i \in \mathbb{N}} f_i \right)^* ([d]) = \left( \bigsqcup_{i \in \mathbb{N}} f_i \right)(d)
\]

by def of lifting

\[
= \bigsqcup_{i \in \mathbb{N}} f_i(d)
\]

by lub in a functional domain

\[
= \bigsqcup_{i \in \mathbb{N}} f_i^*([d])
\]

by def of lifting

\[
= \left( \bigsqcup_{i \in \mathbb{N}} f_i^* \right)([d])
\]

by lub in a functional domain
Let notation (de-lifting)

\[ (E, \sqsubseteq_E) \text{ CPO}_\perp \lambda x. e \in [D \to E] \quad t \in D_\perp \]

\[
\text{let } x \leftarrow t. e \triangleq (\lambda x. e)^* \frac{t}{D_\perp} = \begin{cases} \\
\downarrow_E & \text{if } t = \downarrow_D_\perp \\
\left[ d/x \right] e & \text{if } t = [d] \\
\end{cases}
\]

intuitively:

if \( t \) is a lifted value \([d]\) then we de-lift the value and assign it to \( x \) in \( e \)

otherwise returns \( \downarrow_E \)
Continuity theorems
TH. \((D, \sqsubseteq_D)\) CPO \((E_i, \sqsubseteq_{E_i})\)

\[ f : D \rightarrow E_1 \times E_2 \quad g_i \triangleq \pi_i \circ f \]

\(f\) is continuous \iff \(g_1, g_2\) are continuous

**proof.** \(\Rightarrow\) \(f\) is continuous \implies \(g_i\) is continuous

\(\pi_i\) is continuous

\(\Leftarrow\) note that \(\forall d \in D. \ f(d) = (g_1(d), g_2(d))\)

assume \(g_1, g_2\) are continuous

we want to prove \(f\) is continuous

take a chain \(\{d_i\}_{i \in \mathbb{N}}\) in \(D\)

we must prove \(f \left( \bigsqcup_{i \in \mathbb{N}} d_i \right) = \bigsqcup_{i \in \mathbb{N}} f(d_i)\)

(see next slide)
(continue) \[ f \left( \bigcup_{i \in \mathbb{N}} d_i \right) = \bigcup_{i \in \mathbb{N}} f(d_i) \]

\[ f \left( \bigcup_{i \in \mathbb{N}} d_i \right) = \left( g_1 \left( \bigcup_{i \in \mathbb{N}} d_i \right), g_2 \left( \bigcup_{i \in \mathbb{N}} d_i \right) \right) \text{ by def } g_1, g_2 \]

\[ = \left( \bigcup_{i \in \mathbb{N}} g_1(d_i), \bigcup_{i \in \mathbb{N}} g_2(d_i) \right) \quad g_1, g_2 \text{ are continuous} \]

\[ = \bigcup_{i \in \mathbb{N}} (g_1(d_i), g_2(d_i)) \quad \text{by def of lub of pairs} \]

\[ = \bigcup_{i \in \mathbb{N}} f(d_i) \quad \text{by def } g_1, g_2 \]
**TH.**

\[(D, \sqsubseteq_D) \quad (E, \sqsubseteq_E) \quad (F, \sqsubseteq_F)\]

CPO

\(f : D \times E \to F\)

\(f_d : E \to F\)

\(f_e : D \to F\)

\(f_d \triangleq \lambda e. f(d, e)\)

\(f_e \triangleq \lambda d. f(d, e)\)

\(f\) is continuous \iff \(\forall d \in D. f_d\) are continuous

\(\forall e \in E. f_e\) are continuous

**proof.** \(\Rightarrow\) assume \(f\) is continuous

take a generic \(d \in D\)

we want to prove \(f_d\) is continuous

take a chain \(\{e_i\}_{i \in \mathbb{N}}\) in \(E\)

we prove \(f_d \left( \bigsqcup_{i \in \mathbb{N}} e_i \right) = \bigsqcup_{i \in \mathbb{N}} f_d(e_i)\)

(see next slide)
\[(continue)\]

\[
f_d \left( \bigsqcup_{i \in \mathbb{N}} e_i \right) = \bigsqcup_{i \in \mathbb{N}} f_d(e_i)
\]

\[
f_d \left( \bigsqcup_{i \in \mathbb{N}} e_i \right) = f \left( d, \bigsqcup_{i \in \mathbb{N}} e_i \right)
\]

\[
= f \left( \bigsqcup_{i \in \mathbb{N}} d, \bigsqcup_{i \in \mathbb{N}} e_i \right)
\]

\[
= f \left( \bigsqcup_{i \in \mathbb{N}} (d, e_i) \right)
\]

\[
= \bigsqcup_{i \in \mathbb{N}} f(d, e_i)
\]

\[
= \bigsqcup_{i \in \mathbb{N}} f_d(e_i)
\]

by def of \( f_d \)

by lub of constant chain

by lub of pairs

by continuity of \( f \)

by def of \( f_d \)
(D, \sqsubseteq_D) \quad (E, \sqsubseteq_E) \quad (F, \sqsubseteq_F)

\text{TH. CPO} \quad f : D \times E \rightarrow F

\begin{align*}
\forall d \in D. \quad & f_d : E \rightarrow F \\
\forall e \in E. \quad & f_e : D \rightarrow F
\end{align*}

\begin{align*}
& f_d \triangleq \lambda e. \ f(d, e) \\
& f_e \triangleq \lambda d. \ f(d, e)
\end{align*}

\begin{align*}
\forall d \in D. \quad & f_d \text{ are continuous} \\
\forall e \in E. \quad & f_e \text{ are continuous}
\end{align*}

\langle \quad \text{assume } f_d, f_e \text{ are continuous for all } d, e \quad \rangle

\langle \quad \text{we want to prove } f \text{ is continuous} \quad \rangle

\langle \quad \text{take a chain } \{(d_k, e_k)\}_{k \in \mathbb{N}} \text{ in } D \times E \quad \rangle

\langle \quad \text{we prove } f \left( \bigsqcup_{k \in \mathbb{N}} (d_k, e_k) \right) = \bigsqcup_{k \in \mathbb{N}} f(d_k, e_k) \quad \rangle

\langle \quad \text{(see next slide)} \quad \rangle
(continue) \[ f \left( \bigsqcup_{k \in \mathbb{N}} (d_k, e_k) \right) = \bigsqcup_{k \in \mathbb{N}} f(d_k, e_k) \]

\[ f(\bigsqcup_k (d_k, e_k)) = f(\bigsqcup_i d_i, \bigsqcup_j e_j) \quad \text{by def of lub of pairs} \]

\[ = f_d(\bigsqcup_j e_j) \quad \text{by def of } f_d \text{ with } d \triangleq \bigsqcup_i d_i \]

\[ = \bigsqcup_j f_d(e_j) \quad \text{by continuity of } f_d \]

\[ = \bigsqcup_j f(d, e_j) \quad \text{by def of } f_d \]

\[ = \bigsqcup_j f_{e_j}(d) \quad \text{by def of } f_{e_j} \]

\[ = \bigsqcup_j f_{e_j}(\bigsqcup_i d_i) \quad \text{by def of } d \triangleq \bigsqcup_i d_i \]

\[ = \bigsqcup_j \bigsqcup_i f_{e_j}(d_i) \quad \text{by continuity of } f_{e_j} \]

\[ = \bigsqcup_j \bigsqcup_i f(d_i, e_j) \quad \text{by def of } f_{e_j} \]

\[ = \bigsqcup_k f(d_k, e_k) \quad \text{by switch lemma (applicable?)} \]
(continue) \[ f \left( \bigcup_{k \in \mathbb{N}} (d_k, e_k) \right) = \bigcup_{k \in \mathbb{N}} f(d_k, e_k) \]

\[
\text{if } i \leq n \land j \leq m \text{ then } f(d_i, e_j) \sqsubseteq f(d_n, e_m) \quad \checkmark
\]

\[ \downarrow \]
\[
d_i \sqsubseteq_D d_n \land e_j \sqsubseteq_E e_m
\]

\[
f(d_i, e_j) = f_{d_i}(e_j) \sqsubseteq f_{d_i}(e_m) = f(d_i, e_m) = f_{e_m}(d_i) \sqsubseteq f_{e_m}(d_n) = f(d_n, e_m)
\]

\[
f_{d_i} \quad \text{monotone} \quad f_{e_m} \quad \text{monotone}
\]

\[
= \bigcup_j \bigcup_i f(d_i, e_j)
\]

\[
= \bigcup_k f(d_k, e_k) \quad \text{by switch lemma (applicable?)}
\]
Some important functions
Apply

\[(D, \sqsubseteq_D) \quad \text{CPO} \quad \text{apply} : [D \rightarrow E] \times D \rightarrow E\]

\[\text{apply}(f, d) \triangleq f(d)\]

**TH.** \textit{apply} is monotone

(try to prove on your own)

**TH.** \textit{apply} is continuous

\textit{proof.} from a previous theorem, we prove continuity on each parameter separately \(apply_f\quad apply_d\)

1. for any \(f \in [D \rightarrow E]\) \(apply_f \triangleq \lambda d. \ f(d)\) is continuous
2. for any \(d \in D\) \(apply_d \triangleq \lambda f. \ f(d)\) is continuous

(see next slides)
1. for any $f \in [D \to E]$ \quad apply_f \triangleq \lambda d. \ f(d) \quad \text{is continuous}

take \quad f \in [D \to E] \quad \text{and a chain} \quad \{d_i\}_{i \in \mathbb{N}} \quad \text{in} \quad D

we want to prove \quad apply_f \left( \bigsqcup_i d_i \right) = \bigsqcup_i apply_f(d_i)

apply_f(\bigsqcup_i d_i) = apply(f, \bigsqcup_i d_i) \quad \text{by def of} \quad apply_f

= f(\bigsqcup_i d_i) \quad \text{by def of} \quad apply

= \bigsqcup_i f(d_i) \quad \text{by continuity of} \quad f

= \bigsqcup_i apply(f, d_i) \quad \text{by def of} \quad apply

= \bigsqcup_i apply_f(d_i) \quad \text{by def of} \quad apply_f
2. for any $d \in D$ \( \text{apply}_d \triangleq \lambda f. f(d) \) is continuous

take $d \in D$ and a chain \( \{f_i\}_{i \in \mathbb{N}} \) in \([D \to E]\)

we want to prove \( \text{apply}_d \left( \bigsqcup_i f_i \right) = \bigsqcup_i \text{apply}_d(f_i) \)

\[
\text{apply}_d(\bigsqcup_i f_i) = \text{apply}(\bigsqcup_i f_i, d) \quad \text{by def of } \text{apply}_d \\
= (\bigsqcup_i f_i)(d) \quad \text{by def of } \text{apply} \\
= \bigsqcup_i f_i(d) \quad \text{by def of lub of functions} \\
= \bigsqcup_i \text{apply}(f_i, d) \quad \text{by def of } \text{apply} \\
= \bigsqcup_i \text{apply}_d(f_i) \quad \text{by def of } \text{apply}_d \]
Apply: recap

\[(D, \sqsubseteq_D) \quad \text{CPO} \quad (E, \sqsubseteq_E)\]

apply : \([D \to E] \times D \to E\)

apply(f, d) ≜ f(d)

apply ∈ \([D \to E] \times D \to E\)
Fix

\[(D, \sqsubseteq_D) \text{ CPO}_\bot\]

\[
\text{fix} : [D \to D] \to D
\]

\[
\text{fix} \triangleq \lambda f. \bigsqcup_{n \in \mathbb{N}} f^n(\bot_D)
\]

**TH.** \text{fix} is monotone

(try to prove on your own)

**TH.** \text{fix} is continuous

**proof.** \text{fix} \triangleq \lambda f. \bigsqcup_{n \in \mathbb{N}} f^n(\bot_D) = \bigsqcup_{n \in \mathbb{N}} \lambda f. f^n(\bot_D)

by def of lub in functional domains

we prove that \( \forall n. \lambda f. f^n(\bot_D) \) is continuous

(by mathematical induction on \( n \))

then \text{fix} is continuous because lub of continuous functions

(see next slides)
(continue) \[\forall n. \lambda f. f^n(\bot_D)\]

base case: \[\lambda f. f^0(\bot_D) = \lambda f. \bot_D\]

is a constant function (continuous)

inductive case: assume \[g \triangleq \lambda f. f^n(\bot_D)\] is continuous
we want to prove \[h \triangleq \lambda f. f^{n+1}(\bot_D)\] is continuous

take a chain \(\{f_i\}_{i \in \mathbb{N}}\) in \([D \rightarrow D]\)
we want to prove \[h \left( \bigsqcup_{i \in \mathbb{N}} f_i \right) = \bigsqcup_{i \in \mathbb{N}} h(f_i)\]

(see next slide)
(continue) \( \forall n. \lambda f. f^n(\bot_D) \)

\[
g \triangleq \lambda f. f^n(\bot_D) \\
h \triangleq \lambda f. f^{n+1}(\bot_D) \\
h \left( \bigsqcup_{i \in \mathbb{N}} f_i \right) = \bigsqcup_{i \in \mathbb{N}} h(f_i)
\]

\[
h(\bigsqcup_i f_i) = (\bigsqcup_i f_i)^{n+1}(\bot_D) \quad \text{by def of } h
\]

\[
= (\bigsqcup_j f_j)((\bigsqcup_i f_i)^n(\bot_D)) \quad \text{by def of } (\cdot)^{n+1}
\]

\[
= (\bigsqcup_j f_j)(g(\bigsqcup_i f_i)) \quad \text{by def of } g
\]

\[
= (\bigsqcup_j f_j)(\bigsqcup_i g(f_i)) \quad \text{by ind. hyp } (g \text{ continuous})
\]

\[
= (\bigsqcup_j f_j)(\bigsqcup_i f_i^n(\bot_D)) \quad \text{by def of } g
\]

\[
= \bigsqcup_j \bigsqcup_i f_j(f_i^n(\bot_D)) \quad \text{by def of lub in functional CPO}
\]

\[
= \bigsqcup_j \bigsqcup_i f_j f_i^n(\bot_D) \quad \text{by continuity of } f_j
\]

\[
= \bigsqcup_i \bigsqcup_j f_i f_j^n(\bot_D) \quad \text{by switch lemma}
\]

\[
= \bigsqcup_i h(f_i^n) \quad \text{by def of } (\cdot)^{n+1}
\]

\[
= \bigsqcup_k h(f_k) \quad \text{by def of } h
\]
Fix: recap

\[(D, \sqsubseteq_D) \text{ CPO}_\bot\]

\[\text{fix} : [D \to D] \to D\]

\[\text{fix} \triangleq \lambda f. \bigsqcup_{n \in \mathbb{N}} f^n(\bot_D)\]

\[\text{fix} \in [[D \to D] \to D]\]
Curry

\( (D, \subseteq_D) \) CPO \( (E, \subseteq_E) \) \( (F, \subseteq_F) \)

\[
\text{curry} : (D \times E \rightarrow F) \rightarrow D \rightarrow E \rightarrow F
\]

\[
\text{curry } f \ d \ e \triangleq f(d, e)
\]

**TH.** \( f \) continuous \( \Rightarrow \) \( \text{curry}(f) \) continuous

(try to prove on your own)
Uncurry

\((D, \subseteq D)\) \hspace{1cm} \text{uncurry} : (D \to E \to F) \to (D \times E) \to F

\((E, \subseteq E)\) CPO \hspace{1cm} \text{uncurry } f \ (d, e) \triangleq f \ d \ e

\((F, \subseteq F)\)

**TH.** \(f\) continuous \(\Rightarrow\) \text{uncurry}(f) continuous

(try to prove on your own)

**TH.** \text{uncurry} is the inverse of \text{curry}

(try to prove on your own)
Disjoint Union

$$D = (D, \sqsubseteq_D)$$
$$E = (E, \sqsubseteq_E)$$

$\mathsf{CPO}_\perp$ \implies $D + E = (D \sqcup E, \sqsubseteq_{D \sqcup E})$

$D \sqcup E \triangleq \{(0, d) \mid d \in D\} \cup \{(1, e) \mid e \in E\}$

how to order elements?

is there a bottom element?

is it a complete order?

how to define (continuous) injections?

$$\iota_D : D \to D \sqcup E$$
$$\iota_E : E \to D \sqcup E$$