Overview

Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

Computation-Tree Logic

Equivalences and Abstraction

Temporal logics

extend propositional or predicate logic by
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$\Diamond\varphi$	"φ holds now or eventually in the future"	

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here: two propositional temporal logics:

LTL: linear temporal logic

CTL: computation tree logic

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Linear Temporal Logic (LTL)

- syntax and semantics of LTL ⟵
- automata-based LTL model checking
- complexity of LTL model checking

Computation-Tree Logic

Equivalences and Abstraction

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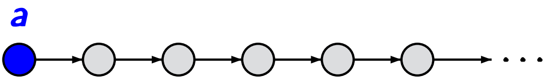
atomic proposition $a \in AP$



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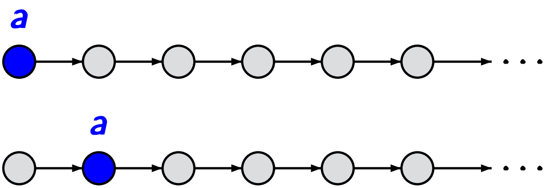

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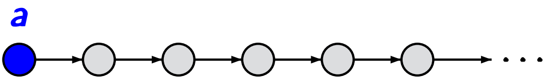
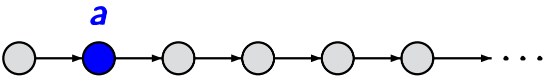
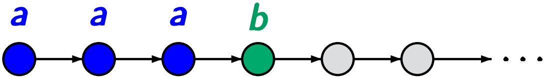


next operator
$\bigcirc a$



until operator
$a \, \mathsf{U} \, b$



Derived operators in LTL

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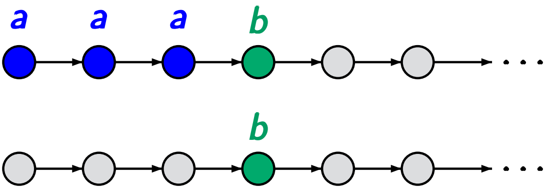
eventually $\Diamond b$



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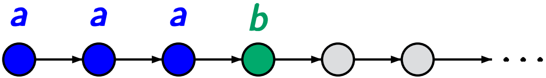
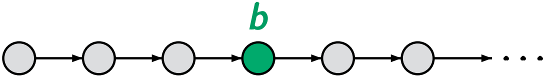
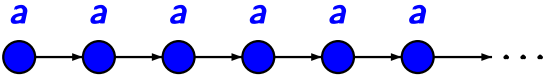


eventually
$\Diamond b$

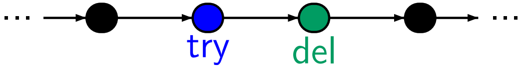


always
$\Box a$



Next $\bigcirc$, until U and eventually $\Diamond$

$\Box\,(\text{try\_to\_send} \rightarrow \bigcirc\ \text{delivered})$



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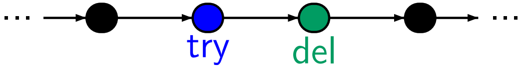

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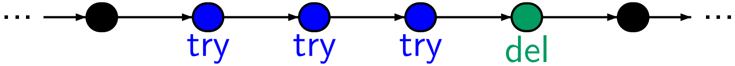

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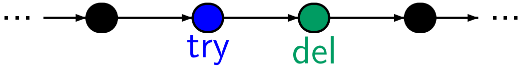

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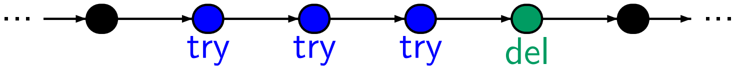

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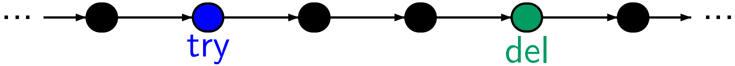

Examples for LTL formulas

$$\varphi \ ::= \ \mathit{true} \ \big| \ a \ \big| \ \varphi_1 \wedge \varphi_2 \ \big| \ \neg\varphi \ \big| \ \bigcirc \varphi \ \big| \ \varphi_1 \,\mathsf{U}\, \varphi_2$$

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traffic light: $\Box(\mathit{yellow} \vee \bigcirc\neg \mathit{red})$

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LTL-semantics

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interpretation of **LTL** formulas over traces, i.e., infinite words over 2^{AP}

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formalized by a satisfaction relation $\models$ for

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LT property of formula φ:

$$Words(\varphi) \stackrel{\text{def}}{=} \left\{\sigma \in \left(2^{AP}\right)^{\omega} : \sigma \models \varphi\right\}$$

LTL-semantics of derived operators $\Diamond$ and $\Box$

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\sigma \models \Diamond \varphi & \text{iff} & \text{there exists } j \geq 0 \text{ such that} \\
 & & A_j\, A_{j+1}\, A_{j+2} \ldots \models \varphi \\
\sigma \models \Box \varphi & \text{iff} & \text{for all } j \geq 0 \text{ we have:} \\
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\end{array}$$

LTL semantics over TS

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given a TS $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$

define satisfaction relation $\models$ for

- **LTL** formulas over AP
- the maximal path fragments and states of $\mathcal{T}$

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assumption: $\mathcal{T}$ has no terminal states, i.e.,
all maximal path fragments in $\mathcal{T}$ are infinite

LTL semantics over paths of TS

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LTL formula φ over AP

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interpretation of φ over infinite path fragments

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remind: LT property of an LTL formula:

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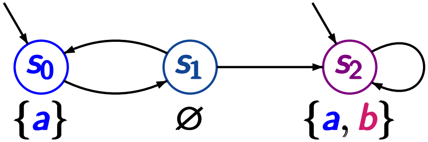
Example: LTL-semantics over paths



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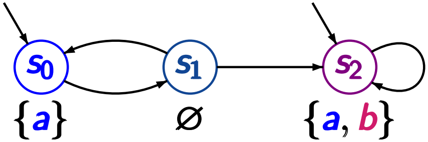

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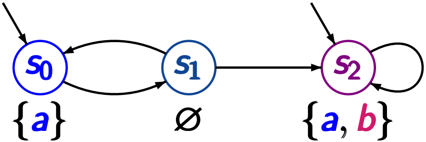
path $\pi = s_0\, s_1\, s_2\, s_2\, s_2\, s_2 \ldots$

Example: LTL-semantics over paths



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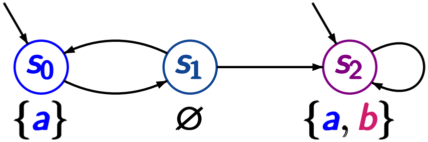

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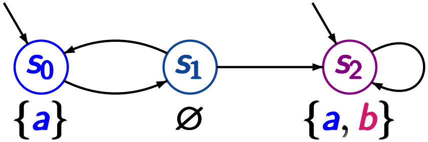

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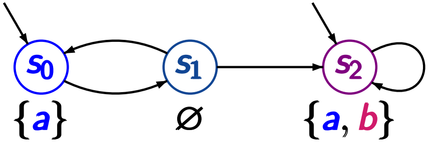

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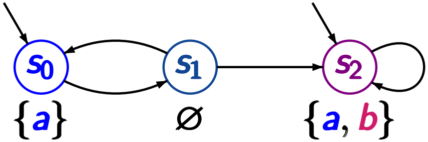

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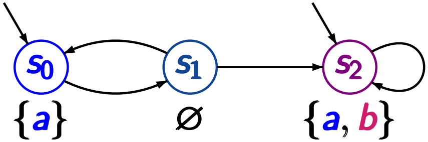

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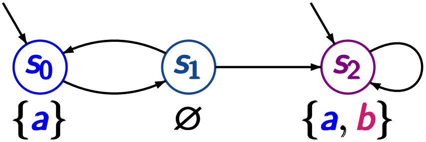

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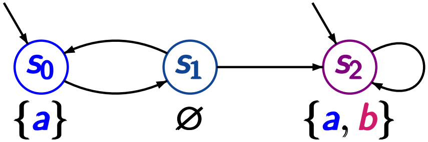

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$\pi \models a$, but $\pi \not\models b$	as $L(s_0) = \{a\}$
$\pi \models \bigcirc(\neg a \wedge \neg b)$	as $L(s_1) = \varnothing$
$\pi \models \bigcirc \bigcirc (a \wedge b)$	as $L(s_2) = \{a, b\}$
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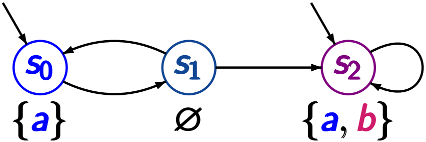

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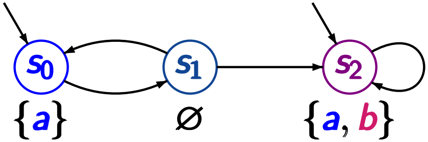
$\pi \models a$, but $\pi \not\models b$	as $L(s_0) = \{a\}$
$\pi \models \bigcirc(\neg a \wedge \neg b)$	as $L(s_1) = \varnothing$
$\pi \models \bigcirc \bigcirc (a \wedge b)$	as $L(s_2) = \{a, b\}$
$\pi \models (\neg b) \mathsf{U} (a \wedge b)$	as $s_0, s_1 \models \neg b$
$\pi \models (\neg b) \mathsf{U} \Box(a \wedge b)$	and $s_2 \models a \wedge b$

Correct or wrong ?



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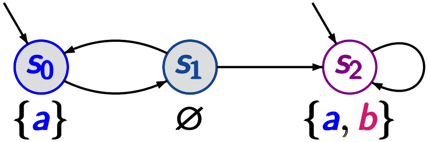

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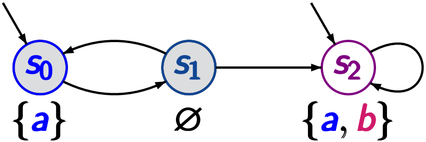

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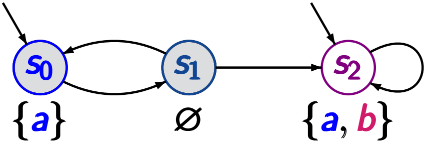

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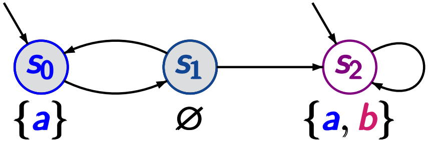

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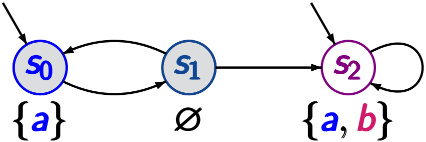
$\pi \models \Diamond b \rightarrow (a \,\mathsf{U}\, b)$?

Correct or wrong ?



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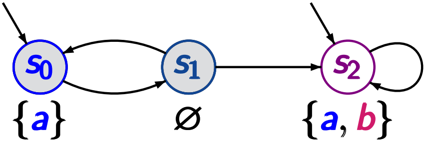

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$\pi \models \bigcirc \bigcirc \neg b$?

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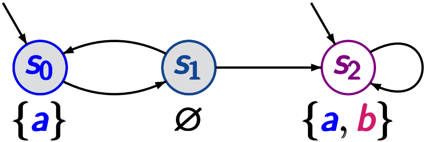

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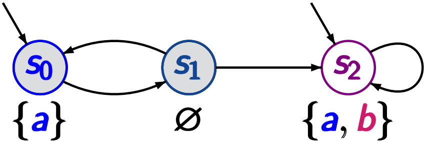

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$\pi \models \Box a$?

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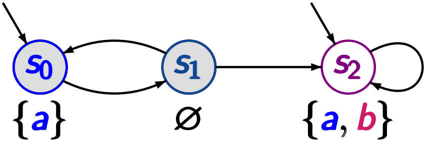

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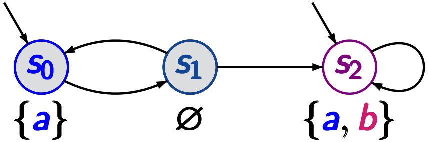

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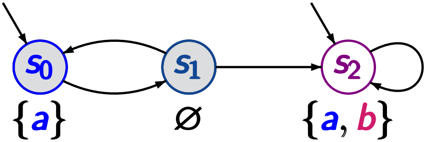
$\pi \models \Box\Diamond a$?

Correct or wrong ?



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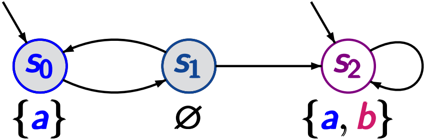
$\pi \not\models \Box a$ $\qquad$ as $s_1 \not\models a$

$\pi \models \Box\Diamond a$ $\qquad$ as $\Box\Diamond \mathrel{\widehat{=}}$ infinitely often

Correct or wrong ?



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$\pi \not\models a \,\mathsf{U}\, b$	as $s_0 \not\models b$ and $s_1 \not\models a \vee b$
$\pi \models \Diamond b \rightarrow (a \,\mathsf{U}\, b)$	as $\pi \not\models \Diamond b$
$\pi \models \bigcirc \bigcirc \neg b$	as $s_0 \models \neg b$
$\pi \not\models \Box a$	as $s_1 \not\models a$
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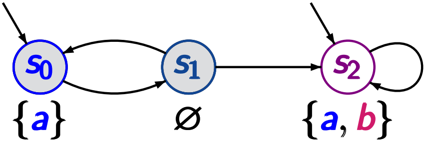

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$\pi \not\models a \,\mathsf{U}\, b$ $\qquad$ as $s_0 \not\models b$ and $s_1 \not\models a \vee b$

$\pi \models \Diamond b \rightarrow (a \,\mathsf{U}\, b)$ $\qquad$ as $\pi \not\models \Diamond b$

$\pi \models \bigcirc \bigcirc \neg b$ $\qquad$ as $s_0 \models \neg b$

$\pi \not\models \Box a$ $\qquad$ as $s_1 \not\models a$

$\pi \models \Box\Diamond a$ $\qquad$ as $\Box\Diamond \mathrel{\widehat{=}}$ infinitely often

$\pi \not\models \Diamond\Box a$ $\qquad$ as $\Diamond\Box \mathrel{\widehat{=}}$ eventually forever

LTL-semantics of derived operators

for $\sigma = A_0\, A_1\, A_2\, \ldots \in \left(2^{AP}\right)^{\omega}$:

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$\sigma \models \Diamond\varphi$	iff	there exists $j \geq 0$ such that $A_j\, A_{j+1}\, A_{j+2} \ldots \models \varphi$
$\sigma \models \Box\varphi$	iff	for all $j \geq 0$ we have: $A_j\, A_{j+1}\, A_{j+2} \ldots \models \varphi$

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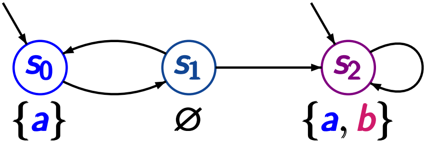
$\sigma \models \Box\Diamond\varphi$	iff	there are infinitely many $j \geq 0$ s.t. $A_j\, A_{j+1}\, A_{j+2} \ldots \models \varphi$
$\sigma \models \Diamond\Box\varphi$	iff	for almost all $j \geq 0$ we have: $A_j\, A_{j+1}\, A_{j+2} \ldots \models \varphi$

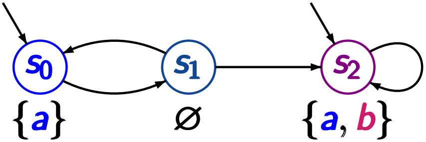
LTL-semantics over paths



$AP = \{a, b\}$

path $\pi = s_0\, s_1\, s_2\, s_2\, s_2\, s_2 \ldots$

LTL-semantics over paths



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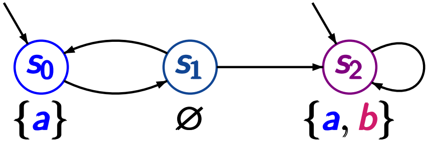

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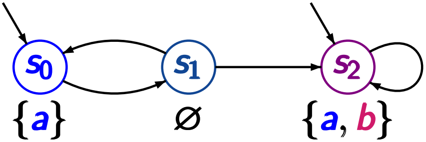
$\pi \models \bigcirc((\neg a \wedge \neg b)\, \mathsf{U}\, (a \wedge b))$?

LTL-semantics over paths



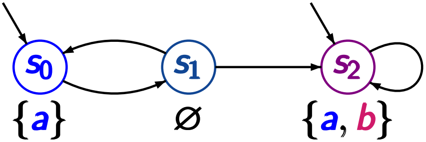
$AP = \{a, b\}$

path $\pi = s_0\, s_1\, s_2\, s_2\, s_2\, s_2 \ldots$ $\quad$ $trace(\pi) = \{a\}\, \varnothing\, \{a, b\}^\omega$

$\pi \models \bigcirc((\neg a \wedge \neg b)\, \mathsf{U}(a \wedge b))$ as $s_1 \models \neg a \wedge \neg b$

$s_2 \models a \wedge b$

LTL-semantics over paths



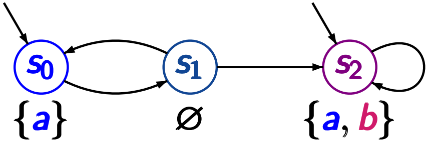
$AP = \{a, b\}$

path $\pi = s_0\, s_1\, s_2\, s_2\, s_2\, s_2 \ldots$ $\quad trace(\pi) = \{a\} \varnothing \{a, b\}^\omega$

$\pi \models \bigcirc((\neg a \wedge \neg b) \, \mathsf{U} (a \wedge b))$ as $s_1 \models \neg a \wedge \neg b$

$s_2 \models a \wedge b$

$\pi \models \bigcirc \Box (a \leftrightarrow b)$?



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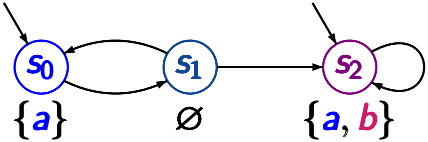
LTL-semantics over paths



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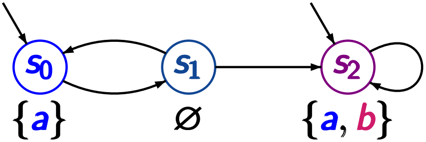

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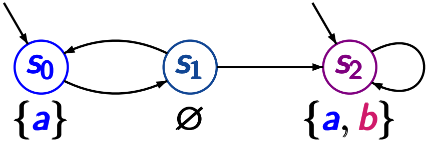

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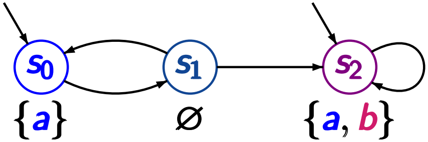

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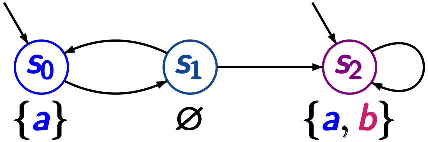

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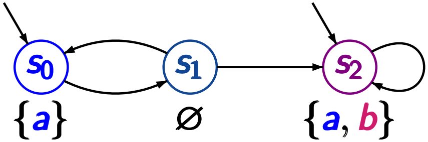

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LTL semantics over the states of a TS

given: TS $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$
without terminal states

LTL formula φ over AP

interpretation of φ over infinite path fragments

$$\pi = s_0\, s_1\, s_2 \ldots \models \varphi \quad \text{iff} \quad trace(\pi) \models \varphi$$

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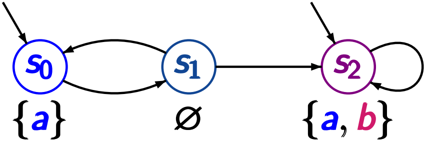

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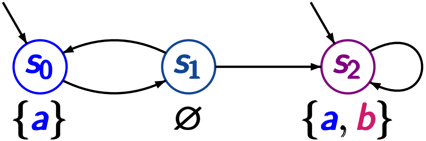

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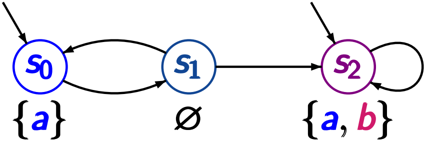

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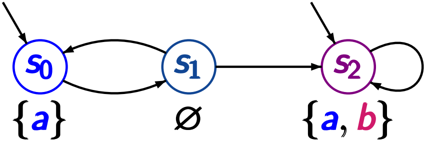

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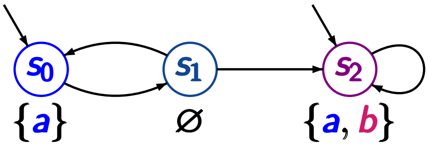

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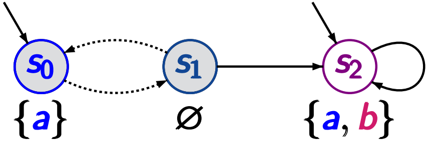

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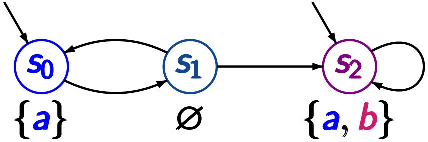

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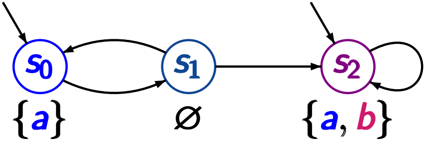

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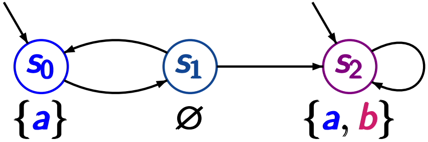

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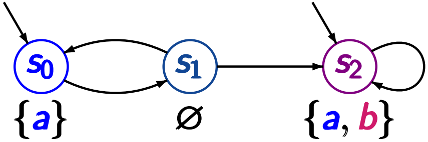

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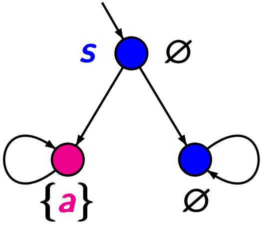

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LTL-formulas for MUTEX protocols

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Claim: $\neg \bigcirc \varphi \equiv \bigcirc \neg \varphi$ "self-duality of next"

Self-duality of the next operator

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Claim: $\neg \bigcirc \varphi \equiv \bigcirc \neg \varphi$ "self-duality of next"

Proof: $A_0 \, A_1 \, A_2 \, A_3 \ldots \models \neg \bigcirc \varphi$

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Proof:

$$\begin{array}{lrcl} & A_0\,A_1\,A_2\,A_3\ldots & \models & \neg\bigcirc\varphi \\ \text{iff} & A_0\,A_1\,A_2\,A_3\ldots & \not\models & \bigcirc\varphi \\ \text{iff} & A_1\,A_2\,A_3\ldots & \not\models & \varphi \end{array}$$

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$$A_0\, A_1\, A_2\, A_3 \ldots \models \neg\bigcirc\varphi$$

iff $A_0\, A_1\, A_2\, A_3 \ldots \not\models \bigcirc\varphi$

iff $A_1\, A_2\, A_3 \ldots \not\models \varphi$

iff $A_1\, A_2\, A_3 \ldots \models \neg\varphi$

Self-duality of the next operator

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$$
\begin{array}{lll}
 & A_0\, A_1\, A_2\, A_3 \ldots & \models \neg \bigcirc \varphi \\
\text{iff} & A_0\, A_1\, A_2\, A_3 \ldots & \not\models \bigcirc \varphi \\
\text{iff} & A_1\, A_2\, A_3 \ldots & \not\models \varphi \\
\text{iff} & A_1\, A_2\, A_3 \ldots & \models \neg \varphi \\
\text{iff} & A_0\, A_1\, A_2\, A_3 \ldots & \models \bigcirc \neg \varphi
\end{array}
$$

Correct or wrong?

$$\Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi$$

Correct or wrong?

$$\Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi$$

correct

Correct or wrong?

$$\Diamond(\varphi \vee \psi) \;\equiv\; \Diamond\varphi \vee \Diamond\psi$$

correct

$$\Diamond(\varphi \wedge \psi) \;\equiv\; \Diamond\varphi \wedge \Diamond\psi$$

$$\Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi$$

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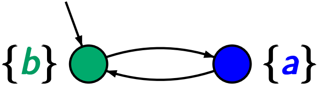
$$\Diamond(\varphi \wedge \psi) \equiv \Diamond\varphi \wedge \Diamond\psi$$

wrong,
e.g.,



$\models \Diamond b \wedge \Diamond a$

$\not\models \Diamond(b \wedge a)$

Correct or wrong?

$$\Diamond(\varphi \vee \psi) \;\equiv\; \Diamond\varphi \vee \Diamond\psi$$

correct

$$\Diamond(\varphi \wedge \psi) \;\equiv\; \Diamond\varphi \wedge \Diamond\psi$$

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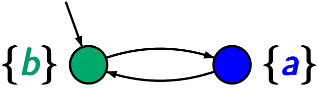

$\models \Diamond b \wedge \Diamond a$

$\not\models \Diamond(b \wedge a)$

---

similarly: $\Box(\varphi \wedge \psi) \;\equiv\; \Box\varphi \wedge \Box\psi$

$\Box(\varphi \vee \psi) \;\not\equiv\; \Box\varphi \vee \Box\psi$

Correct or wrong?

$$\Diamond\Diamond\varphi \equiv \Diamond\varphi$$

Correct or wrong?

$\Diamond\Diamond\varphi \;\equiv\; \Diamond\varphi$

correct Analogous: $\Box\Box\varphi \;\equiv\; \Box\varphi$

Correct or wrong?

$\Diamond\Diamond\varphi \equiv \Diamond\varphi$

correct Analogous: $\Box\Box\varphi \equiv \Box\varphi$

$\bigcirc\Box\varphi \equiv \Box\bigcirc\varphi$

Correct or wrong?

$\Diamond\Diamond\varphi \equiv \Diamond\varphi$

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correct

Correct or wrong?

$\Diamond\Diamond\varphi \equiv \Diamond\varphi$

correct Analogous: $\Box\Box\varphi \equiv \Box\varphi$

$\bigcirc\Box\varphi \equiv \Box\bigcirc\varphi \stackrel{\text{def}}{=} \psi$

correct

note that:

$A_0\, A_1\, A_2 \ldots \models \psi$ iff $A_i\, A_{i+1} \ldots \models \varphi$ for all $i \geq 1$

Correct or wrong?

$\Diamond\Diamond\varphi \equiv \Diamond\varphi$

correct Analogous: $\Box\Box\varphi \equiv \Box\varphi$

$\bigcirc\Box\varphi \equiv \Box\bigcirc\varphi$

correct

$\Diamond\Box\varphi \equiv \Box\Diamond\varphi$

Correct or wrong?

$\Diamond\Diamond\varphi \equiv \Diamond\varphi$

correct Analogous: $\Box\Box\varphi \equiv \Box\varphi$

$\bigcirc\Box\varphi \equiv \Box\bigcirc\varphi$

correct

$\Diamond\Box\varphi \equiv \Box\Diamond\varphi$

$\Box\Diamond \;\widehat{=}\;$ infinitely often

$\Diamond\Box \;\widehat{=}\;$ eventually forever

Correct or wrong?

$\Diamond\Diamond\varphi \equiv \Diamond\varphi$

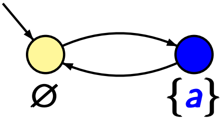
correct Analogous: $\Box\Box\varphi \equiv \Box\varphi$

$\bigcirc\Box\varphi \equiv \Box\bigcirc\varphi$

correct

$\Diamond\Box\varphi \equiv \Box\Diamond\varphi$

wrong



$\Box\Diamond \mathrel{\widehat{=}}$ infinitely often

$\Diamond\Box \mathrel{\widehat{=}}$ eventually forever

$\models \Box\Diamond a$

$\not\models \Diamond\Box a$

Expansion laws

LTLSF3.1-28

Expansion law for U

until: $\varphi \, \mathsf{U} \, \psi \;\equiv\; \psi \;\vee\; (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$

Expansion laws for U and ◊

until: $\varphi \,\mathsf{U}\, \psi \;\equiv\; \psi \;\vee\; (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$

eventually: $\Diamond\psi \;\equiv\; \psi \;\vee\; \bigcirc\Diamond\psi$

Expansion laws for U and $\Diamond$

until: $\varphi \,\mathsf{U}\, \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$

eventually: $\Diamond\psi \equiv \psi \vee \bigcirc\Diamond\psi$

note: $\Diamond\psi = \mathit{true} \,\mathsf{U}\, \psi$

Expansion laws for U and $\Diamond$

until: $\varphi \,\mathsf{U}\, \psi \;\equiv\; \psi \;\vee\; (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$

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note: $\Diamond\psi = \mathit{true} \,\mathsf{U}\, \psi$

$\equiv \psi \;\vee\; (\mathit{true} \;\wedge\; \bigcirc(\mathit{true} \,\mathsf{U}\, \psi))$

Expansion laws for U and $\Diamond$

until: $\varphi \,\mathsf{U}\, \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$

eventually: $\Diamond\psi \equiv \psi \vee \bigcirc\Diamond\psi$

note: $\Diamond\psi = \mathit{true} \,\mathsf{U}\, \psi$

$$\equiv \psi \vee (\mathit{true} \wedge \bigcirc(\underbrace{\mathit{true} \,\mathsf{U}\, \psi}_{=\Diamond\psi}))$$

Expansion laws for U and $\Diamond$

until: $\varphi \,\mathsf{U}\, \psi \;\equiv\; \psi \;\vee\; (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$

eventually: $\Diamond\psi \;\equiv\; \psi \;\vee\; \bigcirc\Diamond\psi$

note:

$$\begin{aligned}\Diamond\psi &= \mathit{true} \,\mathsf{U}\, \psi \\ &\equiv \psi \;\vee\; (\mathit{true} \;\wedge\; \bigcirc(\underbrace{\mathit{true} \,\mathsf{U}\, \psi}_{=\Diamond\psi})) \\ &\equiv \psi \;\vee\; \bigcirc\Diamond\psi\end{aligned}$$

Expansion laws for U, $\Diamond$ and $\Box$

until: $\varphi \mathsf{U} \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \mathsf{U} \psi))$

eventually: $\Diamond\psi \equiv \psi \vee \bigcirc\Diamond\psi$

always: $\Box\psi \equiv$ **?**

Expansion laws for U, $\Diamond$ and $\Box$

until: $\varphi \,\mathsf{U}\, \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$

eventually: $\Diamond\psi \equiv \psi \vee \bigcirc\Diamond\psi$

always: $\Box\psi \equiv \psi \wedge \bigcirc\Box\psi$

Expansion laws for U, $\Diamond$ and $\Box$

until: $\varphi \, \mathsf{U} \, \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$

eventually: $\Diamond \psi \equiv \psi \vee \bigcirc \Diamond \psi$

always: $\Box \psi \equiv \psi \wedge \bigcirc \Box \psi$

$\Box \psi = \neg \Diamond \neg \psi$

until: $\varphi \mathbf{U} \psi \equiv \psi \lor (\varphi \land \bigcirc(\varphi \mathbf{U} \psi))$

eventually: $\Diamond\psi \equiv \psi \lor \bigcirc\Diamond\psi$

always: $\Box\psi \equiv \psi \land \bigcirc\Box\psi$

$$\begin{aligned} \Box\psi &= \neg\Diamond\neg\psi \\ &\equiv \neg(\neg\psi \lor \bigcirc\Diamond\neg\psi) \end{aligned}$$

$\longleftarrow$ expansion law for $\Diamond$

Expansion laws for U, $\Diamond$ and $\Box$

until: $\varphi \, \mathsf{U} \, \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$

eventually: $\Diamond\psi \equiv \psi \vee \bigcirc\Diamond\psi$

always: $\Box\psi \equiv \psi \wedge \bigcirc\Box\psi$

$$
\begin{aligned}
\Box\psi &= \neg\Diamond\neg\psi \\
&\equiv \neg(\neg\psi \vee \bigcirc\Diamond\neg\psi) \\
&\equiv \neg\neg\psi \wedge \neg\bigcirc\Diamond\neg\psi \quad \longleftarrow \boxed{\text{de Morgan}}
\end{aligned}
$$

Expansion laws for U, $\Diamond$ and $\Box$

until: $\varphi \,\mathsf{U}\, \psi \;\equiv\; \psi \vee (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$

eventually: $\Diamond\psi \;\equiv\; \psi \vee \bigcirc\Diamond\psi$

always: $\Box\psi \;\equiv\; \psi \wedge \bigcirc\Box\psi$

$$
\begin{aligned}
\Box\psi &= \neg\Diamond\neg\psi \\
&\equiv \neg(\neg\psi \vee \bigcirc\Diamond\neg\psi) \\
&\equiv \neg\neg\psi \wedge \neg\bigcirc\Diamond\neg\psi \\
&\equiv \psi \wedge \neg\bigcirc\Diamond\neg\psi \quad \longleftarrow \boxed{\text{double negation}}
\end{aligned}
$$

Expansion laws for U, $\Diamond$ and $\Box$

until: $\varphi \mathsf{U} \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \mathsf{U} \psi))$

eventually: $\Diamond\psi \equiv \psi \vee \bigcirc\Diamond\psi$

always: $\Box\psi \equiv \psi \wedge \bigcirc\Box\psi$

$$
\begin{aligned}
\Box\psi &= \neg\Diamond\neg\psi \\
&\equiv \neg(\neg\psi \vee \bigcirc\Diamond\neg\psi) \\
&\equiv \neg\neg\psi \wedge \neg\bigcirc\Diamond\neg\psi \\
&\equiv \psi \wedge \bigcirc\neg\Diamond\neg\psi
\end{aligned}
$$

$\longleftarrow$ self duality of $\bigcirc$

Expansion laws for U, $\Diamond$ and $\Box$

until: $\varphi \mathbin{\mathsf{U}} \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \mathbin{\mathsf{U}} \psi))$

eventually: $\Diamond\psi \equiv \psi \vee \bigcirc\Diamond\psi$

always: $\Box\psi \equiv \psi \wedge \bigcirc\Box\psi$

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\Box\psi &= \neg\Diamond\neg\psi \\
&\equiv \neg(\neg\psi \vee \bigcirc\Diamond\neg\psi) \\
&\equiv \neg\neg\psi \wedge \neg\bigcirc\Diamond\neg\psi \\
&\equiv \psi \wedge \bigcirc\neg\Diamond\neg\psi \\
&\equiv \psi \wedge \bigcirc\Box\psi \qquad \longleftarrow \boxed{\text{definition of } \Box}
\end{aligned}
$$

Expansion laws are fixed point equations

until: $\varphi \mathbf{U} \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \mathbf{U} \psi))$

eventually: $\Diamond\psi \equiv \psi \vee \bigcirc\Diamond\psi$

always: $\Box\psi \equiv \psi \wedge \bigcirc\Box\psi$

Expansion laws are fixed point equations

until: $\boxed{\varphi \, \mathsf{U} \, \psi} \equiv \psi \vee (\varphi \wedge \bigcirc \boxed{\varphi \, \mathsf{U} \, \psi})$

eventually: $\boxed{\lozenge \psi} \equiv \psi \vee \bigcirc \boxed{\lozenge \psi}$

always: $\boxed{\square \psi} \equiv \psi \wedge \bigcirc \boxed{\square \psi}$

Expansion laws are fixed point equations

until: $\boxed{\varphi \,\mathsf{U}\, \psi} \equiv \psi \vee (\varphi \wedge \bigcirc \boxed{\varphi \,\mathsf{U}\, \psi})$

eventually: $\boxed{\Diamond\psi} \equiv \psi \vee \bigcirc\boxed{\Diamond\psi}$

always: $\boxed{\Box\psi} \equiv \psi \wedge \bigcirc\boxed{\Box\psi}$

... don't yield a complete characterization, e.g.,

$$\begin{array}{lcl} \mathit{false} & \equiv & a \wedge \bigcirc \mathit{false} \\ \Box a & \equiv & a \wedge \bigcirc \Box a \end{array}$$

consider $\psi = a$

Expansion laws are fixed point equations

until: $\boxed{\varphi \,\mathsf{U}\, \psi} \equiv \psi \vee (\varphi \wedge \bigcirc \boxed{\varphi \,\mathsf{U}\, \psi})$

eventually: $\boxed{\Diamond\psi} \equiv \psi \vee \bigcirc \boxed{\Diamond\psi}$

always: $\boxed{\Box\psi} \equiv \psi \wedge \bigcirc \boxed{\Box\psi}$

...don't yield a complete characterization, e.g.,

$$\boxed{\begin{array}{lcl} \mathit{false} & \equiv & a \wedge \bigcirc \mathit{false} \\ \Box a & \equiv & a \wedge \bigcirc \Box a \end{array}}$$

although $\Box a \not\equiv \mathit{false}$

Expansion laws are fixed point equations

until: $\varphi \, \mathsf{U} \, \psi \;\equiv\; \psi \;\vee\; (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$

least fixed point

eventually: $\Diamond\psi \;\equiv\; \psi \;\vee\; \bigcirc\Diamond\psi$

least fixed point

always: $\Box\psi \;\equiv\; \psi \;\wedge\; \bigcirc\Box\psi$

...don't yield a complete characterization, e.g.,

$$\mathit{false} \;\equiv\; a \wedge \bigcirc \mathit{false}$$
$$\Box a \;\equiv\; a \wedge \bigcirc \Box a$$

although
$\Box a \;\not\equiv\; \mathit{false}$

Expansion laws are fixed point equations

until: $\varphi \mathbf{U} \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \mathbf{U} \psi))$

least fixed point

eventually: $\Diamond \psi \equiv \psi \vee \bigcirc \Diamond \psi$

least fixed point

always: $\Box \psi \equiv \psi \wedge \bigcirc \Box \psi$

greatest fixed point

... don't yield a complete characterization, e.g.,

$$\mathit{false} \equiv a \wedge \bigcirc \mathit{false}$$
$$\Box a \equiv a \wedge \bigcirc \Box a$$

although $\Box a \not\equiv \mathit{false}$

Expansion law for U

The LTL formula $\chi = \varphi \, \mathbf{U} \, \psi$ is the least solution of

$$\chi \equiv \psi \lor (\varphi \land \bigcirc \chi)$$

Expansion law for U

The LTL formula $\chi = \varphi \mathbf{U} \psi$ is the least solution of

$$\chi \equiv \psi \vee (\varphi \wedge \bigcirc \chi)$$

i.e., $Words(\varphi \mathbf{U} \psi)$ least LT-property E s.t.

$$E = Words(\psi) \cup \{A_0 A_1 A_2 \ldots \in Words(\varphi) : A_1 A_2 \ldots \in E\}$$

Expansion law for U

The LTL formula $\chi = \varphi \, \mathbf{U} \, \psi$ is the least solution of

$$\chi \equiv \psi \vee (\varphi \wedge \bigcirc \chi)$$

i.e., $Words(\varphi \, \mathbf{U} \, \psi)$ least LT-property E s.t.

$$E = Words(\psi) \cup \{A_0 A_1 A_2 \ldots \in Words(\varphi) : A_1 A_2 \ldots \in E\}$$

It even holds that $Words(\varphi \, \mathbf{U} \, \psi)$ least LT-property E s.t.

(1) $Words(\psi) \subseteq E$

(2) $\{A_0 A_1 A_2 \ldots \in Words(\varphi) : A_1 A_2 \ldots \in E\} \subseteq E$

The weak until operator W

The weak until operator W

$$\varphi \, \mathsf{W} \, \psi \;\overset{\text{def}}{=}\; (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$$

The weak until operator W

$$\varphi \, \mathsf{W} \, \psi \;\stackrel{\text{def}}{=}\; (\varphi \, \mathsf{U} \, \psi) \lor \Box\varphi$$

deriving "always" and "until" from "weak until":

$$\Box\varphi \quad \equiv \quad ?$$

The weak until operator W

$$\varphi \, \mathsf{W} \, \psi \ \stackrel{\text{def}}{=} \ (\varphi \, \mathsf{U} \, \psi) \lor \Box\varphi$$

deriving "always" and "until" from "weak until":

$$\Box\varphi \quad \equiv \quad \varphi \, \mathsf{W} \, \mathit{false}$$

The weak until operator W

$$\varphi \, \mathsf{W} \, \psi \;\stackrel{\text{def}}{=}\; (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$$

deriving "always" and "until" from "weak until":

$$\Box\varphi \quad \equiv \quad \varphi \, \mathsf{W} \, \textit{false}$$

$$\varphi \, \mathsf{U} \, \psi \quad \equiv \quad ?$$

The weak until operator W

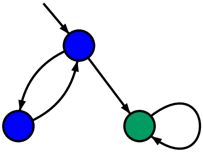
$$\varphi \, \mathsf{W} \, \psi \;\stackrel{\text{def}}{=}\; (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$$

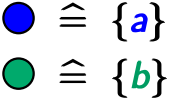
deriving "always" and "until" from "weak until":

$$\Box\varphi \equiv \varphi \, \mathsf{W} \, \mathit{false}$$

$$\varphi \, \mathsf{U} \, \psi \equiv (\varphi \, \mathsf{W} \, \psi) \wedge \Diamond\psi$$

Does $\mathcal{T} \models a \, \mathsf{W} \, b$ hold?





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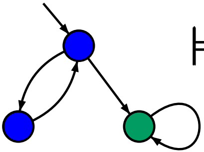
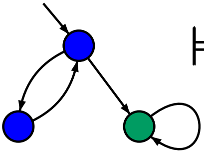


$\models a \mathsf{W} b$

● $\widehat{=}$ $\{a\}$

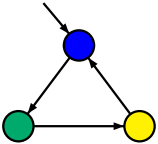
● $\widehat{=}$ $\{b\}$

Does $\mathcal{T} \models a \mathsf{W} b$ hold?

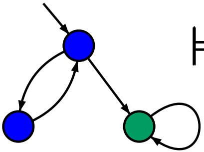


$\models a \mathsf{W} b$

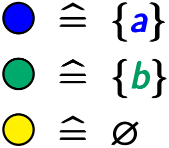
- blue $\widehat{=} \{a\}$
- green $\widehat{=} \{b\}$
- yellow $\widehat{=} \varnothing$

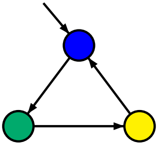


Does $\mathcal{T} \models a \, \mathsf{W} \, b$ hold?



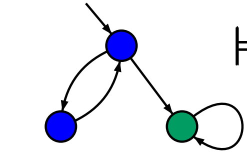
$\models a \, \mathsf{W} \, b$



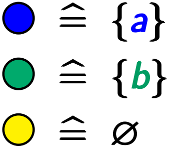


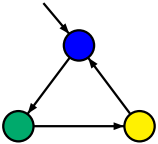
$\models \; a \, \mathsf{W} \, b \quad$ (even $a \, \mathsf{U} \, b$)

Does $\mathcal{T} \models a \, \mathsf{W} \, b$ hold?

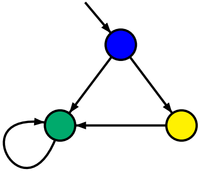


$\models a \, \mathsf{W} \, b$

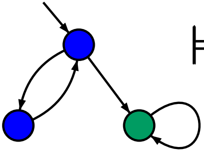




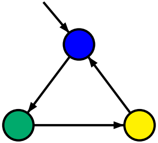
$\models a \, \mathsf{W} \, b$ (even $a \, \mathsf{U} \, b$)



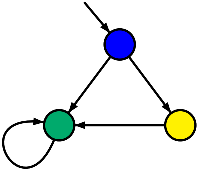
Does $\mathcal{T} \models a\,\mathsf{W}\,b$ hold?



$\models a\,\mathsf{W}\,b$

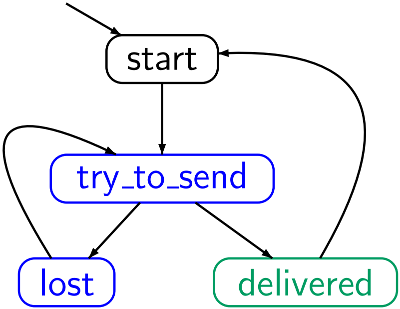


$\models a\,\mathsf{W}\,b$ (even $a\,\mathsf{U}\,b$)

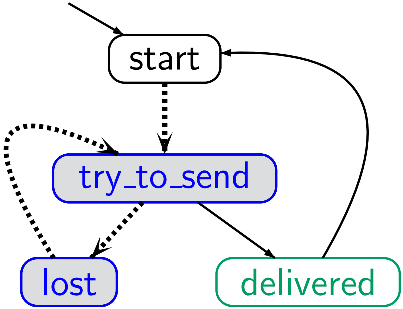


$\not\models a\,\mathsf{W}\,b$

Example: simple communication protocol

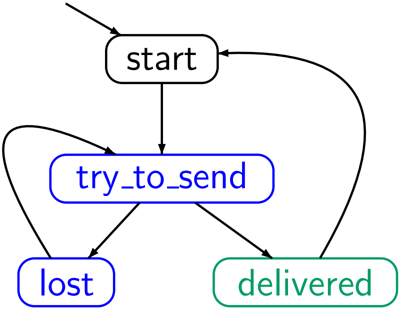


Example: simple communication protocol



$$\mathcal{T} \not\models \Box(\textit{blue} \longrightarrow \textit{blue}\ \mathsf{U}\ \textit{delivered})$$

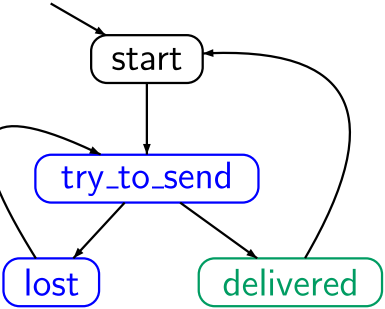
Example: until versus weak until



$$\mathcal{T} \not\models \Box(\textit{blue} \longrightarrow \textit{blue}\, \mathsf{U}\, \textit{delivered})$$

$$\mathcal{T} \models \Box(\textit{blue} \longrightarrow \textit{blue}\, \mathsf{W}\, \textit{delivered})$$

Example: until versus weak until



constrained liveness:

$$\mathcal{T} \not\models \Box(\mathit{blue} \longrightarrow \mathit{blue}\ \mathsf{U}\ \mathit{delivered})$$

safety:

$$\mathcal{T} \models \Box(\mathit{blue} \longrightarrow \mathit{blue}\ \mathsf{W}\ \mathit{delivered})$$

Duality of U and W

$$\varphi \, \mathsf{W} \, \psi \stackrel{\text{def}}{=} (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$$

goal: express $\neg(\varphi \, \mathsf{U} \, \psi)$ via **W**, and vice versa

Duality of U and W

$$\varphi \, \mathsf{W} \, \psi \;\stackrel{\text{def}}{=}\; (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$$

$$\neg(\varphi \, \mathsf{U} \, \psi)$$

$$\equiv \; ((\varphi \wedge \neg\psi) \; \mathsf{U} \; (\neg\varphi \wedge \neg\psi)) \; \vee \; \Box(\varphi \wedge \neg\psi)$$

Duality of U and W

$$\varphi \, \mathsf{W} \, \psi \;\stackrel{\text{def}}{=}\; (\varphi \, \mathsf{U} \, \psi) \;\vee\; \Box\varphi$$

$$\begin{aligned} & \neg(\varphi \, \mathsf{U} \, \psi) \\ \equiv \;& \big((\varphi \wedge \neg\psi) \;\mathsf{U}\; (\neg\varphi \wedge \neg\psi)\big) \;\vee\; \Box(\varphi \wedge \neg\psi) \\ \equiv \;& (\varphi \wedge \neg\psi) \,\mathsf{W}\, (\neg\varphi \wedge \neg\psi) \end{aligned}$$

Duality of U and W

$$\varphi \,\mathsf{W}\, \psi \;\stackrel{\text{def}}{=}\; (\varphi \,\mathsf{U}\, \psi) \vee \Box\varphi$$

$$\begin{aligned}
& \neg(\varphi \,\mathsf{U}\, \psi) \\
\equiv\; & ((\varphi \wedge \neg\psi) \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \vee \Box(\varphi \wedge \neg\psi) \\
\equiv\; & (\varphi \wedge \neg\psi) \,\mathsf{W}\, (\neg\varphi \wedge \neg\psi) \\
\equiv\; & (\neg\psi) \,\mathsf{W}\, (\neg\varphi \wedge \neg\psi)
\end{aligned}$$

Duality of U and W

$$\varphi \, \mathsf{W} \, \psi \;\stackrel{\text{def}}{=}\; (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$$

$$
\begin{aligned}
& \neg(\varphi \, \mathsf{U} \, \psi) \\
\equiv \; & ((\varphi \wedge \neg\psi) \; \mathsf{U} \; (\neg\varphi \wedge \neg\psi)) \vee \Box(\varphi \wedge \neg\psi) \\
\equiv \; & (\varphi \wedge \neg\psi) \; \mathsf{W} \; (\neg\varphi \wedge \neg\psi) \\
\equiv \; & (\neg\psi) \; \mathsf{W} \; (\neg\varphi \wedge \neg\psi)
\end{aligned}
$$

$$\neg(\varphi \, \mathsf{U} \, \psi) \equiv (\neg\psi) \, \mathsf{W} \, (\neg\varphi \wedge \neg\psi)$$

$$\neg(\varphi \, \mathsf{W} \, \psi) \equiv \; ?$$

$$\varphi \,\mathsf{W}\, \psi \stackrel{\text{def}}{=} (\varphi \,\mathsf{U}\, \psi) \vee \Box\varphi$$

$$\begin{aligned}
& \neg(\varphi \,\mathsf{U}\, \psi) \\
\equiv\ & ((\varphi \wedge \neg\psi) \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \vee \Box(\varphi \wedge \neg\psi) \\
\equiv\ & (\varphi \wedge \neg\psi) \,\mathsf{W}\, (\neg\varphi \wedge \neg\psi) \\
\equiv\ & (\neg\psi) \,\mathsf{W}\, (\neg\varphi \wedge \neg\psi)
\end{aligned}$$

$$\neg(\varphi \,\mathsf{U}\, \psi) \equiv (\neg\psi) \,\mathsf{W}\, (\neg\varphi \wedge \neg\psi)$$

$$\neg(\varphi \,\mathsf{W}\, \psi) \equiv (\neg\psi) \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)$$

Expansion laws for U and W

Expansion laws for U and W

$$\varphi \,\mathsf{U}\, \psi \quad \equiv \quad \psi \vee (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$$

$$\varphi \,\mathsf{W}\, \psi \quad \equiv \quad ?$$

Expansion laws for U and W

$$\varphi \, \mathsf{U} \, \psi \quad \equiv \quad \psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$$

$$\varphi \, \mathsf{W} \, \psi \quad \equiv \quad \psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{W} \, \psi))$$

Expansion laws for U and W

$$\varphi \ \mathsf{U} \ \psi \quad \equiv \quad \psi \ \vee \ (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$$

smallest solution

$$\varphi \ \mathsf{W} \ \psi \quad \equiv \quad \psi \ \vee \ (\varphi \wedge \bigcirc(\varphi \, \mathsf{W} \, \psi))$$

Expansion laws for U and W

$\varphi \,\mathsf{U}\, \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$	smallest solution
$\varphi \,\mathsf{W}\, \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \,\mathsf{W}\, \psi))$	largest solution

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Words$(\varphi \,\mathsf{U}\, \psi)$ smallest LT-property E s.t.

Expansion laws for U and W

$\varphi \, \mathsf{U} \, \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$	smallest solution
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$Words(\varphi \, \mathsf{U} \, \psi)$ smallest LT-property E s.t.

$$
\begin{array}{ll}
(1) & Words(\psi) \subseteq E \\
(2) & \{A_0 A_1 A_2 \ldots \in Words(\varphi) : A_1 A_2 \ldots \in E\} \subseteq E
\end{array}
$$

Expansion laws for U and W

$\varphi \,\mathsf{U}\, \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$	smallest solution
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$$(1) \quad Words(\psi) \subseteq E$$

$$(2) \quad \{A_0A_1A_2\ldots \in Words(\varphi) : A_1A_2\ldots \in E\} \subseteq E$$

$$\Updownarrow$$

$$Words(\psi) \cup \{A_0A_1A_2\ldots \in Words(\varphi) : A_1A_2\ldots \in E\} \subseteq E$$

Expansion laws for U and W

$\varphi \mathsf{U} \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \mathsf{U} \psi))$	smallest solution
$\varphi \mathsf{W} \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \mathsf{W} \psi))$	largest solution

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$Words(\varphi \, \mathsf{W} \, \psi)$ largest LT-property E s.t.

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Expansion laws for U and W

$\varphi \,\mathsf{U}\, \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$	smallest solution
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$Words(\varphi \,\mathsf{W}\, \psi)$ largest LT-property E s.t.

$$E \subseteq Words(\psi) \cup \{A_0A_1A_2\ldots \in Words(\varphi) : A_1A_2\ldots \in E\}$$

Expansion laws for U and W

$$\varphi \,\mathsf{U}\, \psi \quad \equiv \quad \psi \ \vee \ (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$$

smallest solution

$$\varphi \,\mathsf{W}\, \psi \quad \equiv \quad \psi \ \vee \ (\varphi \wedge \bigcirc(\varphi \,\mathsf{W}\, \psi))$$

largest solution

Expansion laws for U, W, $\Diamond$, and $\Box$

$$\varphi \,\mathsf{U}\, \psi \quad \equiv \quad \psi \;\vee\; (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$$

smallest solution

$$\Diamond\psi \quad \equiv \quad \psi \;\vee\; \bigcirc\Diamond\psi$$

smallest solution

$$\varphi \,\mathsf{W}\, \psi \quad \equiv \quad \psi \;\vee\; (\varphi \wedge \bigcirc(\varphi \,\mathsf{W}\, \psi))$$

largest solution

$$\Box\varphi \quad \equiv \quad \varphi \wedge \bigcirc\Box\varphi$$

largest solution

---

remind: $\Diamond\psi \;=\; \mathit{true} \,\mathsf{U}\, \psi, \quad \Box\varphi \;\equiv\; \varphi \,\mathsf{W}\, \mathit{false}$

Positive normal form (PNF)

Positive normal form (PNF)

- negation only on the level of literals
- uses for each operator its dual

PNF for propositional logic

- negation only on the level of literals
- uses for each operator its dual

syntax of propositional formulas in PNF:

$$\varphi \ ::= \ \mathit{true} \mid \mathit{false} \mid a \mid \neg a \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2$$

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syntax of propositional formulas in PNF:

$$\varphi \ ::= \ \mathit{true} \mid \mathit{false} \mid a \mid \neg a \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2$$

$\neg \mathit{true} \ \equiv \ \mathit{false}$	duality of the constant truth values
$\neg(\varphi_1 \wedge \varphi_2) \ \equiv \ \neg\varphi_1 \vee \neg\varphi_2$	duality of $\vee$ and $\wedge$ (de Morgan's law)

LTL in positive normal form (PNF)

- negation only on the level of literals
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using duality of constants and duality of $\vee$ and $\wedge$

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$$\varphi \quad ::= \quad \mathit{true} \mid \mathit{false} \mid a \mid \neg a \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid$$

$$\bigcirc \varphi + \text{dual operator for } \bigcirc$$

using duality of constants and duality of $\vee$ and $\wedge$

LTL in positive normal form (PNF)

- negation only on the level of literals
- uses for each operator its dual

$$\varphi \ ::= \ true \mid false \mid a \mid \neg a \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid$$

$$\bigcirc \varphi \longleftarrow$$ no new operator needed for $\neg \bigcirc$

using duality of constants and duality of $\vee$ and $\wedge$

$\neg \bigcirc \varphi \ \equiv \ \bigcirc \neg \varphi$ self-duality of the next operator

LTL in positive normal form (PNF)

- negation only on the level of literals
- uses for each operator its dual

$$\varphi ::= \mathit{true} \mid \mathit{false} \mid a \mid \neg a \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \bigcirc\varphi \mid \varphi_1 \mathbf{U} \varphi_2 \ + \text{dual operator for } \mathbf{U}$$

using duality of constants and duality of $\vee$ and $\wedge$

$\neg \bigcirc \varphi \equiv \bigcirc \neg \varphi$ self-duality of the next operator

LTL in positive normal form (PNF)

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- uses for each operator its dual

$$\varphi \; ::= \; \mathit{true} \mid \mathit{false} \mid a \mid \neg a \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \bigcirc\varphi \mid \varphi_1 \,\mathsf{U}\, \varphi_2 \mid \varphi_1 \,\mathsf{W}\, \varphi_2$$

using duality of constants and duality of $\vee$ and $\wedge$

$\neg \bigcirc \varphi \;\equiv\; \bigcirc \neg \varphi$ self-duality of the next operator

$\neg(\varphi_1 \,\mathsf{U}\, \varphi_2) \;\equiv\; (\neg\varphi_2)\,\mathsf{W}(\neg\varphi_1 \wedge \neg\varphi_2)$

duality of **U** and **W**

Derivation of $\Diamond$ and $\Box$ in LTL-PNF

$$\varphi \quad ::= \quad \mathit{true} \mid \mathit{false} \mid a \mid \neg a \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \bigcirc \varphi \mid \varphi_1 \, \mathsf{U} \, \varphi_2 \mid \varphi_1 \, \mathsf{W} \, \varphi_2$$

$$\varphi \quad ::= \quad \mathit{true} \mid \mathit{false} \mid a \mid \neg a \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \bigcirc\varphi \mid \varphi_1 \mathsf{U} \varphi_2 \mid \varphi_1 \mathsf{W} \varphi_2 \mid \Diamond\varphi \mid \Box\varphi$$

$\Diamond$ and $\Box$ can (still) be derived:

$$\Diamond\varphi \stackrel{\text{def}}{=} \mathit{true}\,\mathsf{U}\,\varphi$$

$$\Box\varphi \stackrel{\text{def}}{=} \varphi\,\mathsf{W}\,\mathit{false}$$

Universality of LTL-PNF

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Each LTL formula can be transformed into an equivalent LTL formula in **PNF**

Universality of LTL-PNF

> Each LTL formula can be transformed into an equivalent LTL formula in **PNF**

LTL formula $\varphi \leadsto$ LTL formula in PNF φ'
by successive application of the following rules:

> Each LTL formula can be transformed into an equivalent LTL formula in **PNF**

LTL formula $\varphi \rightsquigarrow$ LTL formula in PNF φ'
by successive application of the following rules:

$$\begin{array}{lcl}
\neg \mathit{true} & \rightsquigarrow & \mathit{false} \\
\neg\neg\varphi & \rightsquigarrow & \varphi \\
\neg(\varphi_1 \wedge \varphi_2) & \rightsquigarrow & \neg\varphi_1 \vee \neg\varphi_2 \\
\neg \bigcirc \varphi & \rightsquigarrow & \bigcirc\neg\varphi \\
\neg(\varphi_1 \, \mathsf{U} \, \varphi_2) & \rightsquigarrow & (\neg\varphi_2) \, \mathsf{W} (\neg\varphi_1 \wedge \neg\varphi_2)
\end{array}$$

Each LTL formula can be transformed into an equivalent LTL formula in **PNF**

LTL formula $\varphi \rightsquigarrow$ LTL formula in PNF φ'
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$$
\begin{array}{lcl}
\neg \mathit{true} & \rightsquigarrow & \mathit{false} \\
\neg\neg\varphi & \rightsquigarrow & \varphi \\
\neg(\varphi_1 \wedge \varphi_2) & \rightsquigarrow & \neg\varphi_1 \vee \neg\varphi_2 \\
\neg \bigcirc \varphi & \rightsquigarrow & \bigcirc\neg\varphi \\
\neg(\varphi_1 \mathsf{U} \varphi_2) & \rightsquigarrow & (\neg\varphi_2) \mathsf{W} (\neg\varphi_1 \wedge \neg\varphi_2)
\end{array}
$$

exponential-blow up is possible

Example: LTL $\rightsquigarrow$ LTL-PNF

$$\neg \mathit{true} \rightsquigarrow \mathit{false}$$

$$\neg\neg\varphi \rightsquigarrow \varphi$$

$$\neg(\varphi_1 \wedge \varphi_2) \rightsquigarrow \neg\varphi_1 \vee \neg\varphi_2$$

$$\neg \bigcirc \varphi \rightsquigarrow \bigcirc \neg\varphi$$

$$\neg(\varphi_1 \mathbf{U} \varphi_2) \rightsquigarrow (\neg\varphi_2) \mathbf{W} (\neg\varphi_1 \wedge \neg\varphi_2)$$

Example: LTL $\rightsquigarrow$ LTL-PNF

$\neg \textit{true}$	$\rightsquigarrow$	$\textit{false}$	+ analogue rule for $\neg \textit{false}$
$\neg\neg\varphi$	$\rightsquigarrow$	φ	
$\neg(\varphi_1 \wedge \varphi_2)$	$\rightsquigarrow$	$\neg\varphi_1 \vee \neg\varphi_2$	+ analogue rule for $\neg\vee$
$\neg \bigcirc \varphi$	$\rightsquigarrow$	$\bigcirc\neg\varphi$	
$\neg(\varphi_1 \, \mathbf{U} \, \varphi_2)$	$\rightsquigarrow$	$(\neg\varphi_2) \, \mathbf{W} (\neg\varphi_1 \wedge \neg\varphi_2)$	

Example: LTL $\rightsquigarrow$ LTL-PNF

$\neg \mathit{true}$	$\rightsquigarrow$	false	+ analogue rule for $\neg \mathit{false}$
$\neg\neg\varphi$	$\rightsquigarrow$	φ	
$\neg(\varphi_1 \wedge \varphi_2)$	$\rightsquigarrow$	$\neg\varphi_1 \vee \neg\varphi_2$	+ analogue rule for $\neg\vee$
$\neg \bigcirc \varphi$	$\rightsquigarrow$	$\bigcirc\neg\varphi$	
$\neg(\varphi_1 \,\mathsf{U}\, \varphi_2)$	$\rightsquigarrow$	$(\neg\varphi_2)\,\mathsf{W}(\neg\varphi_1 \wedge \neg\varphi_2)$	
$\neg\Diamond\varphi$	$\rightsquigarrow$	$\Box\neg\varphi$	$\neg\Box\varphi \rightsquigarrow \Diamond\neg\varphi$

$\neg \mathit{true}$	$\rightsquigarrow$	false	+ analogue rule for $\neg \mathit{false}$
$\neg\neg\varphi$	$\rightsquigarrow$	φ	
$\neg(\varphi_1 \wedge \varphi_2)$	$\rightsquigarrow$	$\neg\varphi_1 \vee \neg\varphi_2$	+ analogue rule for $\neg\vee$
$\neg \bigcirc \varphi$	$\rightsquigarrow$	$\bigcirc\neg\varphi$	
$\neg(\varphi_1 \mathsf{U} \varphi_2)$	$\rightsquigarrow$	$(\neg\varphi_2) \mathsf{W} (\neg\varphi_1 \wedge \neg\varphi_2)$	
$\neg\Diamond\varphi$	$\rightsquigarrow$	$\Box\neg\varphi$	$\neg\Box\varphi \rightsquigarrow \Diamond\neg\varphi$

$$\neg\Box((a \mathsf{U} b) \vee \bigcirc c)$$

Example: LTL $\rightsquigarrow$ LTL-PNF

$$
\begin{array}{lll}
\neg \mathit{true} & \rightsquigarrow & \mathit{false} \quad + \text{ analogue rule for } \neg \mathit{false} \\
\neg\neg\varphi & \rightsquigarrow & \varphi \\
\neg(\varphi_1 \wedge \varphi_2) & \rightsquigarrow & \neg\varphi_1 \vee \neg\varphi_2 \quad + \text{ analogue rule for } \neg\vee \\
\neg \bigcirc \varphi & \rightsquigarrow & \bigcirc\neg\varphi \\
\neg(\varphi_1 \,\mathsf{U}\, \varphi_2) & \rightsquigarrow & (\neg\varphi_2)\,\mathsf{W}(\neg\varphi_1 \wedge \neg\varphi_2) \\
\neg\Diamond\varphi & \rightsquigarrow & \Box\neg\varphi \qquad \neg\Box\varphi \;\rightsquigarrow\; \Diamond\neg\varphi
\end{array}
$$

$$\neg\Box((a \,\mathsf{U}\, b) \;\vee\; \bigcirc c)$$

$$\equiv \; \Diamond\neg((a \,\mathsf{U}\, b) \;\vee\; \bigcirc c)$$

$\longleftarrow$ duality of $\Diamond$ and $\Box$

Example: LTL $\leadsto$ LTL-PNF

$$
\begin{array}{lll}
\neg \mathit{true} & \leadsto & \mathit{false} \quad + \text{ analogue rule for } \neg \mathit{false} \\
\neg\neg\varphi & \leadsto & \varphi \\
\neg(\varphi_1 \wedge \varphi_2) & \leadsto & \neg\varphi_1 \vee \neg\varphi_2 \quad + \text{ analogue rule for } \neg\vee \\
\neg \bigcirc \varphi & \leadsto & \bigcirc\neg\varphi \\
\neg(\varphi_1 \,\mathsf{U}\, \varphi_2) & \leadsto & (\neg\varphi_2)\,\mathsf{W}(\neg\varphi_1 \wedge \neg\varphi_2) \\
\neg\Diamond\varphi & \leadsto & \Box\neg\varphi \qquad \neg\Box\varphi \leadsto \Diamond\neg\varphi
\end{array}
$$

$$\neg\Box((a \,\mathsf{U}\, b) \vee \bigcirc c)$$

$$\equiv \Diamond\neg((a \,\mathsf{U}\, b) \vee \bigcirc c)$$ ← duality of $\Diamond$ and $\Box$

$$\equiv \Diamond(\neg(a \,\mathsf{U}\, b) \wedge \neg\bigcirc c)$$ ← duality of $\wedge$ and $\vee$

Example: LTL $\rightsquigarrow$ LTL-PNF

$\neg \mathit{true}$	$\rightsquigarrow$	false	+ analogue rule for $\neg \mathit{false}$
$\neg\neg\varphi$	$\rightsquigarrow$	φ	
$\neg(\varphi_1 \wedge \varphi_2)$	$\rightsquigarrow$	$\neg\varphi_1 \vee \neg\varphi_2$	+ analogue rule for $\neg\vee$
$\neg \bigcirc \varphi$	$\rightsquigarrow$	$\bigcirc\neg\varphi$	
$\neg(\varphi_1 \mathsf{U} \varphi_2)$	$\rightsquigarrow$	$(\neg\varphi_2)\mathsf{W}(\neg\varphi_1 \wedge \neg\varphi_2)$	
$\neg\Diamond\varphi$	$\rightsquigarrow$	$\Box\neg\varphi$	$\neg\Box\varphi \rightsquigarrow \Diamond\neg\varphi$

$$\neg\Box((a \mathsf{U} b) \vee \bigcirc c)$$

$$\equiv \Diamond\neg((a \mathsf{U} b) \vee \bigcirc c) \quad \longleftarrow \text{duality of } \Diamond \text{ and } \Box$$

$$\equiv \Diamond(\neg(a \mathsf{U} b) \wedge \neg\bigcirc c) \quad \longleftarrow \text{duality of } \wedge \text{ and } \vee$$

$$\equiv \Diamond(\neg(a \mathsf{U} b) \wedge \bigcirc\neg c) \quad \longleftarrow \text{self-duality of } \bigcirc$$

$\neg \mathit{true}$	$\rightsquigarrow$	false	+ analogue rule for $\neg \mathit{false}$
$\neg\neg\varphi$	$\rightsquigarrow$	φ	
$\neg(\varphi_1 \wedge \varphi_2)$	$\rightsquigarrow$	$\neg\varphi_1 \vee \neg\varphi_2$	+ analogue rule for $\neg\vee$
$\neg \bigcirc \varphi$	$\rightsquigarrow$	$\bigcirc\neg\varphi$	
$\neg(\varphi_1 \mathsf{U} \varphi_2)$	$\rightsquigarrow$	$(\neg\varphi_2) \mathsf{W} (\neg\varphi_1 \wedge \neg\varphi_2)$	
$\neg\Diamond\varphi$	$\rightsquigarrow$	$\Box\neg\varphi$	$\neg\Box\varphi \rightsquigarrow \Diamond\neg\varphi$

$$\neg\Box((a \mathsf{U} b) \vee \bigcirc c)$$

$$\equiv \Diamond\neg((a \mathsf{U} b) \vee \bigcirc c) \quad \longleftarrow \text{duality of } \Diamond \text{ and } \Box$$

$$\equiv \Diamond(\neg(a \mathsf{U} b) \wedge \neg\bigcirc c) \quad \longleftarrow \text{duality of } \wedge \text{ and } \vee$$

$$\equiv \Diamond((\neg b) \mathsf{W} (\neg a \wedge \neg b) \wedge \bigcirc\neg c) \quad \longleftarrow \text{duality of } \mathsf{U} \text{ and } \mathsf{W}$$

Example: LTL $\rightsquigarrow$ LTL-PNF

$\neg \mathit{true}$	$\rightsquigarrow$	false	+ analogue rule for $\neg \mathit{false}$
$\neg\neg\varphi$	$\rightsquigarrow$	φ	
$\neg(\varphi_1 \wedge \varphi_2)$	$\rightsquigarrow$	$\neg\varphi_1 \vee \neg\varphi_2$	+ analogue rule for $\neg\vee$
$\neg \bigcirc \varphi$	$\rightsquigarrow$	$\bigcirc\neg\varphi$	
$\neg(\varphi_1 \, \mathsf{U} \, \varphi_2)$	$\rightsquigarrow$	$(\neg\varphi_2) \, \mathsf{W} (\neg\varphi_1 \wedge \neg\varphi_2)$	
$\neg\Diamond\varphi$	$\rightsquigarrow$	$\Box\neg\varphi$	$\neg\Box\varphi \rightsquigarrow \Diamond\neg\varphi$

$$\neg\Box((a \, \mathsf{U} \, b) \vee \bigcirc c)$$

$$\equiv \Diamond\neg((a \, \mathsf{U} \, b) \vee \bigcirc c)$$

$$\equiv \Diamond(\neg(a \, \mathsf{U} \, b) \wedge \neg\bigcirc c)$$

$$\equiv \Diamond((\neg b) \, \mathsf{W} (\neg a \wedge \neg b) \wedge \bigcirc\neg c) \longleftarrow \text{PNF}$$

Fairness in LTL

LTLSF3.1-38

Recall: action-based fairness

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fairness assumption for TS $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$:

$$\mathcal{F} = (\mathcal{F}_{ucond}, \mathcal{F}_{strong}, \mathcal{F}_{weak})$$

where $\mathcal{F}_{ucond}$, $\mathcal{F}_{strong}$, $\mathcal{F}_{weak} \subseteq 2^{Act}$

- $\mathcal{F}_{ucond}$ unconditional fairness assumption
- $\mathcal{F}_{strong}$ strong fairness assumption
- $\mathcal{F}_{weak}$ weak fairness assumption

Recall: action-based fairness

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execution $s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} s_2 \xrightarrow{\alpha_3} \ldots$ $\mathcal{F}$-fair if

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- for all $A \in \mathcal{F}_{ucond}$: $\overset{\infty}{\exists}\, i \geq 1.\, \alpha_i \in A$

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- for all $A \in \mathcal{F}_{ucond}$: $\overset{\infty}{\exists} i \geq 1. \alpha_i \in A$
- for all $A \in \mathcal{F}_{strong}$:

$$\overset{\infty}{\exists} i \geq 1. A \cap Act(s_i) \neq \varnothing \implies \overset{\infty}{\exists} i \geq 1. \alpha_i \in A$$

fairness assumption for TS $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$:

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- for all $A \in \mathcal{F}_{weak}$:

  $$\overset{\infty}{\forall}\, i \geq 1.\, A \cap Act(s_i) \neq \varnothing \implies \overset{\infty}{\exists}\, i \geq 1.\, \alpha_i \in A$$

Recall: action-based fairness

fairness assumption for TS $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$:

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where $\mathcal{F}_{ucond}$, $\mathcal{F}_{strong}$, $\mathcal{F}_{weak} \subseteq 2^{Act}$

satisfaction relation for LT-properties under fairness:

$\mathcal{T} \models_{\mathcal{F}} E$ iff for all $\mathcal{F}$-fair paths π of $\mathcal{T}$: $trace(\pi) \in E$

Process fairness is LTL-definable

Process fairness is LTL-definable

$$\varphi ::= \mathit{true} \mid a \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \bigcirc \varphi \mid \varphi_1 \, \mathsf{U} \, \varphi_2$$

eventually $\Diamond\varphi \stackrel{\text{def}}{=} \mathit{true} \, \mathsf{U} \, \varphi$

always $\Box\varphi \stackrel{\text{def}}{=} \neg\Diamond\neg\varphi$

infinitely often $\Box\Diamond\varphi$

eventually forever $\Diamond\Box\varphi$

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e.g., unconditional fairness $\Box\Diamond \mathit{crit}_i$

strong fairness $\Box\Diamond \mathit{wait}_i \rightarrow \Box\Diamond \mathit{crit}_i$

Process fairness is LTL-definable

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e.g.,	unconditional fairness	$\Box\Diamond \textit{crit}_i$
	strong fairness	$\Box\Diamond \textit{wait}_i \rightarrow \Box\Diamond \textit{crit}_i$
	weak fairness	$\Diamond\Box \textit{wait}_i \rightarrow \Box\Diamond \textit{crit}_i$

LTL fairness assumptions

LTL fairness assumptions

... are conjunctions of LTL formulas of the form:

- unconditional fairness $\Box\Diamond\phi$
- strong fairness $\Box\Diamond\phi_1 \rightarrow \Box\Diamond\phi_2$
- weak fairness $\Diamond\Box\phi_1 \rightarrow \Box\Diamond\phi_2$

where ϕ_1, ϕ_2, ϕ are propositional formulas

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where ϕ_1, ϕ_2, ϕ are propositional formulas

If *fair* is a LTL fairness assumption, s a state in a TS, and φ an LTL formula then

$s \models_{fair} \varphi$ iff for all $\pi \in Paths(s)$:
if $\pi \models fair$ then $\pi \models \varphi$

LTL fairness assumptions

... are conjunctions of **LTL formulas** of the form:

- unconditional fairness $\Box\Diamond\phi$
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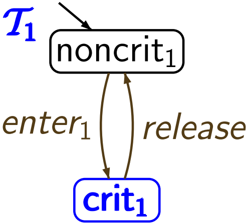
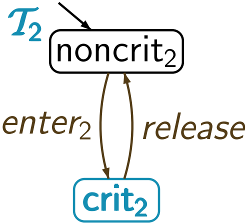
If *fair* is a LTL fairness assumption, s a state in a TS, and φ an LTL formula then

$$s \models_{fair} \varphi \quad \text{iff} \quad \text{for all } \pi \in Paths(s)\text{: if } \pi \models fair \text{ then } \pi \models \varphi$$

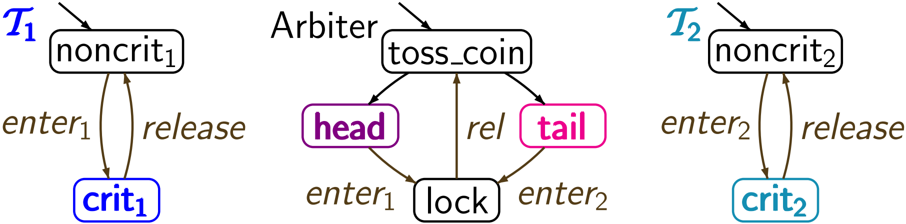
$$\text{iff} \quad s \models fair \rightarrow \varphi$$

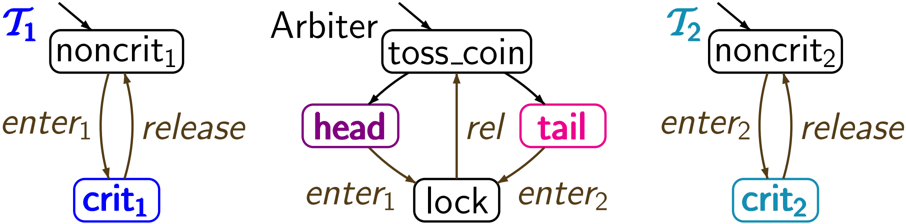
Randomized arbiter for MUTEX



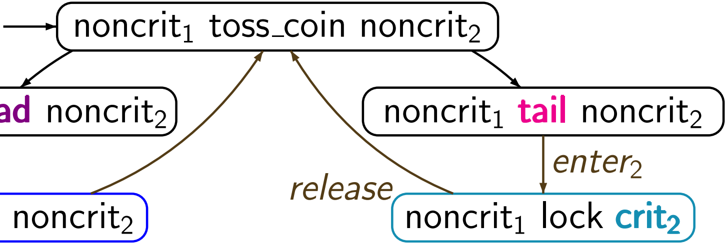


Randomized arbiter for MUTEX

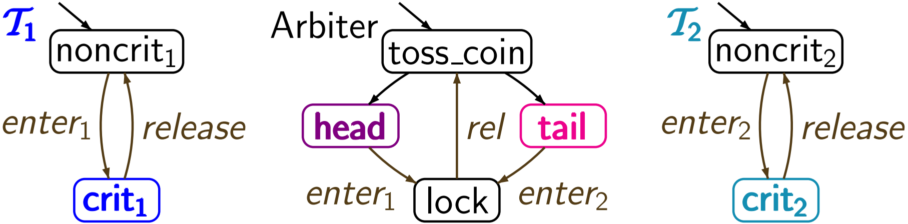




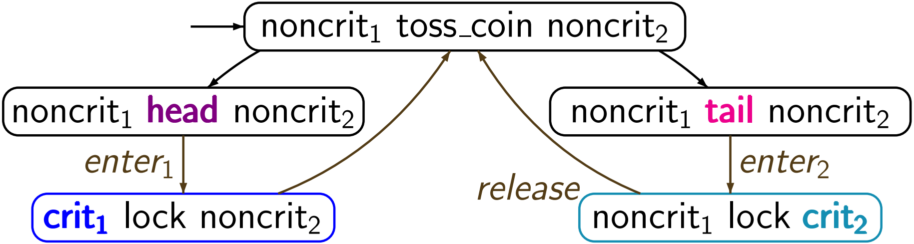
$(\mathcal{T}_1 \,|||\, \mathcal{T}_2) \,\|\, \textit{Arbiter}$



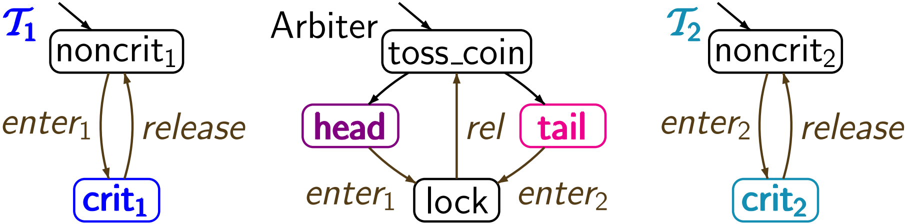
Randomized arbiter for MUTEX



$$(\mathcal{T}_1 \;|||\; \mathcal{T}_2) \parallel \mathit{Arbiter} \not\models \Box\Diamond \mathsf{crit}_1 \wedge \Box\Diamond \mathsf{crit}_2$$



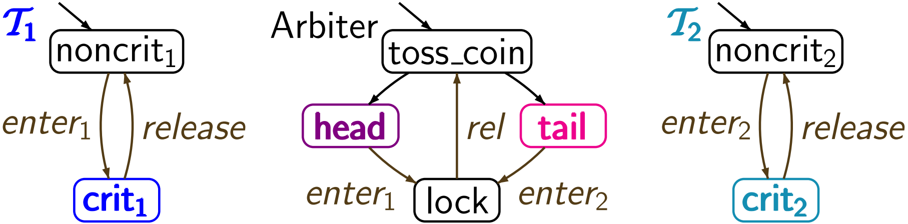
Randomized arbiter for MUTEX



unconditional LTL-fairness:

$$fair \;=\; \Box\Diamond \mathbf{head} \;\wedge\; \Box\Diamond \mathbf{tail}$$

Randomized arbiter for MUTEX



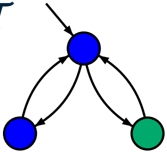
unconditional LTL-fairness:

$$fair = \Box\Diamond \mathsf{head} \wedge \Box\Diamond \mathsf{tail}$$

$$(\mathcal{T}_1 \mid\mid\mid \mathcal{T}_2) \parallel Arbiter \models_{fair} \Box\Diamond \mathsf{crit}_1 \wedge \Box\Diamond \mathsf{crit}_2$$

Correct or wrong?

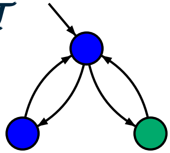
$\mathcal{T}$



LTL fairness assumption
$fair = \Diamond\Box a \rightarrow \Box\Diamond b$

 $\widehat{=} \{a\}$  $\widehat{=} \{b\}$

$\mathcal{T}$



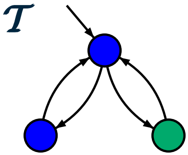
LTL fairness assumption

$fair = \Diamond\Box a \to \Box\Diamond b$

 $\widehat{=}\{a\}$  $\widehat{=}\{b\}$

$\mathcal{T} \models_{fair} \bigcirc b$ **?**

Correct or wrong?



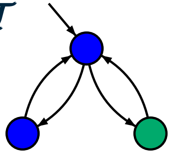
LTL fairness assumption
$fair = \Diamond\Box a \rightarrow \Box\Diamond b$

● $\widehat{=} \{a\}$ ● $\widehat{=} \{b\}$

$\mathcal{T} \not\models_{fair} \bigcirc b$ as ●→●→●→●→●→●→ ... is fair

Correct or wrong?

$\mathcal{T}$



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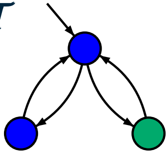
$\mathcal{T} \models_{fair} a \,\mathsf{U}\, b$ **?**

Correct or wrong?

$\mathcal{T}$



LTL fairness assumption

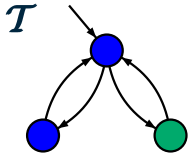
$fair = \Diamond\Box a \rightarrow \Box\Diamond b$

● (blue) $\widehat{=} \{a\}$ ● (green) $\widehat{=} \{b\}$

$\mathcal{T} \not\models_{fair} \bigcirc b$ as ● (blue) → ● (blue) → ● (blue) → ● (green) → ● (blue) → ● (green) → … is fair

$\mathcal{T} \models_{fair} a \, \mathsf{U} \, b$ ✓

Correct or wrong?



LTL fairness assumption

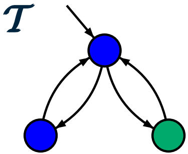
$fair = \Diamond\Box a \rightarrow \Box\Diamond b$

● $\widehat{=} \{a\}$ ● $\widehat{=} \{b\}$

$\mathcal{T} \not\models_{fair} \bigcirc b$ as ●→●→●→●→●→●→ ... is fair

$\mathcal{T} \models_{fair} a \,\mathsf{U}\, b$ √

$\mathcal{T} \models_{fair} a \,\mathsf{U}\, \Box(b \leftrightarrow \bigcirc a)$ **?**



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$\mathcal{T} \models_{fair} a \,\mathsf{U}\, b$ ✓

$\mathcal{T} \not\models_{fair} a \,\mathsf{U}\, \Box(b \leftrightarrow \bigcirc a)$

as ●→●→●→●→●→●→ ... is fair

LTL-fairness assumptions

- can be necessary to prove liveness properties, e.g., mutual exclusion with arbiter/semaphore

$$\mathcal{T}_{sem} \not\models \Box\Diamond crit_1 \wedge \Box\Diamond crit_2$$

$$\mathcal{T}_{sem} \models_{fair} \Box\Diamond crit_1 \wedge \Box\Diamond crit_2$$

for appropriate fairness condition

LTL-fairness assumptions

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$$\mathcal{T}_{sem} \not\models \Box\Diamond crit_1 \wedge \Box\Diamond crit_2$$

$$\mathcal{T}_{sem} \models_{fair} \Box\Diamond crit_1 \wedge \Box\Diamond crit_2$$

for appropriate fairness condition, e.g.,

$$fair = \bigwedge_{i=1,2} \Big((\Box\Diamond wait_i \rightarrow \Box\Diamond crit_i) \wedge (\Diamond\Box noncrit_i \rightarrow \Box\Diamond wait_i) \Big)$$

- can be necessary to prove liveness properties, e.g., mutual exclusion with arbiter/semaphore

$$\mathcal{T}_{sem} \not\models \Box\Diamond crit_1 \wedge \Box\Diamond crit_2$$

$$\mathcal{T}_{sem} \models_{fair} \Box\Diamond crit_1 \wedge \Box\Diamond crit_2$$

for appropriate fairness condition

- can be verifiable system properties

e.g., Peterson algorithm guarantees strong fairness

$$\mathcal{T}_{Pet} \models \Box\Diamond wait_1 \rightarrow \Box\Diamond crit_1$$

LTL-fairness assumptions

- can be necessary to prove liveness properties, e.g.,

  $$\mathcal{T}_{sem} \not\models \Box\Diamond crit_1 \wedge \Box\Diamond crit_2$$

  $$\mathcal{T}_{sem} \models_{fair} \Box\Diamond crit_1 \wedge \Box\Diamond crit_2$$

  for appropriate fairness condition

- can be verifiable system properties, e.g.,

  $$\mathcal{T}_{Pet} \models \Box\Diamond wait_1 \rightarrow \Box\Diamond crit_1$$

- are irrelevant for verifying safety properties

$$\mathcal{T} \models \varphi_{safe} \quad \text{iff} \quad \mathcal{T} \models_{fair} \varphi_{safe}$$

if *fair* is realizable

Correct or wrong?

Each strong **LTL** fairness assumption

$$fair \;=\; \Box\Diamond a \rightarrow \Box\Diamond b$$

is realizable for each TS over $AP = \{a, b, \ldots\}$.

Correct or wrong?

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$$fair \; = \; \Box\Diamond a \rightarrow \Box\Diamond b$$

is realizable for each TS over $AP = \{a, b, \ldots\}$.

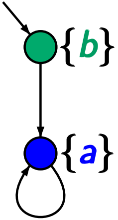
recall: a fairness condition is called realizable if for each reachable state s there exists a fair path starting in s

Each strong **LTL** fairness assumption

$$fair = \Box\Diamond a \rightarrow \Box\Diamond b$$

is realizable for each TS over $AP = \{a, b, \ldots\}$.

wrong



$fair = \Box\Diamond a \rightarrow \Box\Diamond b$

is <u>not</u> realizable

Action-based fairness $\rightsquigarrow$ LTL-fairness

Action-based fairness $\leadsto$ LTL-fairness

idea: use new atomic propositions ***enabled***(A) and ***taken***(A) and extend the labeling function:

$$enabled(A) \in L(s) \quad \text{iff} \quad s \xrightarrow{\alpha} \ldots \text{ for some } \alpha \in A$$

$$taken(A) \in L(s) \quad \text{iff} \quad \text{for all transitions } \ldots \xrightarrow{\alpha} s\text{:}\ \alpha \in A$$

Action-based fairness $\rightsquigarrow$ LTL-fairness

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- unconditional A-fairness: $\Box\Diamond taken(A)$
- strong A-fairness: $\Box\Diamond enabled(A) \rightarrow \Box\Diamond taken(A)$
- weak A-fairness: $\Diamond\Box enabled(A) \rightarrow \Box\Diamond taken(A)$

Action-based fairness $\rightsquigarrow$ LTL-fairness

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$$taken(A) \in L(s) \quad \text{iff} \quad \text{for } \boxed{\text{all}} \text{ transitions } \ldots \xrightarrow{\alpha} s\text{:} \quad \alpha \in A$$

problem: each state s can have several incoming transitions

$$t \xrightarrow{\alpha} s, \quad u \xrightarrow{\beta} s, \ \ldots$$

Action-based fairness $\rightsquigarrow$ LTL-fairness

idea: use new atomic propositions ***enabled***(A) and ***taken***(A) and extend the labeling function:

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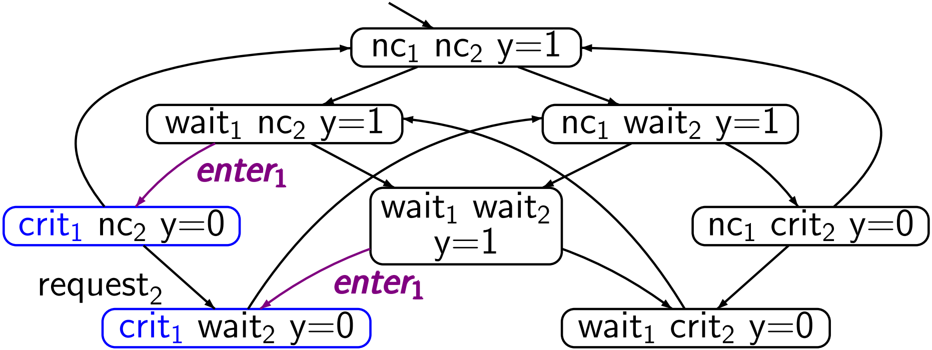
alternative 1: ad-hoc choice of "***taken***-predicate"

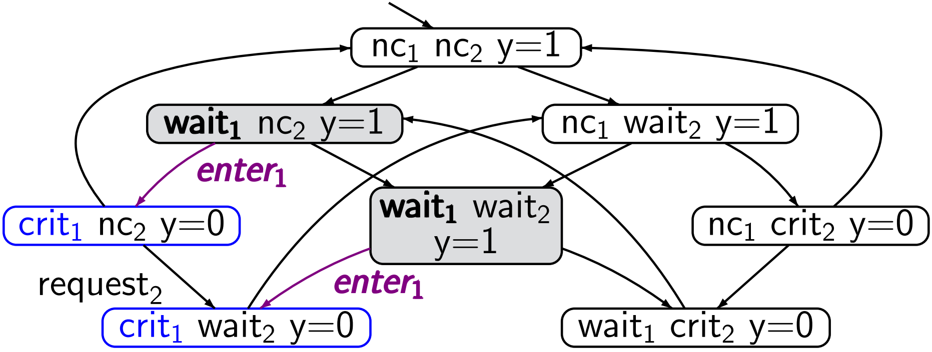
alternative 2: modify the given transition system by adding an action component to the states

Ad-hoc: action fairness ⇝ LTL-fairness



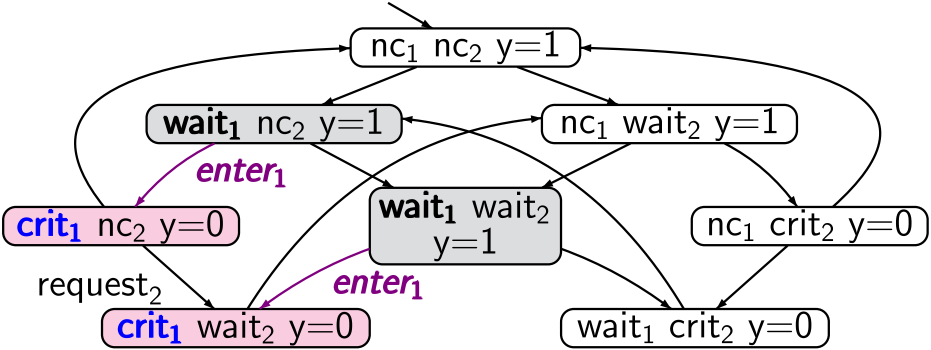
TS for mutual exclusion with semaphore

Ad-hoc: action fairness ⇝ LTL-fairness



□ $\widehat{=}$ *enabled*({*enter*$_1$})

TS for mutual exclusion with semaphore

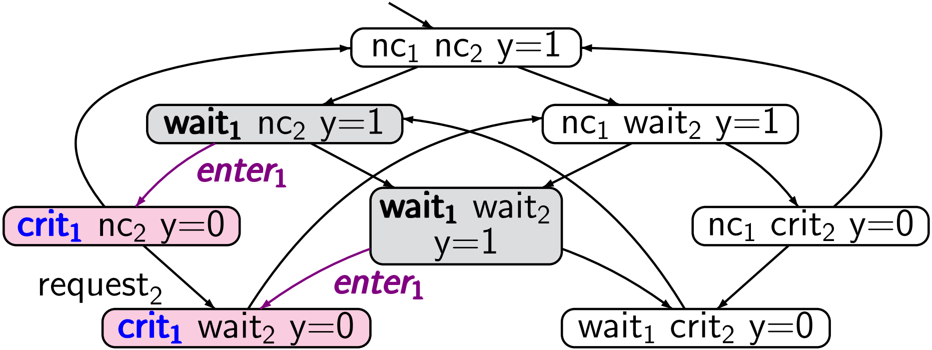


(pink box) $\widehat{=}$ *taken*({*enter*$_1$})

(grey box) $\widehat{=}$ *enabled*({*enter*$_1$})

TS for mutual exclusion with semaphore

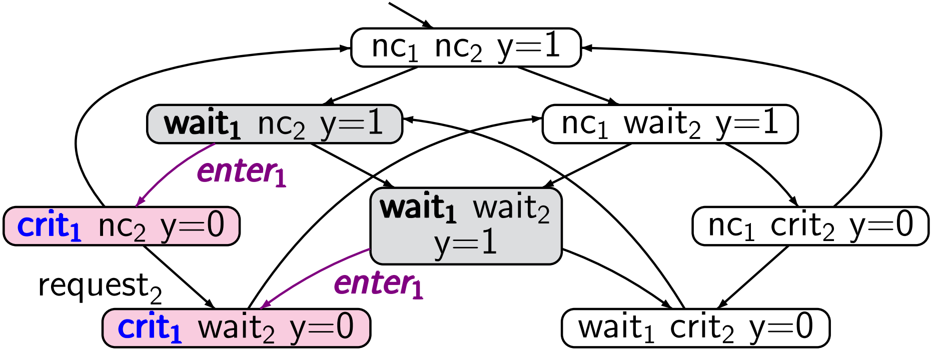
Ad-hoc: action fairness ⇝ LTL-fairness



(pink box) $\widehat{=}$ *taken*({*enter*$_1$}) $\widehat{=}$ **crit$_1$**

(grey box) $\widehat{=}$ *enabled*({*enter*$_1$}) $\widehat{=}$ **wait$_1$** $\wedge \neg$**crit$_2$**

Ad-hoc: action fairness ⇝ LTL-fairness

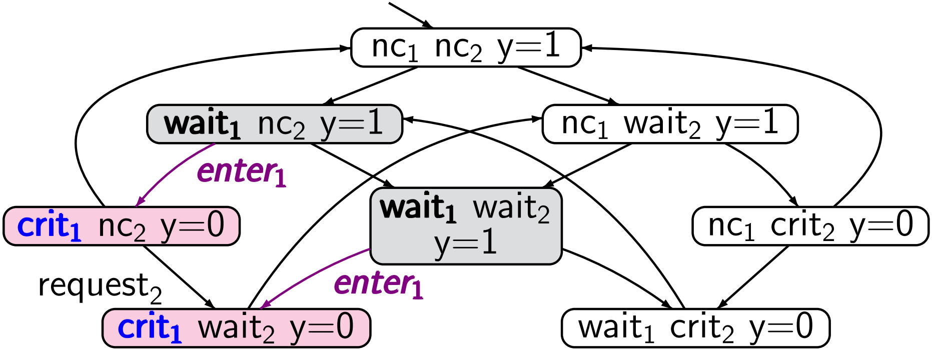


(pink box)	$\widehat{=}$	*taken*({*enter*$_1$})	$\widehat{=}$	**crit$_1$**
(grey box)	$\widehat{=}$	*enabled*({*enter*$_1$})	$\widehat{=}$	**wait$_1$** $\wedge \neg$**crit$_2$**

strong {*enter*$_1$}-fairness: LTL formula

$$\Box\Diamond\, enabled(\{enter_1\}) \rightarrow \Box\Diamond\, taken(\{enter_1\})$$

Ad-hoc: action fairness ⇝ LTL-fairness



(pink box) $\widehat{=}$ *taken*({*enter*$_1$}) $\widehat{=}$ crit$_1$

(grey box) $\widehat{=}$ *enabled*({*enter*$_1$}) $\widehat{=}$ wait$_1 \wedge \neg$crit$_2$

$$\Box\Diamond\, \textit{enabled}(\{\textit{enter}_1\}) \rightarrow \Box\Diamond\, \textit{taken}(\{\textit{enter}_1\})$$

$$\widehat{=}\ \Box\Diamond\, (\text{wait}_1 \wedge \neg\text{crit}_2) \rightarrow \Box\Diamond\, \text{crit}_1$$

Action-based fairness $\rightsquigarrow$ LTL-fairness

idea: use new atomic propositions ***enabled*(*A*)** and ***taken*(*A*)** and extend the labeling function:

$$enabled(A) \in L(s) \text{ iff } s \xrightarrow{\alpha} \ldots \text{ for some } \alpha \in A$$

$$taken(A) \in L(s) \text{ iff for all transitions } \ldots \xrightarrow{\alpha} s\text{: } \alpha \in A$$

alternative 1: **ad-hoc choice** of "***taken***-predicate"

alternative 2: modify the given transition system by adding an action component to the states *idea*: use new atomic propositions ***enabled(A)*** and ***taken(A)*** and extend the labeling function:

$$\mathit{enabled}(A) \in L(s) \quad \text{iff} \quad s \xrightarrow{\alpha} \ldots \text{ for some } \alpha \in A$$

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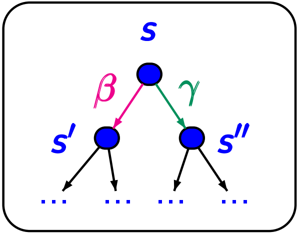
alternative 1: ad-hoc choice of "***taken***-predicate"

alternative 2: modify the given transition system by **adding an action component** to the states

Action-based fairness ⇝ LTL-fairness

transition system

$\mathcal{T} = (S, Act, \rightarrow, \ldots)$



Action-based fairness ⇝ LTL-fairness

transition system
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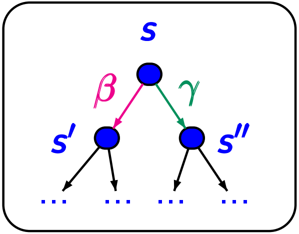
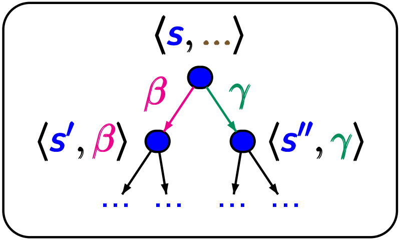


transition system
$\mathcal{T}' = (S \times Act, \ldots, AP', L')$



Action-based fairness ⇝ LTL-fairness

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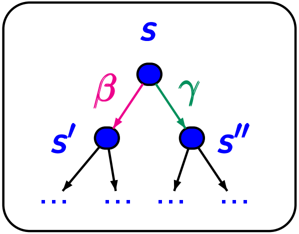

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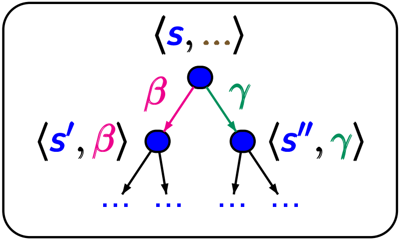
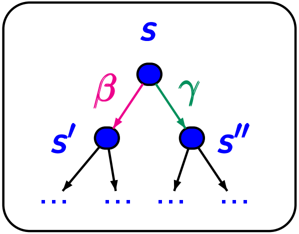


strong A-fairness
for $A \subseteq Act$

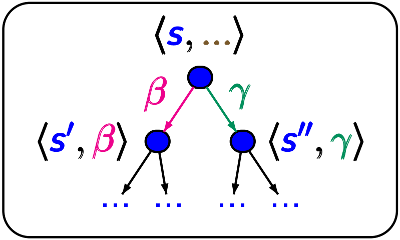
strong **LTL**-fairness
$\Box\Diamond enabled(A) \rightarrow \Box\Diamond taken(A)$

Action-based fairness ⇝ LTL-fairness

transition system
$\mathcal{T} = (S, Act, \rightarrow, \ldots)$



transition system
$\mathcal{T}' = (S \times Act, \ldots, AP', L')$



strong A-fairness
for $A \subseteq Act$

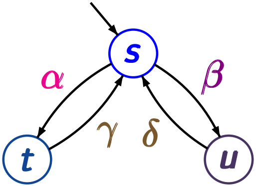
strong **LTL**-fairness
$\Box\Diamond enabled(A) \rightarrow \Box\Diamond taken(A)$

$enabled(A) \in L'(\langle s, \alpha \rangle)$ iff $s \xrightarrow{\beta} \ldots$ for some $\beta \in A$

$taken(A) \in L'(\langle s, \alpha \rangle)$ iff $\alpha \in A$

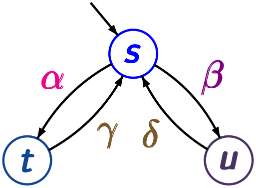
Example: action fairness ⇝ LTL-fairness

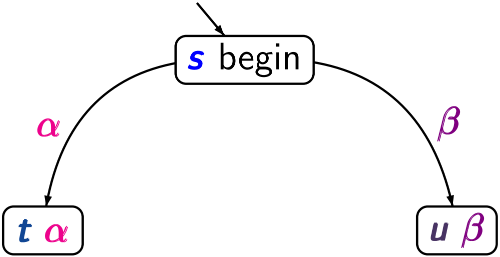
action-based fairness ⇝ LTL-fairness



Example: action fairness $\rightsquigarrow$ LTL-fairness

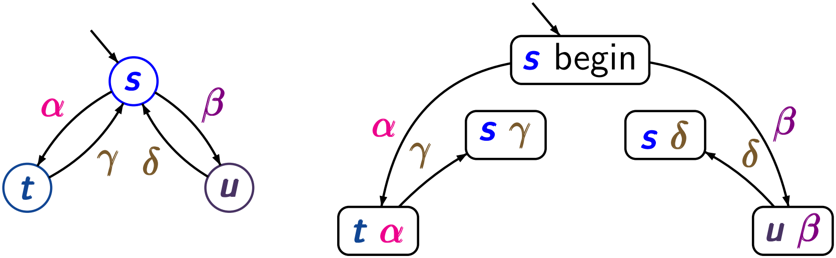
action-based fairness $\rightsquigarrow$ LTL-fairness





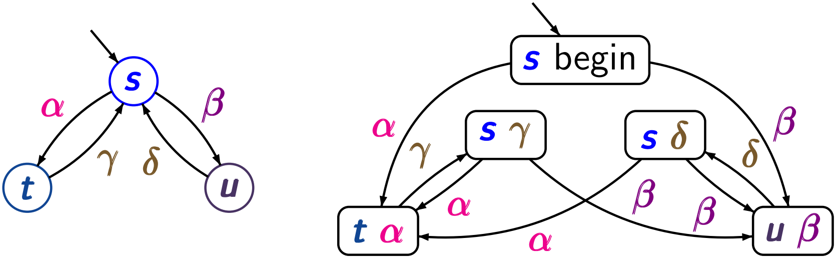
Example: action fairness ⇝ LTL-fairness

action-based fairness ⇝ LTL-fairness



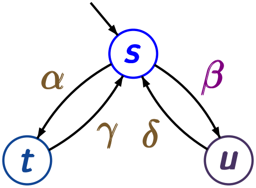
Example: action fairness ⇝ LTL-fairness

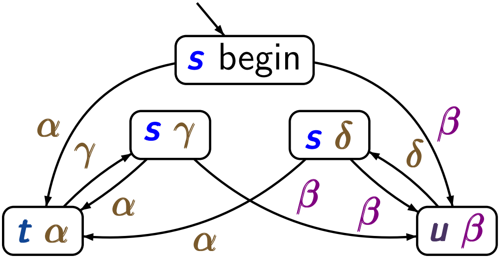
action-based fairness ⇝ LTL-fairness



Example: action fairness ⇝ LTL-fairness

action-based fairness ⇝ LTL-fairness

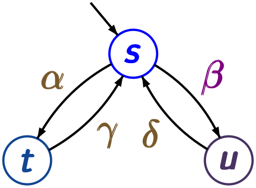


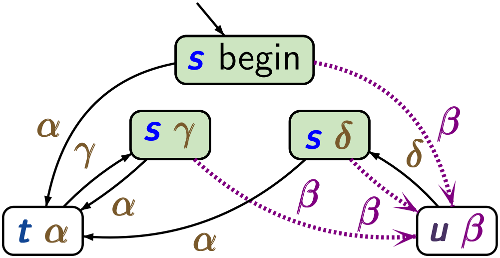


strong fairness for $\{\beta\}$:

$$\Box\Diamond\, \mathit{enabled}(\beta) \rightarrow \Box\Diamond\, \mathit{taken}(\beta)$$

action-based fairness ⇝ LTL-fairness





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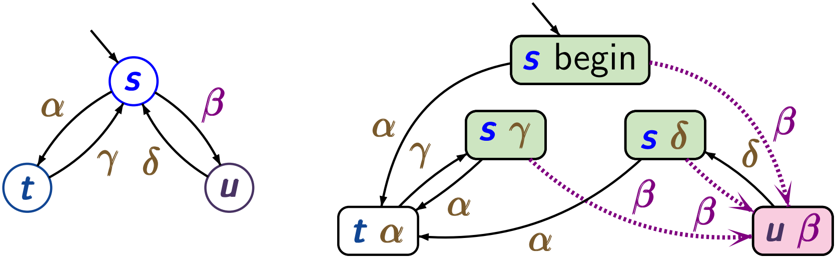
Example: action fairness $\rightsquigarrow$ LTL-fairness

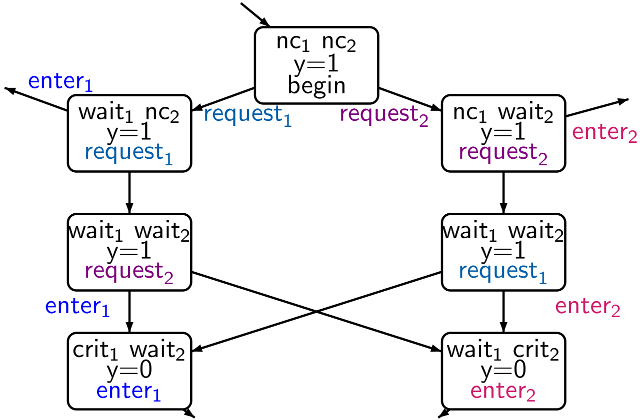
action-based fairness $\rightsquigarrow$ LTL-fairness



strong fairness for $\{\beta\}$:

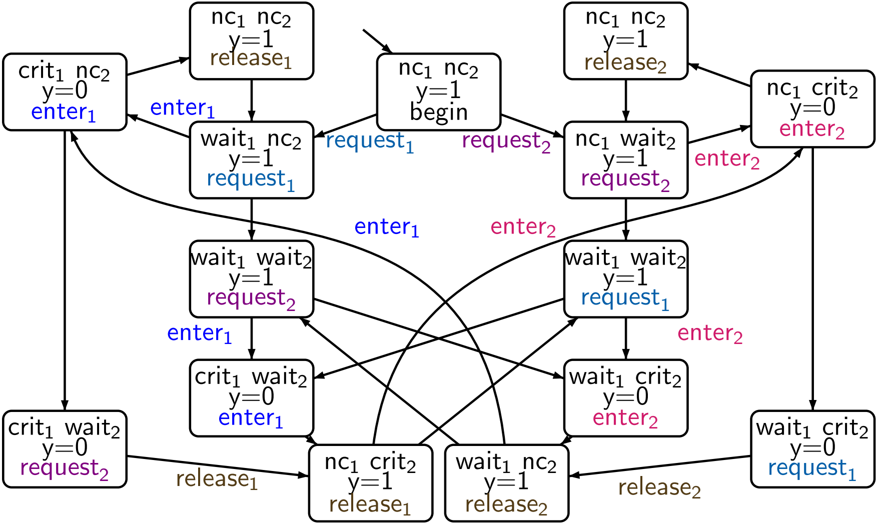
$$\Box\Diamond\, \textit{enabled}(\beta) \rightarrow \Box\Diamond\, \textit{taken}(\beta)$$

add additional variable last_action with domain Act $\cup$ {begin}



Example: mutual exclusion with semaphore

add additional variable last_action with domain Act ∪ {begin}



add additional variable last_action with domain Act ∪ {begin}